INTRODUCTION TO THEORETICAL GEOMECHANICS - SOLUTIONS MANUAL

Chloé Arson
Georgia Institute of Technology

CONTENTS

PART I GEOMECHANICS IN ELASTICITY

1 Review of tensor algebra **3**

Problems 3

2 Elements of Continuum Mechanics **19**

Problems 19

3 Analytical solutions of boundary value problems in linear elasticity **49**

Problems 49

4 Finite Element Method in linear elasticity **77**

Problems 77

5 Elements of poro-elasticity **125**

Problems 125

6 Finite Element Method for poro-elasticity **185**

Problems 185

7 Fundamental principles of plasticity **219**

Problems 219

8 Perfect plasticity in geomechanics 223

Problems 223

PART I

GEOMECHANICS IN ELASTICITY

CHAPTER 1

REVIEW OF TENSOR ALGEBRA

PROBLEMS

1.1 Prove the following equations:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$$
$$(\boldsymbol{a} \times \boldsymbol{d}) \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{c} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d})$$
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0}$$

Solution:
See Figure 1.1 for proof of:

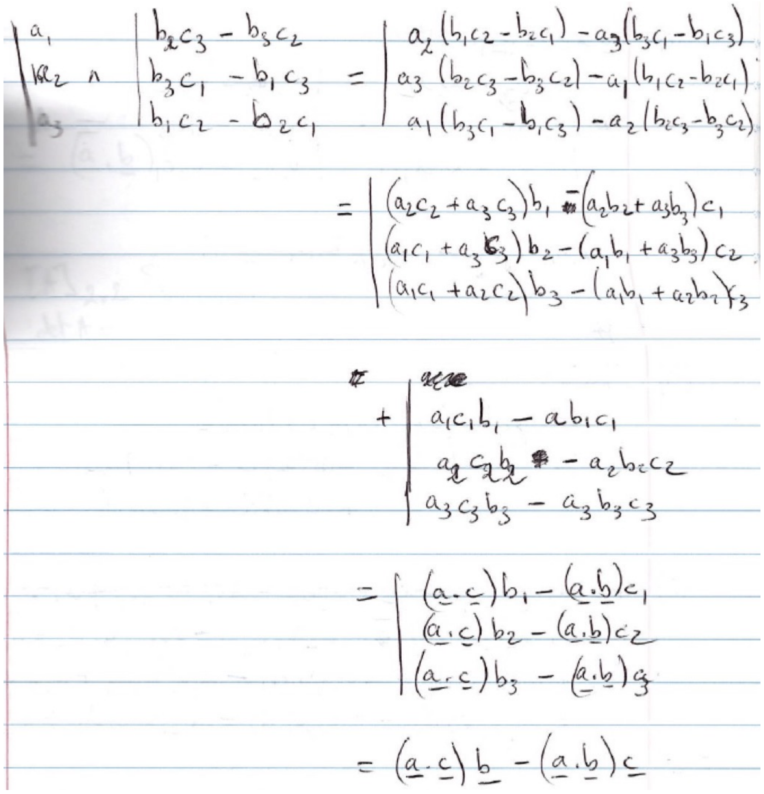
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$$

See Figure 1.2 for proof of:

$$(\boldsymbol{a} \times \boldsymbol{d}) \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{c} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d})$$

See Figure 1.3 for proof of:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0}$$



$$\left|\begin{array}{l} a_1 \\ a_2 \\ a_3 \end{array}\right. \wedge \left|\begin{array}{l} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{array}\right. = \left|\begin{array}{l} a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) \\ a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1) \\ a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2) \end{array}\right.$$

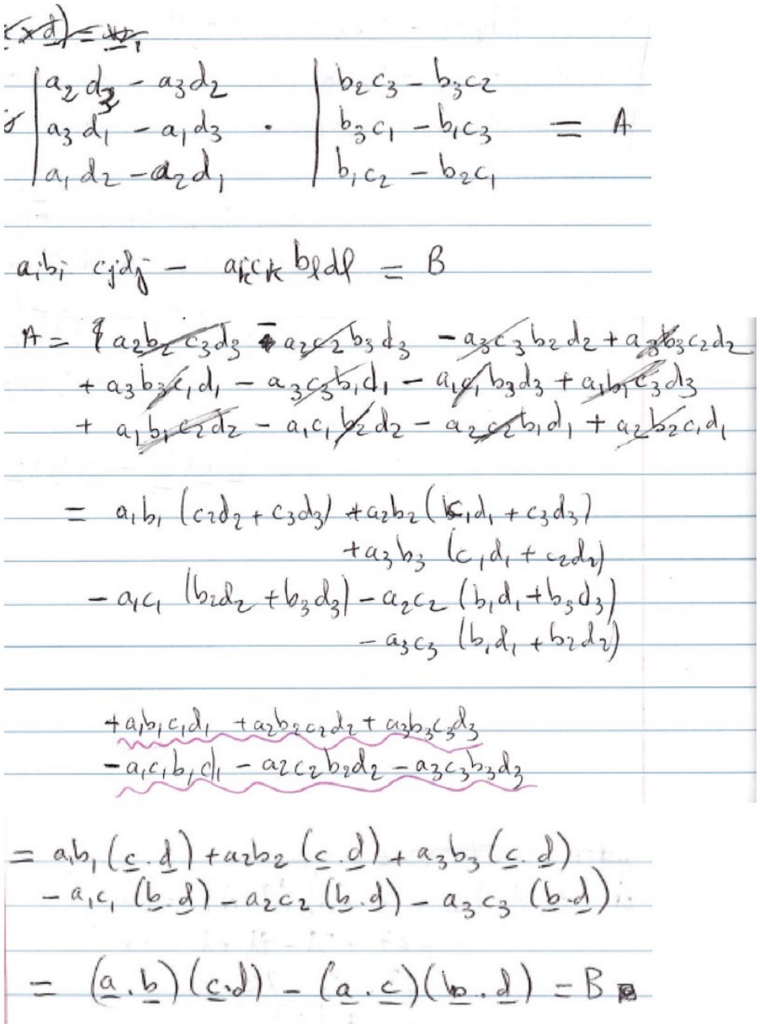
$$= \left|\begin{array}{l} (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1 \\ (a_1c_1 + a_3c_3)b_2 - (a_1b_1 + a_3b_3)c_2 \\ (a_1c_1 + a_2c_2)b_3 - (a_1b_1 + a_2b_2)c_3 \end{array}\right.$$

$$+ \left|\begin{array}{l} a_1c_1b_1 - ab_1c_1 \\ a_2c_2b_2 - a_2b_2c_2 \\ a_3c_3b_3 - a_3b_3c_3 \end{array}\right.$$

$$= \left|\begin{array}{l} (\underline{a}\cdot\underline{c})b_1 - (\underline{a}\cdot\underline{b})c_1 \\ (\underline{a}\cdot\underline{c})b_2 - (\underline{a}\cdot\underline{b})c_2 \\ (\underline{a}\cdot\underline{c})b_3 - (\underline{a}\cdot\underline{b})c_3 \end{array}\right.$$

$$= (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$$

Figure 1.1 Proof of Problem 1.1 - Equation 1.



$\times \underline{d}) =$ [illegible]

$$\begin{vmatrix} a_2 d_3 - a_3 d_2 \\ a_3 d_1 - a_1 d_3 \\ a_1 d_2 - a_2 d_1 \end{vmatrix} \cdot \begin{vmatrix} b_2 c_3 - b_3 c_2 \\ b_3 c_1 - b_1 c_3 \\ b_1 c_2 - b_2 c_1 \end{vmatrix} = A$$

$$a_i b_i \; c_j d_j - a_k c_k \, b_l d_l = B$$

$$\begin{aligned} A = \{ & a_2 \cancel{b_2} c_3 d_3 - a_2 \cancel{c_2} b_3 d_3 - \cancel{a_3} c_3 b_2 d_2 + a_3 \cancel{b_3} c_2 d_2 \\ & + a_3 b_3 \cancel{c_1} d_1 - a_3 \cancel{c_3} b_1 d_1 - a_1 \cancel{c_1} b_3 d_3 + a_1 \cancel{b_1} c_3 d_3 \\ & + a_1 b_1 \cancel{c_2} d_2 - a_1 c_1 \cancel{b_2} d_2 - a_2 \cancel{c_2} b_1 d_1 + a_2 \cancel{b_2} c_1 d_1 \end{aligned}$$

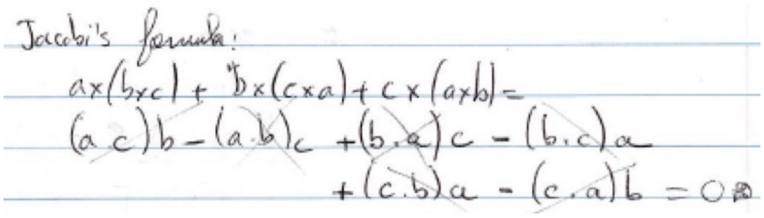
$$\begin{aligned} = \; & a_1 b_1 (c_2 d_2 + c_3 d_3) + a_2 b_2 (c_1 d_1 + c_3 d_3) \\ & \qquad + a_3 b_3 (c_1 d_1 + c_2 d_2) \\ & - a_1 c_1 (b_2 d_2 + b_3 d_3) - a_2 c_2 (b_1 d_1 + b_3 d_3) \\ & \qquad - a_3 c_3 (b_1 d_1 + b_2 d_2) \end{aligned}$$

$$\begin{aligned} & + a_1 b_1 c_1 d_1 + a_2 b_2 c_2 d_2 + a_3 b_3 c_3 d_3 \\ & - a_1 c_1 b_1 d_1 - a_2 c_2 b_2 d_2 - a_3 c_3 b_3 d_3 \end{aligned}$$

$$\begin{aligned} = \; & a_1 b_1 (\underline{c} \cdot \underline{d}) + a_2 b_2 (\underline{c} \cdot \underline{d}) + a_3 b_3 (\underline{c} \cdot \underline{d}) \\ & - a_1 c_1 (\underline{b} \cdot \underline{d}) - a_2 c_2 (\underline{b} \cdot \underline{d}) - a_3 c_3 (\underline{b} \cdot \underline{d}) \end{aligned}$$

$$= (\underline{a} \cdot \underline{b})(\underline{c} \cdot \underline{d}) - (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) = B \; ■$$

Figure 1.2 Proof of Problem 1.1 - Equation 2.



Jacobi's formula:

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) =$$

$$(a \cdot c) b - (a \cdot b) c + (b \cdot a) c - (b \cdot c) a$$

$$+ (c \cdot b) a - (c \cdot a) b = 0 \; ■$$

Figure 1.3 Proof of Problem 1.1 - Equation 3.

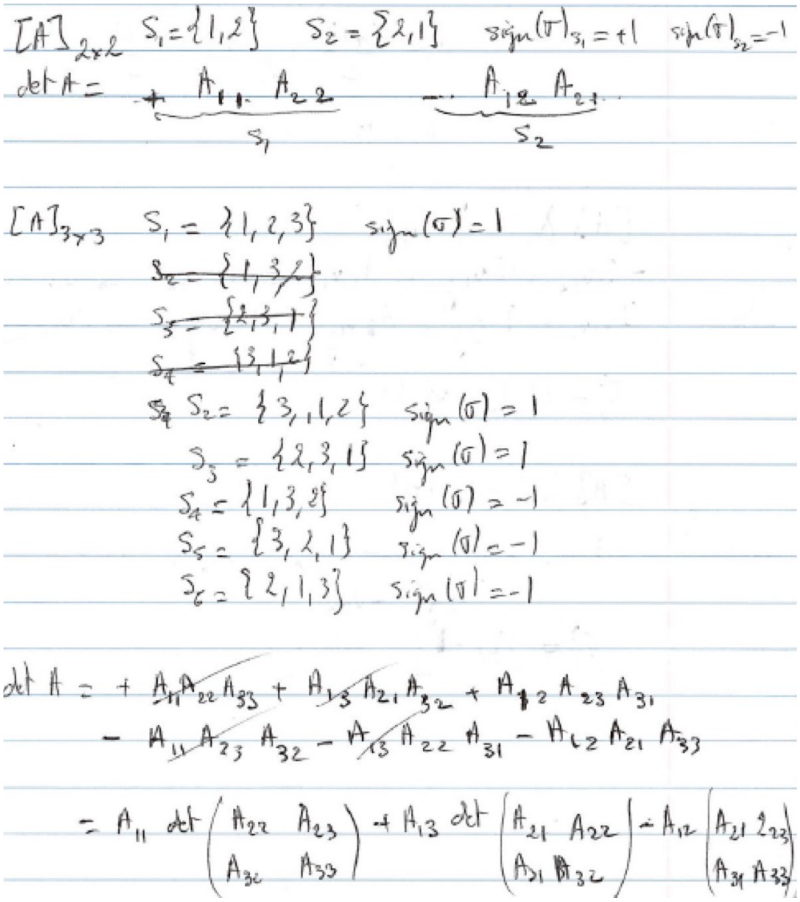
1.2 For a 2 x 2 matrix $[A]$, show that:

$$\det(A) = A_{11}A_{22} - A_{12}A_{21}$$

For a 3 x 3 matrix, show that:

$$\begin{aligned}\det(A) \quad &= \quad A_{11}\, det\left(\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}\right) \\ &\quad -A_{12}\, det\left(\begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix}\right) + A_{13}\, det\left(\begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}\right)\end{aligned}$$

Solution: see Figure 1.4.



$[A]_{2\times2}$ $S_1=\{1,2\}$ $S_2=\{2,1\}$ $sign(\sigma)_{S_1}=+1$ $sign(\sigma)_{S_2}=-1$

$\det A = \underbrace{+A_{11}A_{22}}_{S_1} \quad \underbrace{-A_{12}A_{21}}_{S_2}$

$[A]_{3\times3}$ $S_1=\{1,2,3\}$ $sign(\sigma)=1$

~~$S_2=\{1,3,2\}$~~

~~$S_3=\{2,3,1\}$~~

~~$S_4=\{3,1,2\}$~~

$S_2=\{3,1,2\}$ $sign(\sigma)=1$

$S_3=\{2,3,1\}$ $sign(\sigma)=1$

$S_4=\{1,3,2\}$ $sign(\sigma)=-1$

$S_5=\{3,2,1\}$ $sign(\sigma)=-1$

$S_6=\{2,1,3\}$ $sign(\sigma)=-1$

$\det A = +A_{11}A_{22}A_{33} + A_{13}A_{21}A_{32} + A_{12}A_{23}A_{31}$

$\quad - A_{11}A_{23}A_{32} - A_{13}A_{22}A_{31} - A_{12}A_{21}A_{33}$

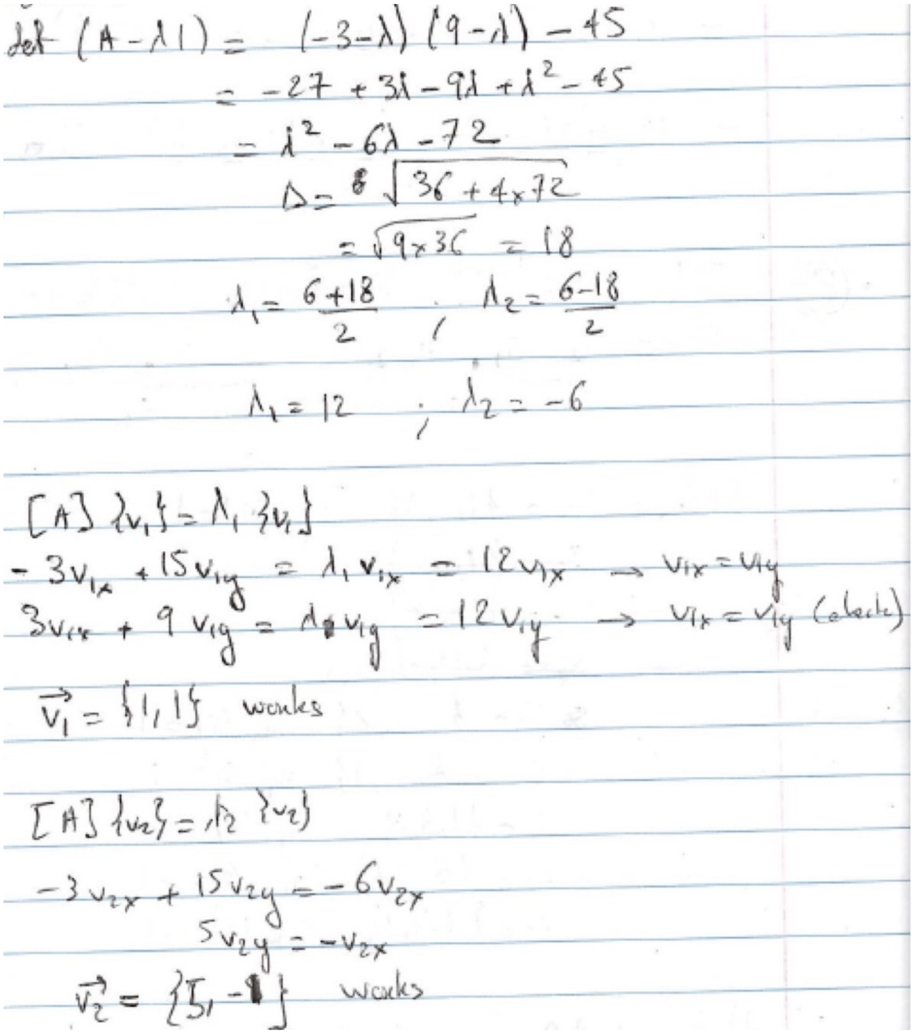
$= A_{11}\det\begin{pmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{pmatrix} + A_{13}\det\begin{pmatrix}A_{21} & A_{22}\\ A_{31} & A_{32}\end{pmatrix} - A_{12}\begin{pmatrix}A_{21} & A_{23}\\ A_{31} & A_{33}\end{pmatrix}$

Figure 1.4 Solution of Problem 1.2.

1.3 Find the eigenvalues of $[A]$, and for each eigenvalue, find an eigenvector, where:

$$[A] = \begin{bmatrix} -3 & 15 \\ 3 & 9 \end{bmatrix}$$

Solution: see Figure 1.5.



$$\det(A - \lambda 1) = (-3-\lambda)(9-\lambda) - 45$$
$$= -27 + 3\lambda - 9\lambda + \lambda^2 - 45$$
$$= \lambda^2 - 6\lambda - 72$$
$$\Delta = \sqrt{36 + 4 \times 72}$$
$$= \sqrt{9 \times 36} = 18$$
$$\lambda_1 = \frac{6+18}{2} \quad , \quad \lambda_2 = \frac{6-18}{2}$$
$$\lambda_1 = 12 \quad ; \quad \lambda_2 = -6$$

$$[A]\{v_1\} = \lambda_1 \{v_1\}$$
$$-3v_{1x} + 15v_{1y} = \lambda_1 v_{1x} = 12v_{1x} \rightarrow v_{1x} = v_{1y}$$
$$3v_{1x} + 9v_{1y} = \lambda_1 v_{1y} = 12v_{1y} \rightarrow v_{1x} = v_{1y} \text{ (check)}$$
$\vec{v_1} = \{1, 1\}$ works

$$[A]\{v_2\} = \lambda_2 \{v_2\}$$
$$-3v_{2x} + 15v_{2y} = -6v_{2x}$$
$$5v_{2y} = -v_{2x}$$
$\vec{v_2} = \{5, -1\}$ works

Figure 1.5 Solution of Problem 1.3.

1.4 Let us consider an Euclidian space of dimension 3, with the Cartesian coordinate system ($\boldsymbol{e}_1$,$\boldsymbol{e}_2$,$\boldsymbol{e}_3$). In the following, tensors noted in lower case are vectors, tensors noted with capital letters are tensors of order 2 and tensors noted in calligraphic font are tensors of order 4. Develop the following expressions in index notation:

$$\boldsymbol{a}\cdot\mathcal{T}\cdot\boldsymbol{b},\quad \boldsymbol{a}\otimes\boldsymbol{b}:\mathcal{T},\quad \mathcal{T}\otimes\boldsymbol{A}\,\vdots\,\boldsymbol{B}\otimes\boldsymbol{c}$$

$$\boldsymbol{a}\otimes\boldsymbol{b}:\mathcal{T}:\boldsymbol{c}\otimes\boldsymbol{d},\quad \mathcal{T}\,\vdots\,\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c}\otimes\boldsymbol{d}$$

Solution:

$$\boldsymbol{a}\cdot\mathcal{T}\cdot\boldsymbol{b} = a_i\mathcal{T}_{ijkl}b_l\,\boldsymbol{e}_j\otimes\boldsymbol{e}_k$$

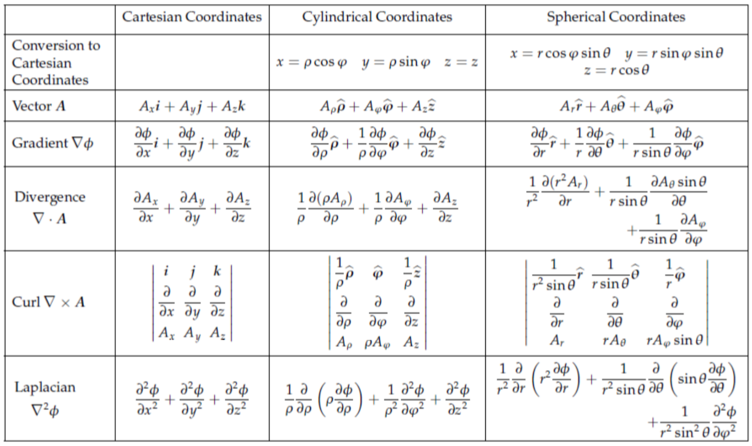
$$\boldsymbol{a}\otimes\boldsymbol{b}:\mathcal{T} = a_ib_j\mathcal{T}_{jik}\,\boldsymbol{e}_j\otimes\boldsymbol{e}_k$$

$$\mathcal{T}\otimes\boldsymbol{A}\,\vdots\,\boldsymbol{B}\otimes\boldsymbol{c} = \mathcal{T}_{ijkl}A_{mn}B_{nm}\mathcal{T}_{lopq}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_o\otimes\boldsymbol{e}_p\otimes\boldsymbol{e}_q$$

$$\boldsymbol{a}\otimes\boldsymbol{b}:\mathcal{T}:\boldsymbol{c}\otimes\boldsymbol{d} = a_ib_j\mathcal{T}_{jikl}c_ld_k$$

$$\mathcal{T}\,\vdots\,\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c}\otimes\boldsymbol{d} = \mathcal{T}_{ijkl}a_lb_kc_jd_i$$

1.5 Prove the formulas given in the following Table:



	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Cartesian Coordinates		$x=\rho\cos\varphi\quad y=\rho\sin\varphi\quad z=z$	$x=r\cos\varphi\sin\theta\quad y=r\sin\varphi\sin\theta$ $z=r\cos\theta$
Vector A	$A_xi+A_yj+A_zk$	$A_\rho\widehat{\rho}+A_\varphi\widehat{\varphi}+A_z\widehat{z}$	$A_r\widehat{r}+A_\theta\widehat{\theta}+A_\varphi\widehat{\varphi}$
Gradient $\nabla\phi$	$\frac{\partial\phi}{\partial x}i+\frac{\partial\phi}{\partial y}j+\frac{\partial\phi}{\partial z}k$	$\frac{\partial\phi}{\partial\rho}\widehat{\rho}+\frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\widehat{\varphi}+\frac{\partial\phi}{\partial z}\widehat{z}$	$\frac{\partial\phi}{\partial r}\widehat{r}+\frac{1}{r}\frac{\partial\phi}{\partial\theta}\widehat{\theta}+\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\widehat{\varphi}$
Divergence $\nabla\cdot A$	$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial A_\theta\sin\theta}{\partial\theta}$ $+\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times A$	$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{\rho}\widehat{\rho} & \widehat{\varphi} & \frac{1}{\rho}\widehat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{r^2\sin\theta}\widehat{r} & \frac{1}{r\sin\theta}\widehat{\theta} & \frac{1}{r}\widehat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian $\nabla^2\phi$	$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2}+\frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)$ $+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

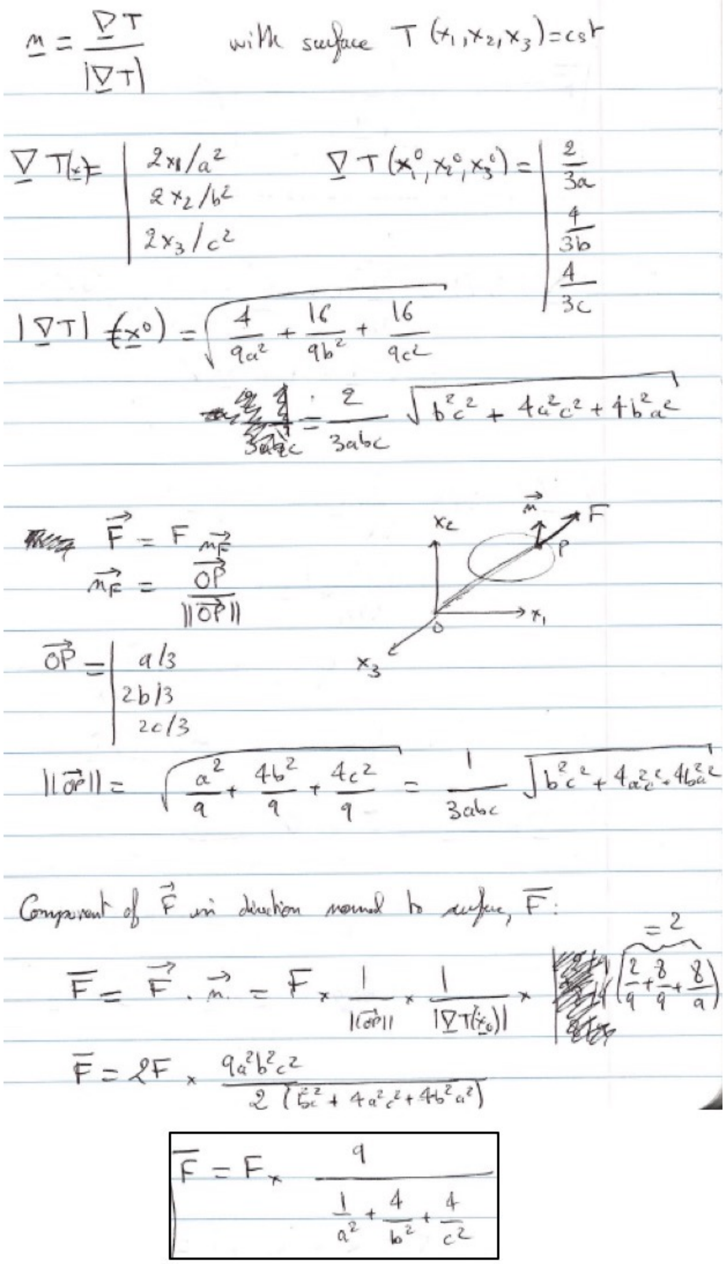
Solution:

1.6 A force of magnitude F acts in a direction radially away from the axes origin, at a point with coordinates ($a/3$, $2b/3$, $2c/3$) on the surface of the ellipsoid of equation:

$$\left(\frac{x_1}{a}\right)^2+\left(\frac{x_2}{b}\right)^2+\left(\frac{x_3}{c}\right)^2=1$$

Determine the component of the force in the direction normal to the surface.

Solution: see Figure 1.6.



$\underline{n} = \frac{\underline{\nabla} T}{|\underline{\nabla} T|}$ with surface $T(x_1,x_2,x_3)=cst$

$\underline{\nabla} T(\underline{x}) = \left| \begin{array}{l} 2x_1/a^2 \\ 2x_2/b^2 \\ 2x_3/c^2 \end{array} \right.$ $\underline{\nabla} T(x_1^0,x_2^0,x_3^0) = \left| \begin{array}{l} \frac{2}{3a} \\ \frac{4}{3b} \\ \frac{4}{3c} \end{array} \right.$

$|\underline{\nabla} T|(\underline{x}^0) = \sqrt{\frac{4}{9a^2}+\frac{16}{9b^2}+\frac{16}{9c^2}}$

$= \frac{2}{3abc}\sqrt{b^2c^2+4a^2c^2+4b^2a^2}$

$\vec{F} = F\,\vec{n}_F$

$\vec{n}_F = \frac{\vec{OP}}{\|\vec{OP}\|}$

$\vec{OP} = \left| \begin{array}{l} a/3 \\ 2b/3 \\ 2c/3 \end{array} \right.$

$\|\vec{OP}\| = \sqrt{\frac{a^2}{9}+\frac{4b^2}{9}+\frac{4c^2}{9}} = \frac{1}{3abc}\sqrt{b^2c^2+4a^2c^2+4b^2a^2}$

Component of $\vec{F}$ in direction normal to surface, $\bar{F}$:

$\bar{F} = \vec{F}\cdot\vec{n} = F \times \frac{1}{\|\vec{OP}\|} \times \frac{1}{|\underline{\nabla} T(\underline{x}_0)|} \times \overbrace{\left(\frac{2}{9}+\frac{8}{9}+\frac{8}{9}\right)}^{=2}$

$\bar{F} = 2F \times \frac{9a^2b^2c^2}{2\,(b^2c^2+4a^2c^2+4b^2a^2)}$

$\boxed{\bar{F} = F \times \frac{9}{\frac{1}{a^2}+\frac{4}{b^2}+\frac{4}{c^2}}}$

Figure 1.6 Solution of Problem 1.6.

1.7 In the following, U is a scalar and $\boldsymbol{a}$ is a vector. Prove the following equations:

$$div\,(\boldsymbol{\nabla} U) = \Delta U$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} U) = \mathbf{0}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{a}) = \mathbf{0}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) = \boldsymbol{\nabla}\,(\boldsymbol{\nabla} \cdot \boldsymbol{a}) - \Delta \boldsymbol{a}$$

Solution:
See Figure 1.7 for the proofs of:

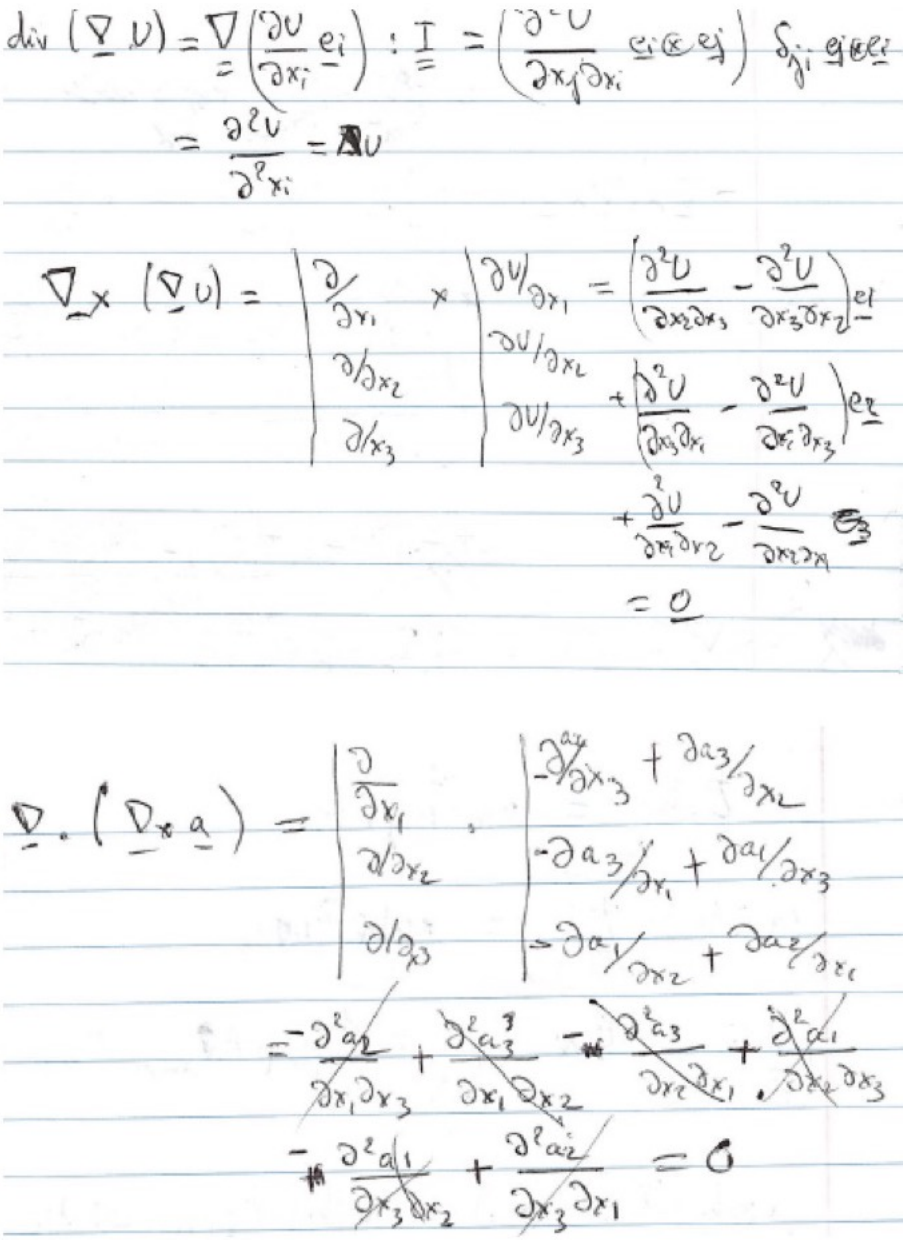
$$div\,(\boldsymbol{\nabla} U) = \Delta U$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} U) = \mathbf{0}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{a}) = \mathbf{0}$$

See Figure 1.8 for the proof of:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) = \boldsymbol{\nabla}\,(\boldsymbol{\nabla} \cdot \boldsymbol{a}) - \Delta \boldsymbol{a}$$



$$\operatorname{div}(\underline{\nabla}U) = \underline{\underline{\nabla}}\left(\frac{\partial U}{\partial x_i}\underline{e_i}\right) : \underline{\underline{I}} = \left(\frac{\partial^2 U}{\partial x_j \partial x_i}\underline{e_i} \otimes \underline{e_j}\right) \delta_{ji}\, \underline{e_j} \otimes \underline{e_i}$$

$$= \frac{\partial^2 U}{\partial^2 x_i} = \Delta U$$

$$\underline{\nabla} \times (\underline{\nabla} U) = \begin{vmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/x_3 \end{vmatrix} \times \begin{vmatrix} \partial U/\partial x_1 \\ \partial U/\partial x_2 \\ \partial U/\partial x_3 \end{vmatrix} = \left(\frac{\partial^2 U}{\partial x_2 \partial x_3} - \frac{\partial^2 U}{\partial x_3 \partial x_2}\right)\underline{e_1}$$

$$+ \left(\frac{\partial^2 U}{\partial x_3 \partial x_1} - \frac{\partial^2 U}{\partial x_1 \partial x_3}\right)\underline{e_2}$$

$$+ \frac{\partial^2 U}{\partial x_1 \partial x_2} - \frac{\partial^2 U}{\partial x_2 \partial x_1}\, e_3$$

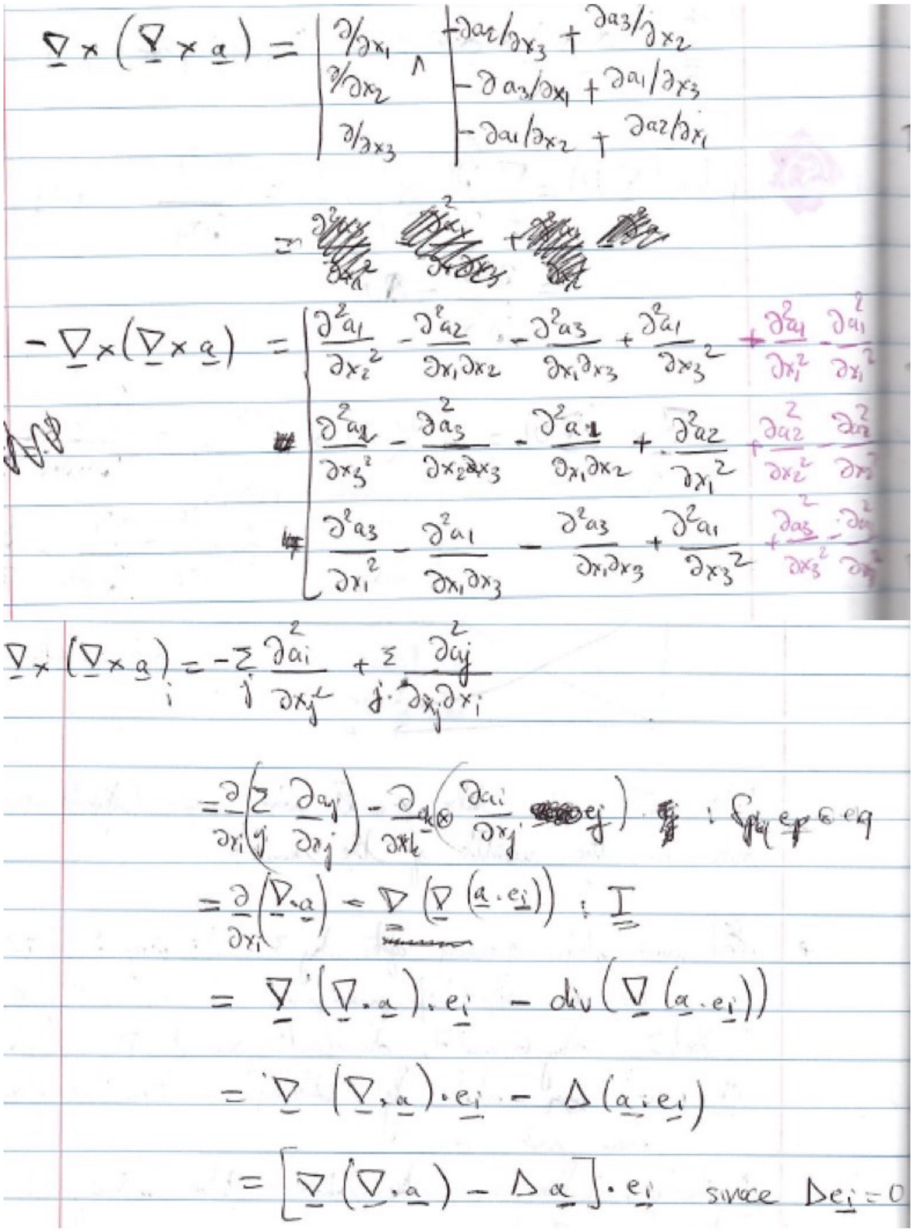
$$= \underline{0}$$

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{a}) = \begin{vmatrix} \dfrac{\partial}{\partial x_1} \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{vmatrix} \cdot \begin{vmatrix} -\partial a_2/\partial x_3 + \partial a_3/\partial x_2 \\ -\partial a_3/\partial x_1 + \partial a_1/\partial x_3 \\ -\partial a_1/\partial x_2 + \partial a_2/\partial x_1 \end{vmatrix}$$

$$= \cancel{-\frac{\partial^2 a_2}{\partial x_1 \partial x_3}} + \cancel{\frac{\partial^2 a_3}{\partial x_1 \partial x_2}} - \cancel{\frac{\partial^2 a_3}{\partial x_2 \partial x_1}} + \cancel{\frac{\partial^2 a_1}{\partial x_2 \partial x_3}}$$

$$- \cancel{\frac{\partial^2 a_1}{\partial x_3 \partial x_2}} + \cancel{\frac{\partial^2 a_2}{\partial x_3 \partial x_1}} = 0$$

Figure 1.7 Solution of Problem 1.7 - Equations 1-3.



$$\underline{\nabla}\times(\underline{\nabla}\times\underline{a}) = \begin{vmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{vmatrix} \wedge \begin{vmatrix} -\partial a_2/\partial x_3 + \partial a_3/\partial x_2 \\ -\partial a_3/\partial x_1 + \partial a_1/\partial x_3 \\ -\partial a_1/\partial x_2 + \partial a_2/\partial x_1 \end{vmatrix}$$

$$-\underline{\nabla}\times(\underline{\nabla}\times\underline{a}) = \left[\begin{array}{l} \frac{\partial^2 a_1}{\partial x_2^2} - \frac{\partial^2 a_2}{\partial x_1 \partial x_2} - \frac{\partial^2 a_3}{\partial x_1 \partial x_3} + \frac{\partial^2 a_1}{\partial x_3^2} + \frac{\partial^2 a_1}{\partial x_1^2} - \frac{\partial^2 a_1}{\partial x_1^2} \\ \frac{\partial^2 a_2}{\partial x_3^2} - \frac{\partial^2 a_3}{\partial x_2 \partial x_3} - \frac{\partial^2 a_1}{\partial x_1 \partial x_2} + \frac{\partial^2 a_2}{\partial x_1^2} + \frac{\partial^2 a_2}{\partial x_2^2} - \\ \frac{\partial^2 a_3}{\partial x_1^2} - \frac{\partial^2 a_1}{\partial x_1 \partial x_3} - \frac{\partial^2 a_3}{\partial x_1 \partial x_3} + \frac{\partial^2 a_1}{\partial x_3^2} + \frac{\partial^2 a_3}{\partial x_3^2} - \end{array}\right.$$

$$\underline{\nabla}\times(\underline{\nabla}\times\underline{a})_i = -\sum_j \frac{\partial^2 a_i}{\partial x_j^2} + \sum_j \frac{\partial^2 a_j}{\partial x_j \partial x_i}$$

$$= \frac{\partial}{\partial x_i}\left(\sum_j \frac{\partial a_j}{\partial x_j}\right) - \frac{\partial}{\partial x_k}\otimes\left(\frac{\partial a_i}{\partial x_j}\, e_j\right) : \delta_{pq}\, e_p \otimes e_q$$

$$= \frac{\partial}{\partial x_i}(\underline{\nabla}\cdot\underline{a}) - \underline{\underline{\nabla}}\left(\underline{\nabla}(\underline{a}\cdot\underline{e_i})\right) : \underline{\underline{I}}$$

$$= \underline{\nabla}(\underline{\nabla}\cdot\underline{a})\cdot\underline{e_i} - \mathrm{div}\left(\underline{\nabla}(\underline{a}\cdot\underline{e_i})\right)$$

$$= \underline{\nabla}(\underline{\nabla}\cdot\underline{a})\cdot\underline{e_i} - \Delta(\underline{a}\cdot\underline{e_i})$$

$$= \left[\underline{\nabla}(\underline{\nabla}\cdot\underline{a}) - \Delta\underline{a}\right]\cdot\underline{e_i} \quad \text{since } \Delta\underline{e_i} = 0$$

Figure 1.8 Solution of Problem 1.7 - Equation 4.

1.8 In the following, U and V are scalars and $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors. Prove the following equations:

$$\boldsymbol{\nabla}(U\,V) = V\,\boldsymbol{\nabla}(U) + U\,\boldsymbol{\nabla}(V)$$

$$\boldsymbol{\nabla}\cdot(U\,\boldsymbol{a}) = \boldsymbol{a}\cdot\boldsymbol{\nabla}(U) + U\,(\boldsymbol{\nabla}\cdot\boldsymbol{a})$$

$$\boldsymbol{\nabla}\times(U\,\boldsymbol{a}) = \boldsymbol{\nabla}(U)\times\boldsymbol{a} + U\,(\boldsymbol{\nabla}\times\boldsymbol{a})$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a}) - \boldsymbol{a}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})$$

$$\boldsymbol{\nabla}\times(\boldsymbol{a}\times\boldsymbol{b}) = (\boldsymbol{\nabla}\cdot\boldsymbol{b})\,\boldsymbol{a} - (\boldsymbol{\nabla}\cdot\boldsymbol{a})\,\boldsymbol{b} + (\boldsymbol{b}\cdot\boldsymbol{\nabla})\,\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{\nabla})\,\boldsymbol{b}$$

$$\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{b}) = \boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{b}) + \boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{a}) + (\boldsymbol{b}\cdot\boldsymbol{\nabla})\,\boldsymbol{a} + (\boldsymbol{a}\cdot\boldsymbol{\nabla})\,\boldsymbol{b}$$

Solution:

See Figure 1.9 for the proofs of:

$$\boldsymbol{\nabla}(U\,V) = V\,\boldsymbol{\nabla}(U) + U\,\boldsymbol{\nabla}(V)$$

$$\boldsymbol{\nabla}\cdot(U\,\boldsymbol{a}) = \boldsymbol{a}\cdot\boldsymbol{\nabla}(U) + U\,(\boldsymbol{\nabla}\cdot\boldsymbol{a})$$

$$\boldsymbol{\nabla}\times(U\,\boldsymbol{a}) = \boldsymbol{\nabla}(U)\times\boldsymbol{a} + U\,(\boldsymbol{\nabla}\times\boldsymbol{a})$$

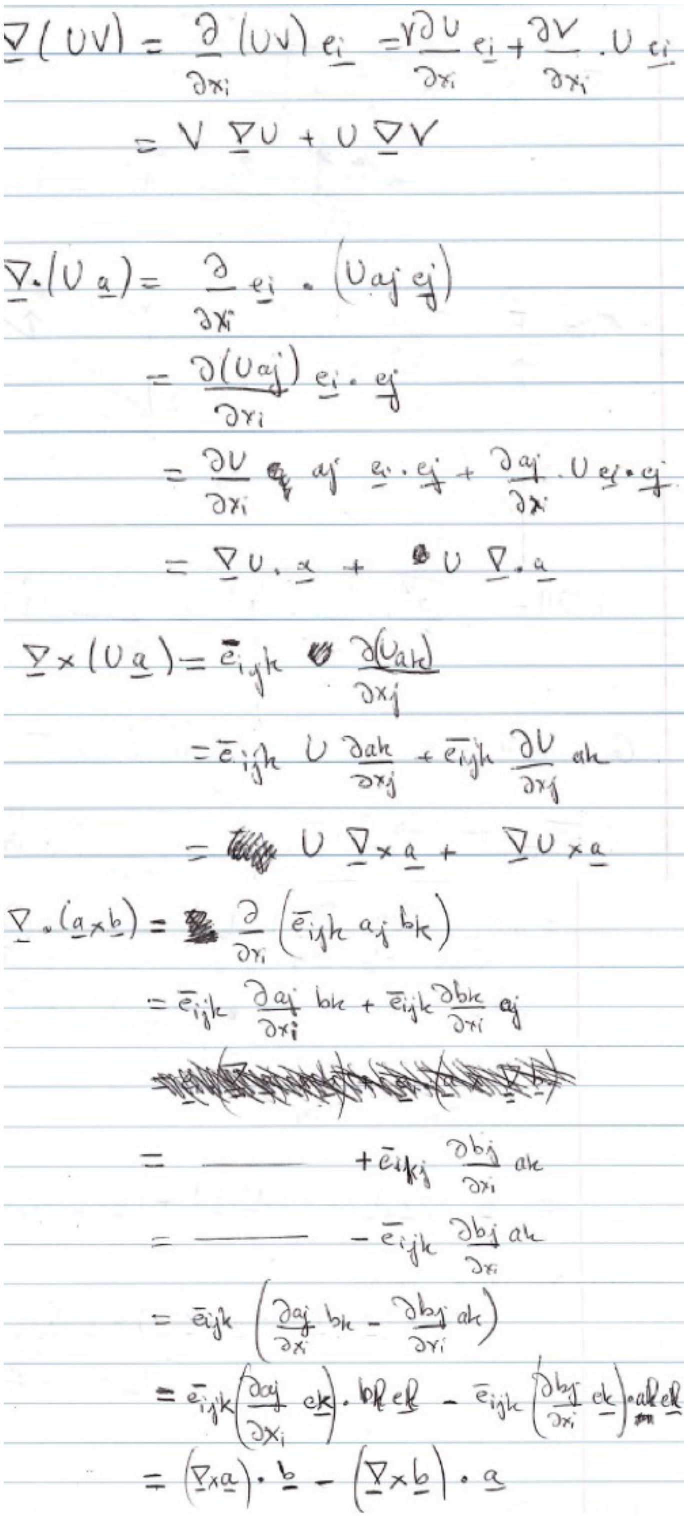
$$\boldsymbol{\nabla}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a}) - \boldsymbol{a}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})$$

See Figure 1.10 for the proof of:

$$\boldsymbol{\nabla}\times(\boldsymbol{a}\times\boldsymbol{b}) = (\boldsymbol{\nabla}\cdot\boldsymbol{b})\,\boldsymbol{a} - (\boldsymbol{\nabla}\cdot\boldsymbol{a})\,\boldsymbol{b} + (\boldsymbol{b}\cdot\boldsymbol{\nabla})\,\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{\nabla})\,\boldsymbol{b}$$

See Figure 1.11 for the proof of:

$$\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{b}) = \boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{b}) + \boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{a}) + (\boldsymbol{b}\cdot\boldsymbol{\nabla})\,\boldsymbol{a} + (\boldsymbol{a}\cdot\boldsymbol{\nabla})\,\boldsymbol{b}$$



$$\underline{\nabla}(UV) = \frac{\partial}{\partial x_i}(UV)\,\underline{e_i} = V\frac{\partial U}{\partial x_i}\underline{e_i} + \frac{\partial V}{\partial x_i}.U\,\underline{e_i}$$

$$= V\,\underline{\nabla}U + U\,\underline{\nabla}V$$

$$\underline{\nabla}\cdot(U\,\underline{a}) = \frac{\partial}{\partial x_i}\underline{e_i}\cdot(U a_j\,\underline{e_j})$$

$$= \frac{\partial(U a_j)}{\partial x_i}\underline{e_i}\cdot\underline{e_j}$$

$$= \frac{\partial U}{\partial x_i}\,a_j\,\underline{e_i}\cdot\underline{e_j} + \frac{\partial a_j}{\partial x_i}.U\,\underline{e_i}\cdot\underline{e_j}$$

$$= \underline{\nabla}U\cdot\underline{a} + U\,\underline{\nabla}\cdot\underline{a}$$

$$\underline{\nabla}\times(U\,\underline{a}) = \bar{e}_{ijk}\frac{\partial(U a_k)}{\partial x_j}$$

$$= \bar{e}_{ijk}\,U\frac{\partial a_k}{\partial x_j} + \bar{e}_{ijk}\frac{\partial U}{\partial x_j}a_k$$

$$= U\,\underline{\nabla}\times\underline{a} + \underline{\nabla}U\times\underline{a}$$

$$\underline{\nabla}\cdot(\underline{a}\times\underline{b}) = \frac{\partial}{\partial x_i}\left(\bar{e}_{ijk}\,a_j\,b_k\right)$$

$$= \bar{e}_{ijk}\frac{\partial a_j}{\partial x_i}b_k + \bar{e}_{ijk}\frac{\partial b_k}{\partial x_i}a_j$$

$$= \underline{\qquad\qquad} + \bar{e}_{ikj}\frac{\partial b_j}{\partial x_i}a_k$$

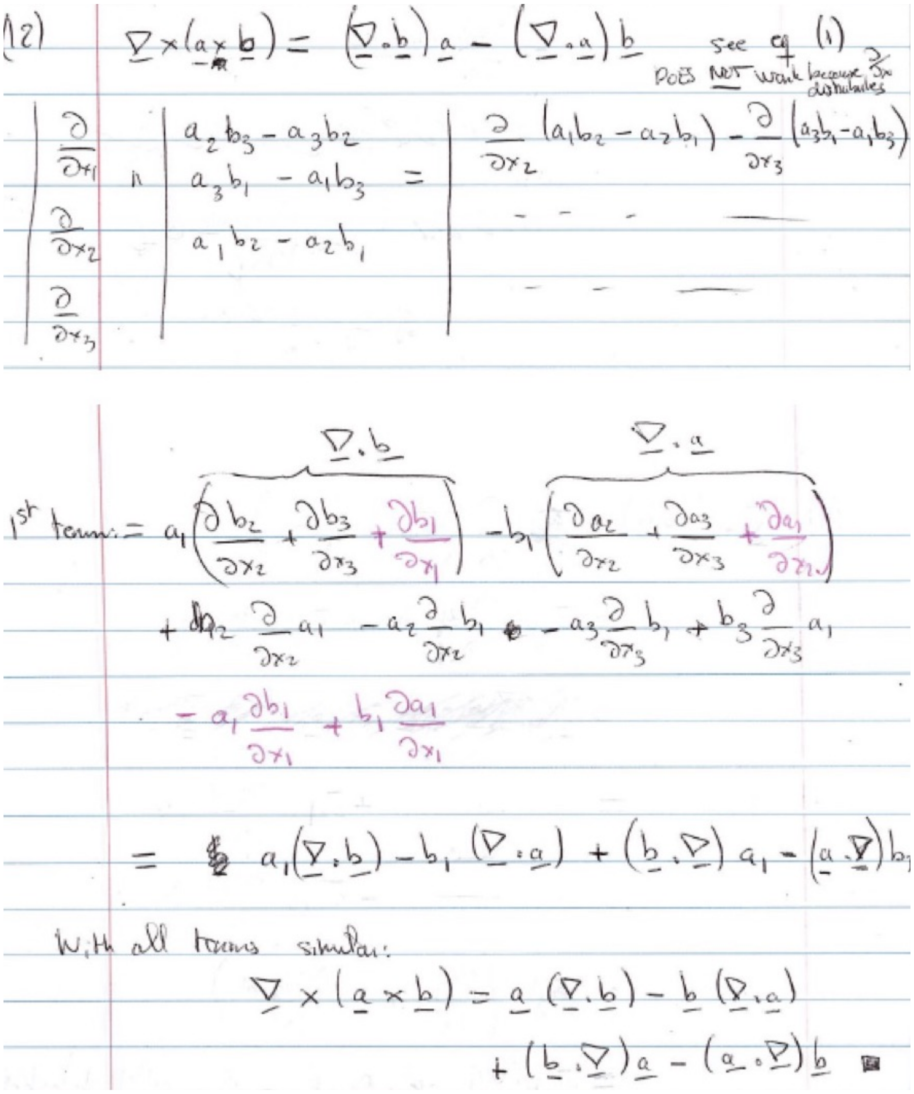
$$= \underline{\qquad\qquad} - \bar{e}_{ijk}\frac{\partial b_j}{\partial x_i}a_k$$

$$= \bar{e}_{ijk}\left(\frac{\partial a_j}{\partial x_i}b_k - \frac{\partial b_j}{\partial x_i}a_k\right)$$

$$= \bar{e}_{ijk}\left(\frac{\partial a_j}{\partial x_i}\underline{e_k}\right)\cdot b_k\underline{e_k} - \bar{e}_{ijk}\left(\frac{\partial b_j}{\partial x_i}\underline{e_k}\right)\cdot a_k\underline{e_k}$$

$$= (\underline{\nabla}\times\underline{a})\cdot\underline{b} - (\underline{\nabla}\times\underline{b})\cdot\underline{a}$$

Figure 1.9 Solution of Problem 1.8 - Equations 1-4.



12) $\underline{\nabla} \times (\underline{a} \times \underline{b}) = (\underline{\nabla} \cdot \underline{b})\,\underline{a} - (\underline{\nabla} \cdot \underline{a})\,\underline{b}$ see eq (1)

DOES NOT work because $\frac{\partial}{\partial x}$ distributes

$$\left|\begin{array}{c} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{array}\right| \wedge \left|\begin{array}{l} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{array}\right| = \left|\begin{array}{l} \frac{\partial}{\partial x_2}(a_1 b_2 - a_2 b_1) - \frac{\partial}{\partial x_3}(a_3 b_1 - a_1 b_3) \\ - - - \quad — \\ - - — \end{array}\right.$$

$$1^{st}\ \text{term} = a_1 \overbrace{\left(\frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3} + \frac{\partial b_1}{\partial x_1}\right)}^{\underline{\nabla} \cdot \underline{b}} - b_1 \overbrace{\left(\frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} + \frac{\partial a_1}{\partial x_1}\right)}^{\underline{\nabla} \cdot \underline{a}}$$

$$+ b_2 \frac{\partial}{\partial x_2} a_1 - a_2 \frac{\partial}{\partial x_2} b_1 - a_3 \frac{\partial}{\partial x_3} b_1 + b_3 \frac{\partial}{\partial x_3} a_1$$

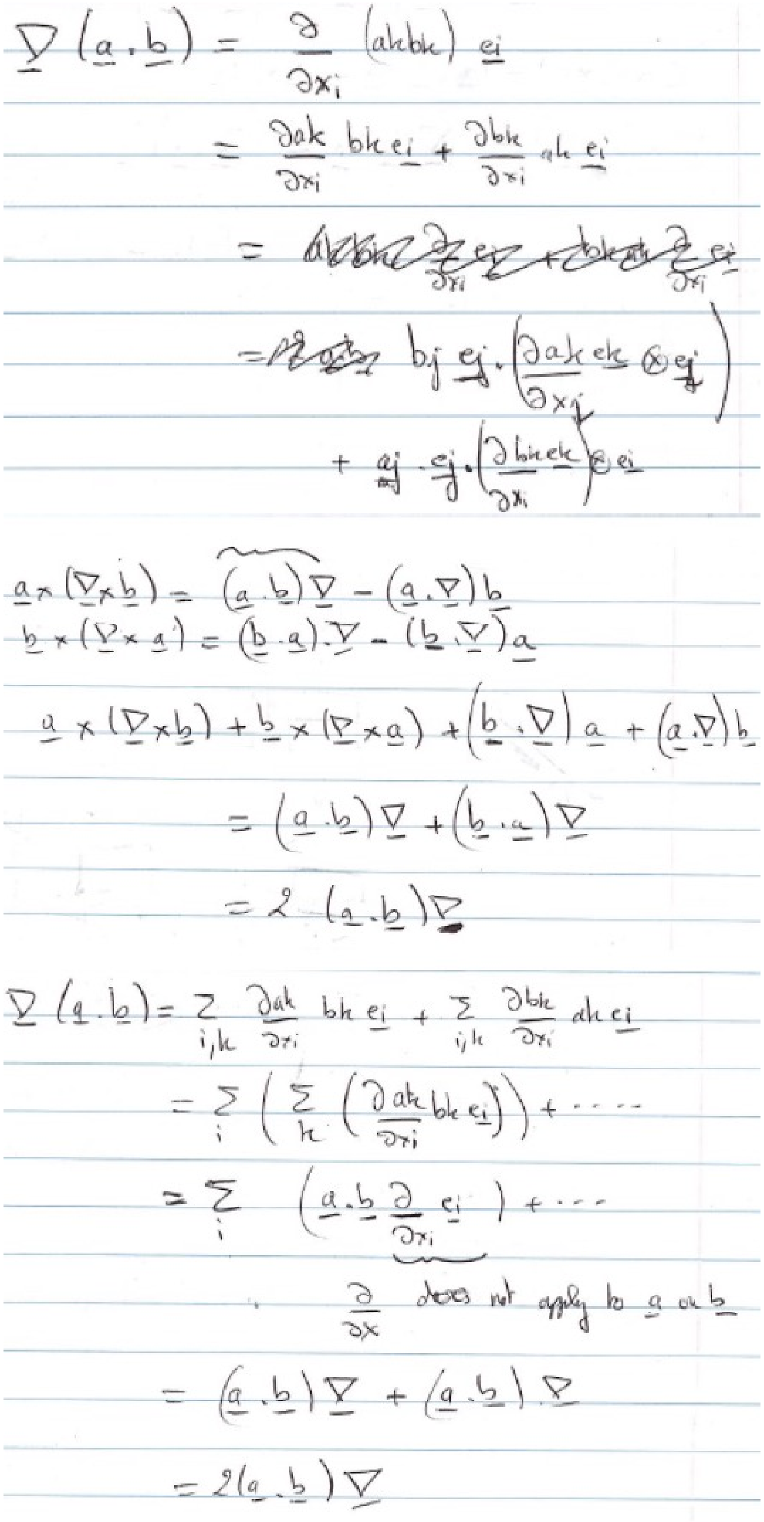
$$- a_1 \frac{\partial b_1}{\partial x_1} + b_1 \frac{\partial a_1}{\partial x_1}$$

$$= a_1 (\underline{\nabla} \cdot \underline{b}) - b_1 (\underline{\nabla} \cdot \underline{a}) + (\underline{b} \cdot \underline{\nabla})\, a_1 - (\underline{a} \cdot \underline{\nabla})\, b$$

With all terms similar:

$$\underline{\nabla} \times (\underline{a} \times \underline{b}) = \underline{a}\,(\underline{\nabla} \cdot \underline{b}) - \underline{b}\,(\underline{\nabla} \cdot \underline{a})$$
$$+ (\underline{b} \cdot \underline{\nabla})\,\underline{a} - (\underline{a} \cdot \underline{\nabla})\,\underline{b} \quad ■$$

Figure 1.10 Solution of Problem 1.8 - Equation 5.



$$\underline{\nabla}(\underline{a}\cdot\underline{b}) = \frac{\partial}{\partial x_i}(a_k b_k)\,\underline{e_i}$$

$$= \frac{\partial a_k}{\partial x_i} b_k \underline{e_i} + \frac{\partial b_k}{\partial x_i} a_k \underline{e_i}$$

$$= [illegible]$$

$$= b_j\,\underline{e_j}\cdot\left(\frac{\partial a_k \underline{e_k}}{\partial x_j}\otimes\underline{e_j}\right)$$

$$+ \underline{a_j}\cdot\underline{e_j}\cdot\left(\frac{\partial b_k \underline{e_k}}{\partial x_i}\right)\otimes\underline{e_i}$$

$$\underline{a}\times(\underline{\nabla}\times\underline{b}) = \overbrace{(\underline{a}\cdot\underline{b})\underline{\nabla}} - (\underline{a}\cdot\underline{\nabla})\underline{b}$$

$$\underline{b}\times(\underline{\nabla}\times\underline{a}) = (\underline{b}\cdot\underline{a})\cdot\underline{\nabla} - (\underline{b}\cdot\underline{\nabla})\underline{a}$$

$$\underline{a}\times(\underline{\nabla}\times\underline{b}) + \underline{b}\times(\underline{\nabla}\times\underline{a}) + (\underline{b}\cdot\underline{\nabla})\,\underline{a} + (\underline{a}\cdot\underline{\nabla})\underline{b}$$

$$= (\underline{a}\cdot\underline{b})\underline{\nabla} + (\underline{b}\cdot\underline{a})\underline{\nabla}$$

$$= 2\,(\underline{a}\cdot\underline{b})\underline{\nabla}$$

$$\underline{\nabla}(\underline{a}\cdot\underline{b}) = \sum_{i,k}\frac{\partial a_k}{\partial x_i} b_k \underline{e_i} + \sum_{i,k}\frac{\partial b_k}{\partial x_i} a_k \underline{e_i}$$

$$= \sum_i\left(\sum_k\left(\frac{\partial a_k}{\partial x_i} b_k \underline{e_i}\right)\right) + \cdots$$

$$= \sum_i\left(\underline{a}\cdot\underline{b}\,\underbrace{\frac{\partial}{\partial x_i}\underline{e_i}}\right) + \cdots$$

$\frac{\partial}{\partial x}$ does not apply to $\underline{a}$ or $\underline{b}$

$$= (\underline{a}\cdot\underline{b})\underline{\nabla} + (\underline{a}\cdot\underline{b})\cdot\underline{\nabla}$$

$$= 2(\underline{a}\cdot\underline{b})\underline{\nabla}$$

Figure 1.11 Solution of Problem 1.8 - Equation 6.

1.9 Homework 1 - Problem 1

We define the third-order permutation tensor $\overline{e}$ as follows:

$$\forall i, j, k, \quad \overline{e}_{ijk} = \begin{cases} 0 \text{ when two indices are equal} \\ +1 \text{ when the indices are 1,2,3 or an even permutation of 1,2,3} \\ -1 \text{ when the indices are an odd permutation of 1,2,3} \end{cases}$$

1. Show that the components of the permutation tensor can be calculated as follows:

$$\overline{e}_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$$

2. Calculate hte numerical expression of the following:

$$\delta_{ij}\delta_{ij}; \quad \overline{e}_{ijk}\overline{e}_{ijk}; \quad \delta_{ij}\overline{e}_{ijk}; \quad \overline{e}_{ijk}\overline{e}_{kji}$$

Solution:

1. Let us pose $A = (i-j)(j-k)(k-i)/2$. By construction, $(i-j)(j-k)(k-i)$ is equal to zero whenever two indices are equal. So $A = 0$ whenever two indices are equal. Now suppose that i, j and k are all different from one another. There are six possible sets of indices, including three even permutations: (1,2,3); (3,1,2) and (2,3,1), and including three odd permutations: (1,3,2); (2,1,3) and (3,2,1). If we test each of the six sets in the expression of A, we find that $A = 1$ when permutations are even and $A = -1$ for odd permutations. As a result, $A = \overline{e}_{ijk}$.

2. Let us calculate the given expressions.

$$\delta_{ij}\delta_{ij} = \delta_{ii} = \sum_{i=1}^{1}(1) = 3$$

$$\overline{e}_{ijk}\overline{e}_{ijk} = \sum_{i,j,k}\left(\overline{e}_{ijk}\right)^2 = \sum_{i\neq j, j\neq k, k\neq i}(1) = 6$$

since there exists six sets of permutations for which the three indices are different from one another.

$$\delta_{ij}\overline{e}_{ijk} = \overline{e}_{iik} = 0$$

since the permutation tensor is zero whenever two indices are equal.

$$\overline{e}_{ijk}\overline{e}_{kji} = \overline{e}_{ijk} \times \left(-\overline{e}_{ijk}\right) = -\overline{e}_{ijk}\overline{e}_{ijk} = -6$$

according to the previous calculations.

1.10 Homework 1 - Problem 2

In the following, $\underline{v}$ and $\underline{r}$ are vectors and ϕ is a scalar. Prove the following relations:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) = \underline{\nabla} \cdot (\underline{\nabla} \cdot \underline{v}) - \underline{\nabla}^2 (\underline{v})$$

$$\underline{\nabla} \times (\phi \underline{v}) = \phi\left(\underline{\nabla} \times \underline{v}\right) - \underline{v} \times \underline{\nabla}\phi$$

$$\underline{\nabla} \cdot \underline{r} = 3$$

$$\underline{\nabla} \times \underline{r} = \underline{0}$$

in which $\underline{r} = x_i \underline{e}_i$ is the position vector, and $r = \sqrt{x_i x_i}$ is the magnitude of the position vector. *Hint: Use index notations to prove the relations above.*

Solution:

For the first relationship:

$$\begin{aligned}
\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) \cdot \underline{e}_1 &= \left(\frac{\partial^2 v_2}{\partial x_1 \partial x_2} - \frac{\partial^2 v_1}{\partial x_2^2}\right) - \left(\frac{\partial^2 v_1}{\partial x_3^2} - \frac{\partial^2 v_3}{\partial x_1 \partial x_3}\right) \\
&= \frac{\partial}{\partial x_1}\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) - \left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_3^2}\right) \\
&= \frac{\partial}{\partial x_1}\left(\underline{\nabla} \cdot \underline{v}\right) - \underline{\nabla}^2\left(\underline{v}\right) \cdot \underline{e}_1 \\
&= \underline{\nabla}\left(\underline{\nabla} \cdot \underline{v}\right) \cdot \underline{e}_1 - \underline{\nabla}^2\left(\underline{v}\right) \cdot \underline{e}_1
\end{aligned}$$

and the same result can be obtained for the other components, from which we deduce:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) = \underline{\nabla} \cdot \left(\underline{\nabla} \cdot \underline{v}\right) - \underline{\nabla}^2\left(\underline{v}\right)$$

For the second relationship, we can use the distributivity of the tensorial product:

$$\underline{\nabla} \times (\phi \underline{v}) = \underline{\nabla}\phi \times \underline{v} + \phi\left(\underline{\nabla} \times \underline{v}\right) = \phi\left(\underline{\nabla} \times \underline{v}\right) - \underline{v} \times \underline{\nabla}\phi$$

For the third relationship:

$$\underline{\nabla} \cdot \underline{r} = \sum_{i=1}^{3}\sum_{i=j}^{3} \frac{\partial x_i}{\partial x_j} \underline{e}_i \cdot \underline{e}_j = \sum_{i=1}^{3}\sum_{i=j}^{3} \frac{\partial x_i}{\partial x_j}\delta_{ij} = \sum_{i=1}^{3}\frac{\partial x_i}{\partial x_i}\sum_{i=1}^{3}(1) = 3$$

For the fourth relationship:

$$\underline{\nabla} \times \underline{r} = \begin{cases} \frac{\partial x_3}{\partial x_2} - \frac{\partial x_2}{\partial x_3} \\ \frac{\partial x_1}{\partial x_3} - \frac{\partial x_3}{\partial x_1} \\ \frac{\partial x_2}{\partial x_1} - \frac{\partial x_1}{\partial x_2} \end{cases} = \begin{cases} 0 \\ 0 \\ 0 \end{cases}$$

CHAPTER 2

ELEMENTS OF CONTINUUM MECHANICS

PROBLEMS

2.1 We consider the 2D domain $0 < x_1 < L$, $-c < x_2 < c$ (in Cartesian coordinates). We define: $I = 4c^3/3$. The state of stress in the domain is given as follows:

$$\sigma_{11} = \frac{p}{I}\left(x_1^2 x_2 - \frac{2}{3}x_2^3\right), \quad \sigma_{22} = \frac{p}{I}\left(\frac{1}{3}x_2^3 - c^2 x_2 + \frac{2}{3}c^3\right), \quad \sigma_{12} = \frac{p}{I}\left(\left(c^2 - x_2^2\right)x_1\right)$$

a. Show that the state of stress is in equilibrium, i.e., $\underline{div}\left(\underline{\underline{\sigma}}\right) = \underline{0}$.
b. Calculate the state of stress on each of the four sides of the domain.
c. Calculate the resulting force that is applied on the face $x_1 = x_{10}$ by the part of the domain $x_1 \geq x_{10}$. Calculate the resulting moment at the point $(x_{10}, 0)$.
d. Suppose that $c << L$. Give a loading boundary condition that approximates the stress field given in the equation above.
e. Numerical application: consider a plane wing (shaped as a parallelepiped) that is 20m long (i.e., L=20m), and that has a half width c=1 cm. The wing is subjected to a uniformly distributed lifting surface force $p = C_z \rho_a V^2/2$. We give: C_z=0.8; V=200 m/s; ρ_a=1 kg/m^3. Calculate σ_{11}^{max}.

Solution:
a. Calculate:

$$\frac{\partial \sigma_{ij}(x_1, x_2)}{\partial x_j} \quad \text{for i=1,2}$$

and verify that $\underline{div}\left(\underline{\underline{\sigma}}\right) = \underline{0}$.

b. On the top face, $x_2 = +c$ and the normal vector is $\boldsymbol{e}_2$, therefore the stress tensor components are:

$$\sigma_{12}(x_1, x_2 = +c) = 0, \quad \sigma_{22}(x_1, x_2 = +c) = 0$$

On the right face, $x_1 = +L$ and the normal vector is $\boldsymbol{e}_1$, therefore the stress tensor components are:

$$\sigma_{11}(x_1 = L, x_2) = \frac{3p}{4}\left(\frac{L^2 x_2}{c^3} - \frac{2x_2^3}{3c^3}\right), \quad \sigma_{21}(x_1 = L, x_2) = \sigma_{12}(L, x_2) = \frac{3pL}{4c}\left(1 - \frac{x_2^2}{c^2}\right)$$

On the bottom face, $x_2 = -c$ and the normal vector is $-\boldsymbol{e}_2$, therefore the stress tensor components are:

$$\sigma_{12}(x_1, x_2 = -c) = 0, \quad \sigma_{22}(x_1, x_2 = -c) = p$$

On the left face, $x_1 = 0$ and the normal vector is $-\boldsymbol{e}_1$, therefore the stress tensor components are:

$$\sigma_{12}(x_1 = 0, x_2) = 0, \quad \sigma_{22}(x_1 = 0, x_2) = -\frac{p\,x_2^3}{2\,c^3}$$

c. The traction $\underline{F}$ on the face of equation $x_1 = x_{10}$ and extending from $x_2 = -C$ to $x_2 = +c$ can be written in terms of the surface distribution of tractions $\underline{t}$ that are applied on that face, as follows:

$$\underline{F} = \int_{-c}^{+c} \underline{t} dx_2$$

Note that here, a ”surface” distribution is actually a line distribution, since the problem is 2D. Using the definition of Cauchy’s stress tensor:

$$\underline{F} = \int_{-c}^{+c} \underline{\underline{\sigma}} \cdot \underline{n} dx_2$$

in which here, $\underline{n} = \underline{e}_1$, and so:

$$\forall i = 1, 2, \quad F_i = \int_{-c}^{+c} \sigma_{i1} dx_2$$

We find:

$$F_1 = 0, \quad F_2 = p\,x_{10}$$

The moment about the point of coordinates (x_{10},0), is, by definition:

$$\underline{\mathcal{M}} = \int_{-c}^{+c} \underline{r} \times \underline{t} dx_2$$

in which $\underline{r} = (x_1 - x_{10})\underline{e}_1 + x_2\underline{e}_2$ is the vector position. Since $x_1 = x_{10}$ on the face:

$$\int_{-c}^{+c} (x_1 - x_{10})\sigma_{21} dx_2 = 0$$

So the moment reduces to:

$$\mathcal{M} = -\boldsymbol{e_3}\int_{-c}^{+c} x_2\sigma_{11}dx_2 = -\boldsymbol{e_3}\frac{p}{I}\int_{-c}^{+c}\left((x_{10})^2(x_2)^2 - \frac{2}{3}(x_2)^4\right)$$

$$\mathcal{M} = -\boldsymbol{e_3}\frac{2}{3}\frac{p}{I}(x_{10})^2c^3 + \boldsymbol{e_3}\frac{4}{15}\frac{p}{I}(c)^5$$

$$\mathcal{M} = -\boldsymbol{e_3}\frac{p}{2}(x_{10})^2 + \boldsymbol{e_3}\frac{p}{5}c^2$$

d. If the beam is slender, $c << L$, $\sigma_{11}(0, x_2)$ is of order 0 in L/c while $\sigma_{11}(L, x_2)$ and $\sigma_{12}(L, x_2)$ are of order $(L/c)^2$ and L/c, respectively. The force that acts on the face $x_1 = 0$ becomes negligible. Stresses reach a maximum at $x_1 = L$. Stresses are almost uniaxial since σ_{12} is one order of magnitude less than σ_{11}. It can also be noted that σ_{11} is linear in x_2.

e. We find p= 16,000 Pa and:

$$\sigma_{11}^{max} = \sigma_{11}(L, c) \simeq \frac{3pL^2}{4c^2} \simeq 5\,\text{MPa}$$

2.2 By considering the initial state of stress σ_{ij} at a point and a general increment of stress $\Delta\sigma_{ij}$, determine which of the stress invariants I allows superposition such that: $I_{\text{final}} = I_{\text{initial}} + I_{\text{incremental}}$.

Solution:
The first invariant is the only invariant that is linear in stress and that can satisfy the superposition equation.

2.3 The stress state at a point is given as:

$$[\sigma] = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 5 & 1 \\ 2 & 1 & 6 \end{bmatrix} \quad \text{[kPa]}$$

a. Determine the stress invariants I_1, I_2 and I_3 at the point.
b. Determine the invariants J_1, J_2 and J_3 of the deviatoric stress tensor s_{ij}.

Solution:
a. Invariants of $[\sigma]$, calculated with MATLAB:

$$I_1 = Tr\,(\boldsymbol{\sigma}) = 15, \quad I_2 = \frac{1}{2}\left((Tr\,(\boldsymbol{\sigma}))^2 - Tr\,(\boldsymbol{\sigma}^2)\right) = 60, \quad I_3 = \det(\boldsymbol{\sigma}) = 54$$

b. Deviatoric stress tensor $[s] = [\sigma] - p[I]$ with $p = Tr([\sigma])/3$:

$$[s] = \begin{bmatrix} -1 & 3 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{[kPa]}$$

Invariants of the deviatoric stress, calculated with MATLAB:

$$J_1 = Tr\left(\boldsymbol{\sigma}\right) = 0 \quad J_2 = \frac{1}{2} Tr\left(\boldsymbol{s}^2\right) = 15, \quad J_3 = \det\left(\boldsymbol{s}\right) = 4$$

2.4 Consider a point in plane stress, at which the state of stress is given by: $\sigma_{xx} = 2\,$MPa, $\sigma_{yy} = -1\,$MPa and $\sigma_{xy} = 0.5\,$MPa. For the given state of stress:
a. Draw Mohr's circle.
b. Determine the orientation of the principal planes and the corresponding principal stresses.
c. Determine the state of stress after the element has been rotated through and angle of 30^o *clockwise*.

Solution:
a. To draw Mohr's circle (not represented here), calculate the coordinates of the center C(σ_{avg},0) and calculate the radius R:

$$\sigma_{avg} = \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right) = 0.5\,\text{MPa}, \quad R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \left(\sigma_{xy}\right)^2} \simeq 1.58\,\text{MPa}$$

b. Principal stresses, from Mohr's circle:

$$\sigma_1 = OP_1 = 0C + CP_1 = \sigma_{avg} + R = 2.08\,\text{MPa}$$

$$\sigma_2 = OP_2 = 0C - CP_2 = \sigma_{avg} - R = -1.08\,\text{MPa}$$

Orientation of the principal planes, from Mohr's circle:

$$\tan 2\theta_p = \frac{AA'}{CA} = \frac{\sigma_{xy}}{(\sigma_{xx} - \sigma_{yy})/2} = \frac{1}{3}, \quad \Rightarrow \theta_p = 9^o + k * 90^o,\ k \in \mathbb{Z}$$

c. State of stress after rotation by -30^o, using Mohr's circle:

$$\sigma_{x'x'} = CD' = OC + CD' = \sigma_{avg} + R\cos\left(2 * 30^o + 2\theta_p\right) = 829\,\text{kPa}$$

$$\sigma_{y'y'} = CE' = OC - CE' = \sigma_{avg} - R\cos\left(2 * 30^o + 2\theta_p\right) = 171\,\text{kPa}$$

$$\sigma_{x'y'} = DD' = R\sin\left(2 * 30^o + 2\theta_p\right) = 1.55\,\text{MPa}$$

2.5 A steel penstock has a 750-mm outer diameter, a 12-mm wall thickness and connects a reservoir at A with a generating station at B, as shown in Figure 2.1. Knowing that the density of water is 1000kg/m^3, determine the maximum normal stress and the maximum shearing stress in the penstock under static conditions. *Hint: in a cylindrical pressurized vessel (i.e., in a cylindrical container that has a radius that is large compared to the thickness of the shell, and in which the pressure inside is larger than the pressure outside), the hoop stress is equal to* pr/t *and the longitudinal stress is equal to* $pr/2t$*, in which* p *is the pressure of the fluid inside the vessel,* r *is the inner radius of the vessel and* t *is the thickness of the shell.*

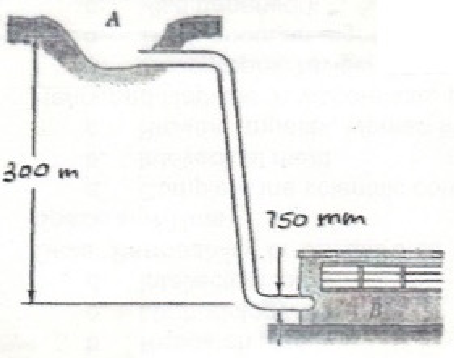


Figure 2.1 Penstock studied in Problem 2.5.

Solution:

$$p = \rho_w\, g\, h = 2.94\,\text{MPa}$$

$$\sigma_1 = \frac{pr}{t} = 89\,\text{MPa}$$

$$\sigma_3 = \sigma_z = 0\,\text{MPa}$$

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = 44.5\,\text{MPa}$$

2.6 Square plates, each of 16-mm thickness, can be bent and welded together in either of the two ways shown to form the cylindrical portion of a compressed air tank, as shown in Figure Figure 2.2. Knowing that the allowable normal stress perpendicular to the weld is 65MPa, determine the largest allowable gage pressure in each case. *Use the same hint as in Problem 2.5.*

Solution:

Configuration (a):

It is required that:

$$\sigma_1 = \frac{pr}{t} \leq \sigma_{allow}, \quad \sigma_2 = \frac{pr}{2t} \leq \sigma_{allow}$$

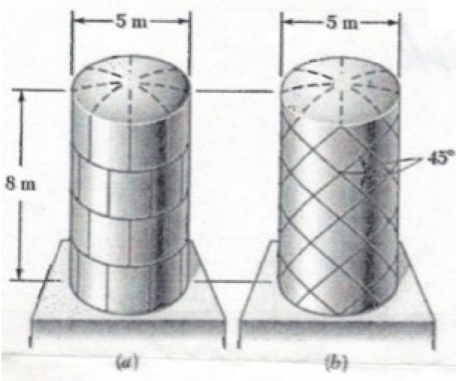


Figure 2.2 Compressed air tanks studied in Problem 2.6.

which is satisfied as long as:

$$\frac{pr}{t} \leq \sigma_{allow}$$

from which we get:

$$p \leq \frac{t\,\sigma_{allow}}{r} = 419\,\text{kPa}$$

Configuration (b):
A rotation by 45^o in the physical plane corresponds to a rotation by an angle of 90^o in the stress plane (in Mohr?s circle). At 90^o from P1 (maximum principal stress), the stress in Mohr?s circle is:

$$\sigma_1' = \sigma_2' = \sigma_{avg} = \frac{1}{2}\left(\frac{pr}{t} + \frac{pr}{2t}\right) = \frac{3pr}{4t}$$

As a result, it is required that:

$$\frac{3pr}{4t} \leq \sigma_{allow}$$

from which we get:

$$\frac{3pr}{4t} \leq \frac{4t\sigma_{allow}}{3r} = 558\,\text{kPa}$$

Configuration (b) is 33% more efficient than configuration (a).

2.7 Show which of the following strain states satisfies the compatibility condition:

$$u_3 = 0, \quad \epsilon_{11} = \frac{\left(x_1^2 + x_2^2\right)}{a^2}, \quad \epsilon_{22} = \frac{x_2^2}{a^2}, \quad \epsilon_{12} = \frac{x_1 x_2}{a^2}$$

$$u_3 = 0, \quad \epsilon_{11} = \frac{x_3\left(x_1^2 + x_2^2\right)}{a^3}, \quad \epsilon_{22} = \frac{x_2^2 x_3}{a^3}, \quad \epsilon_{12} = \frac{x_1 x_2 x_3}{a^3}$$

Solution:

For the first strain state, we are in plane strain since all the derivatives by x_3 are zero and the displacement along direction 3 is zero. In the expression

$$\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \epsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 \epsilon_{jl}}{\partial x_i \partial x_k} \tag{2.1}$$

the possible combination of indexes (i,j,k,l) are

$$\begin{array}{llll}
(1,1,1,1) & & & \\
(2,1,1,1) & (1,2,1,1) & (1,1,2,1) & (1,1,1,2) \\
(2,2,1,1) & (1,2,2,1) & (1,1,2,2) & (2,1,1,2) \\
(2,2,2,1) & (1,2,2,2) & (2,1,2,2) & (2,2,1,2) \\
(2,2,2,2) & & &
\end{array}$$

and we check that the expression in Equation 2.1 is zero in all cases, which means that the strain state satisfies the compatibility condition.

For the second strain state, we are not in a plane strain condition because not all the derivatives about x_3 are zero. The expression in Equation 2.1 is not zero for all possible combination of indexes. In particular, for (i,j,k,l)=(1,2,3,2), we have:

$$\begin{aligned}
A &= \frac{\partial^2 \epsilon_{12}}{\partial x_3 \partial x_2} + \frac{\partial^2 \epsilon_{32}}{\partial x_1 \partial x_2} - \frac{\partial^2 \epsilon_{13}}{\partial x_2 \partial x_2} - \frac{\partial^2 \epsilon_{22}}{\partial x_1 \partial x_3} \\
&= \frac{x1}{a^3} + 0 - 0 - 0 \neq 0
\end{aligned}$$

The result above shows that the second strain state does not satisfy the compatibility condition.

2.8 Let us consider a vertical beam of length L = 2m and half-width h=0.2 m, as shown in Figure 4.6.a. The position vector is given as $\boldsymbol{OP} = X\boldsymbol{e}_X + Y\boldsymbol{e}_Y$ in the Cartesian coordinate system, with $-h \leq X \leq +h$ and $0 \leq Y \leq L$. Consider the deformed configuration shown in Figure 4.6.b. such that the new position vector is given as:

$$\boldsymbol{OP} = \boldsymbol{x}_0(Y,t) + X\boldsymbol{e}_r\left(\theta(Y,t)\right), \quad \boldsymbol{e}_r = \cos\theta\boldsymbol{e}_X + \sin\theta\boldsymbol{e}_Y, \quad \boldsymbol{e}_\theta(\theta) = -\sin\theta\boldsymbol{e}_X + \cos\theta\boldsymbol{e}_Y$$

in which $\boldsymbol{x}_0(Y,t)$ and $\theta(Y,t)$ will be defined later.

a. Calculate the Green-Lagrange deformation tensor $\boldsymbol{e}$ as a function of $\boldsymbol{x}_0(Y,t)$ and $\theta(Y,t)$.

b. Calculate the elongation in the $\boldsymbol{e}_X$-direction.

c. Calculate the elongation in the $\boldsymbol{e}_Y$-direction, on the vertical axis $(X = 0)$ and on the lateral sides $(X = \pm h)$.

d. Calculate the distortion in the direction ($\boldsymbol{e}_X$,$\boldsymbol{e}_Y$) as a function of Y.

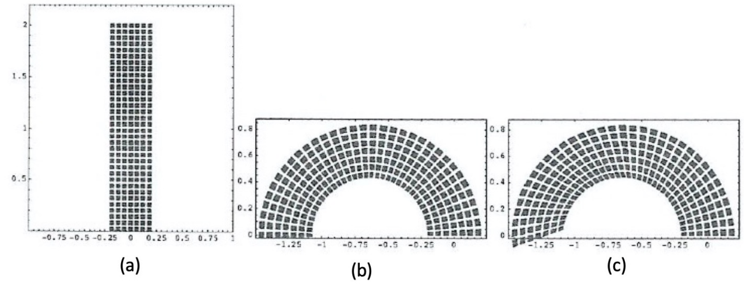
e. Now we pose:

$$\boldsymbol{x}_0(Y,t) = \frac{L}{\pi}\left(-\boldsymbol{e}_X + \boldsymbol{e}_r\left(\theta(Y,t)\right)\right), \quad \theta(Y,t) = \frac{\pi Y}{L}$$

Repeat questions a.-d. above.
f. Now suppose that the deformed configuration is that shown in Figure 4.6.c, in which:

$$\boldsymbol{OP} = \boldsymbol{x}_0(Y,t) \, + X\boldsymbol{e}_r \, (1.1\theta(Y,t))$$

Explain the difference between Figure 4.6.b and Figure 4.6.c.



(a) (b) (c)

Figure 2.3 Beam problem studied in Problem 2.8.

Solution:
a. By definition of the Green-Lagrange deformation tensor:

$$\underline{\underline{e}} = \frac{1}{2}\left(\underline{\underline{\nabla}}\left(\underline{u}\right) + ^T\underline{\underline{\nabla}}\left(\underline{u}\right) + \underline{\underline{\nabla}}\left(\underline{u}\right) \cdot ^T\underline{\underline{\nabla}}\left(\underline{u}\right)\right)$$

We first calculate the gradient of the displacement field:

$$\underline{\underline{\nabla}}\left(\underline{u}\right) = \left(\frac{\partial x_i}{\partial X_j} - \delta_{ij}\right)\underline{e}_i \otimes \underline{e}_j$$

in which $X_i\underline{e}_i$ is the position vector in the undeformed configuration and $x_i\underline{e}_i$ is the position vector in the deformed configuration. Using the notations of the problem, we calculate:

$$\boldsymbol{\nabla}\left(\boldsymbol{u}\right) \;=\; \frac{d\boldsymbol{x}_0(Y,t)}{dY}\otimes\boldsymbol{e}_Y + \boldsymbol{e}_r\left(\theta(Y,t)\right)\otimes\boldsymbol{e}_X + X\frac{d\boldsymbol{e}_r\left(\theta(Y,t)\right)}{dY}\otimes\boldsymbol{e}_Y - \boldsymbol{e}_X\otimes\boldsymbol{e}_X - \boldsymbol{e}_Y\otimes\boldsymbol{e}_Y$$

From there, the Green-Lagrange deformation tensor is obtained:

$$\begin{aligned}\boldsymbol{e} = \quad & \frac{1}{2}\frac{d\boldsymbol{x}_0(Y,t)}{dY}\cdot\boldsymbol{e}_r\,\left(\boldsymbol{e}_X\otimes\boldsymbol{e}_Y + \boldsymbol{e}_Y\otimes\boldsymbol{e}_X\right) \\ & + \left[\frac{\left(|\frac{d\boldsymbol{x}_0(Y,t)}{dY}|^2 - 1\right)}{2} + X\frac{d\theta(Y,t)}{dY}\frac{d\boldsymbol{x}_0(Y,t)}{dY}\cdot\boldsymbol{e}_\theta + \frac{1}{2}X^2\left(\frac{d\theta(Y,t)}{dY}\right)^2\right]\boldsymbol{e}_Y\otimes\boldsymbol{e}_Y\end{aligned}$$

b. From Equation 2.2, we see that $\boldsymbol{e}_X\cdot\boldsymbol{e}\cdot\boldsymbol{e}_X = e_{XX} = 0$.
c. From Equation 2.2, we have:

$$e_{YY} = \frac{\left(|\frac{d\boldsymbol{x}_0(Y,t)}{dY}|^2 - 1\right)}{2} + X\frac{d\theta(Y,t)}{dY}\frac{d\boldsymbol{x}_0(Y,t)}{dY}\cdot\boldsymbol{e}_\theta + \frac{1}{2}X^2\left(\frac{d\theta(Y,t)}{dY}\right)^2$$

On the vertical axis $(X = 0)$:

$$e_{YY}(X=0) = \frac{\left(|\frac{d\boldsymbol{x}_0(Y,t)}{dY}|^2 - 1\right)}{2}$$

On the lateral sides $(X = \pm h)$:

$$e_{YY}(X=\pm h) = \frac{\left(|\frac{d\boldsymbol{x}_0(Y,t)}{dY}|^2 - 1\right)}{2} \pm h\frac{d\theta(Y,t)}{dY}\frac{d\boldsymbol{x}_0(Y,t)}{dY} \cdot \boldsymbol{e}_\theta + \frac{1}{2}h^2\left(\frac{d\theta(Y,t)}{dY}\right)^2$$

If $\frac{d\theta(Y,t)}{dY} > 0$, the maximum elongation is on the side $X = +h$. If $\frac{d\theta(Y,t)}{dY} < 0$, the maximum elongation is on the side $X = -h$.

d. Here, we calculate the distortion of the angle $(\boldsymbol{e}_X, \boldsymbol{e}_Y)$, as follows:

$$\gamma = {}^T\boldsymbol{F}(\boldsymbol{e}_X) \cdot \boldsymbol{F}(\boldsymbol{e}_Y) - \boldsymbol{e}_X \cdot \boldsymbol{e}_Y = {}^T\boldsymbol{F}(\boldsymbol{e}_X) \cdot \boldsymbol{F}(\boldsymbol{e}_Y)$$

in which $\boldsymbol{F}$ is the gradient of the transport function ϕ, which associate a position vector in the deformed configuration to a vector in the undeformed configuration. We have:

$$\boldsymbol{e} = \frac{1}{2}\left({}^T\boldsymbol{F} \cdot \boldsymbol{F} - \boldsymbol{I}\right)$$

from which we get:

$${}^T\boldsymbol{F} \cdot \boldsymbol{F} = 2\boldsymbol{e} + \boldsymbol{I}$$

and in particular:

$${}^T\boldsymbol{F}(\boldsymbol{e}_X) \cdot \boldsymbol{F}(\boldsymbol{e}_Y) = F_{Xj}F_{jY} = 2e_{XY} = \frac{d\boldsymbol{x}_0(Y,t)}{dY} \cdot \boldsymbol{e}_r$$

We find that the distortion is zero if the the tangent to the neutral axis is orthogonal to the deformed section (Bernouilli's assumption).

e. For the proposed functions $\boldsymbol{x}_0$ and θ, we have:

$$\boldsymbol{e} = \frac{\pi X}{L}\left(1 + \frac{\pi X}{2L}\right)\boldsymbol{e}_Y \otimes \boldsymbol{e}_Y$$

We have: $e_{XX} = e_{XY} = 0$ (no elongation in the X-direction and no distortion), $e_{YY} = \frac{\pi X}{L}\left(1 + \frac{\pi X}{2L}\right) \simeq 36\%$ according to the figure. Few materials can resist such an elongation in the direction of the fibers (latex can).

f. In the right figure, the distortion is non-zero: the material points shown in the right figure experience shear deformation. We can indeed check that:

$$\frac{d\boldsymbol{x}_0(Y,t)}{dY} \cdot \boldsymbol{e}_r(1.1\theta) = \boldsymbol{e}_\theta(\theta) \cdot \boldsymbol{e}_r(1.1\theta) \neq 0$$

2.9 A cube of granite with sides of length a = 89 mm (see Figure 2.4) is tested in a laboratory under triaxial stress. Assume E = 80 GPa, ν = 0.25. Gages mounted on the testing machine show that the compressive strains in the material are ϵ_{xx} = -138 x 10^{-5} and $\epsilon_{yy} = \epsilon_{zz}$ = -510 x 10^{-6}. Determine the following quantities:

a. The normal stresses σ_{xx}, σ_{yy}, and σ_{zz} acting on the x, y, and z faces of the cube;

b. The maximum shear stress τ_{max} in the material;

c. The change ΔV in the volume of the cube;

d. The maximum value of σ_{xx} when the change in volume must be limited to -0.11%.

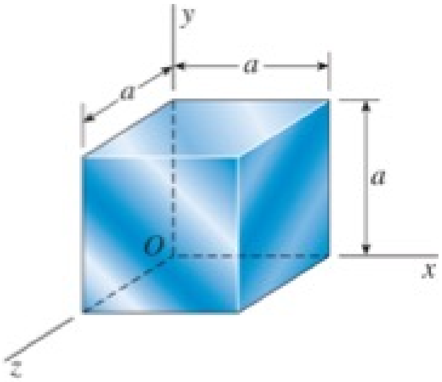


Figure 2.4 Granite sample studied in Problem 2.9.

Solution:

a. Using Hook's law in terms of the Young's modulus and the Poisson's ratio:

$$\sigma_{ii} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_{ii} + \nu(\epsilon_{jj}+\epsilon_{kk})\right], \quad i \neq j, \quad j \neq k, \quad k \neq i$$

in which there is no summation on the indices. Numerically:

$$\sigma_{xx} = -82.6\,\text{MPa}, \quad \sigma_{yy} = -54.7\,\text{MPa}, \quad \sigma_{zz} = -54.7\,\text{MPa}$$

b. The cube is in a state of triaxial stress, therefore there is no shear stress on any of the faces and σ_{xx}, σ_{yy} and σ_{zz} are principal stresses. The maximum shear stress is half of the difference between the major and minor principal stresses, and can therefore be calculated as follows:

$$\tau_{max} = \frac{\sigma_{yy}-\sigma_{xx}}{2} = 13.92\,\text{MPa}$$

c. Change of volume:

$$\begin{aligned}\Delta V &= V_1 - V_0 = (a+\Delta a)(b+\Delta b)(c+\Delta c) - abc \\ &= (a + a\epsilon_{xx})(b+b\epsilon_{yy})(c+c\epsilon_{zz}) - abc \\ &= abc\left[(1+\epsilon_{xx})(1+\epsilon_{yy})(1+\epsilon_{zz}) - 1\right]\end{aligned}$$

Under the small deformation assumption:

$$\Delta V \simeq abc\,(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}) = abc\,\epsilon_v$$

Numerically:

$$\epsilon_v = -1.2\times 10^{-3}, \quad \Delta V = -846\,\text{mm}^3$$

d. We impose:

$$\epsilon_v^{max} = -0.11\%$$

Developing the expression of the volumetric strain:

$$max\,(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}) = -0.11\%$$

Substituting the deformations by their relationship to stress via Hooke's law:

$$\frac{(1-2\nu)}{E} max\,(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = -0.11\%$$

If σ_{yy} and σ_{zz} are fixed, then:

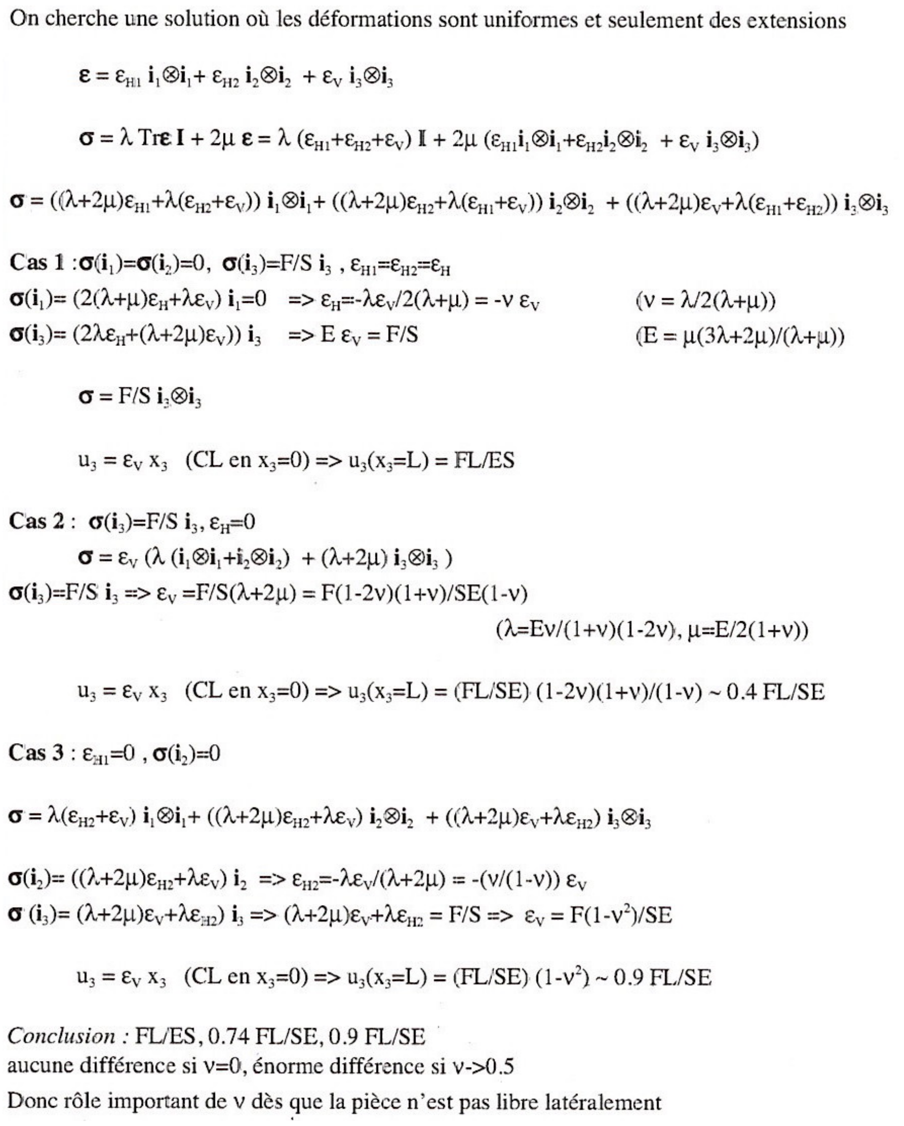
$$\sigma_{xx}^{max} = -\frac{0.11\% E}{(1-2\nu)} - \sigma_{yy} - \sigma_{zz}$$

Numerically:

$$\sigma_{xx}^{max} = -73\,\text{MPa}$$

2.10 Consider a parallelepidic specimen of length L, with a square section of edge length a. The square section of the specimen lies on the plane $(0, x_1, x_2)$. The parallelepiped can slide on the plane but cannot loose contact with the plane. The specimen is loaded at $x_3 = L$ by a uniform compression $-F$ in the $\boldsymbol{e}_3$ direction. The material that makes the specimen is isotropic and linear elastic,with a Young's modulus E and a Poisson's ratio ν. We assume that deformations are very small, and we neglect the gravity and inertia forces. We give: L=20 cm, a=1 cm, E=200 GPa, ν=0.3, F=100 N.
a. Calculate the displacement field at $x_3 = L$ if the lateral faces of the specimen are free of stress.
b. Calculate the displacement field at $x_3 = L$ if the specimen is encased in a very rigid support, within which it can slide. c. Calculate the displacement field at $x_3 = L$ if the the faces $x_1 \pm a/2$ are fixed and if the other two lateral faces are free of stress.
d. Now suppose that the specimen is encased in a very rogod support and that there is an initial misfit of 0.01 mm between the lateral faces. What is the required value of the compression force F for which the lateral faces of the specimen get in contact with the rigid support?

Solution: see Figure 2.5.



On cherche une solution où les déformations sont uniformes et seulement des extensions

$$\boldsymbol{\varepsilon} = \varepsilon_{H1}\ \mathbf{i}_1\otimes\mathbf{i}_1 + \varepsilon_{H2}\ \mathbf{i}_2\otimes\mathbf{i}_2\ + \varepsilon_V\ \mathbf{i}_3\otimes\mathbf{i}_3$$

$$\boldsymbol{\sigma} = \lambda\ \mathrm{Tr}\boldsymbol{\varepsilon}\ \mathbf{I} + 2\mu\ \boldsymbol{\varepsilon} = \lambda\ (\varepsilon_{H1}+\varepsilon_{H2}+\varepsilon_V)\ \mathbb{I} + 2\mu\ (\varepsilon_{H1}\mathbf{i}_1\otimes\mathbf{i}_1+\varepsilon_{H2}\mathbf{i}_2\otimes\mathbf{i}_2\ + \varepsilon_V\ \mathbf{i}_3\otimes\mathbf{i}_3)$$

$$\boldsymbol{\sigma} = ((\lambda+2\mu)\varepsilon_{H1}+\lambda(\varepsilon_{H2}+\varepsilon_V))\ \mathbf{i}_1\otimes\mathbf{i}_1 + ((\lambda+2\mu)\varepsilon_{H2}+\lambda(\varepsilon_{H1}+\varepsilon_V))\ \mathbf{i}_2\otimes\mathbf{i}_2\ + ((\lambda+2\mu)\varepsilon_V+\lambda(\varepsilon_{H1}+\varepsilon_{H2}))\ \mathbf{i}_3\otimes\mathbf{i}_3$$

Cas 1 : $\boldsymbol{\sigma}(\mathbf{i}_1)=\boldsymbol{\sigma}(\mathbf{i}_2)=0$, $\boldsymbol{\sigma}(\mathbf{i}_3)=F/S\ \mathbf{i}_3$, $\varepsilon_{H1}=\varepsilon_{H2}=\varepsilon_H$

$\boldsymbol{\sigma}(\mathbf{i}_1)= (2(\lambda+\mu)\varepsilon_H+\lambda\varepsilon_V)\ \mathbf{i}_1=0 \quad => \varepsilon_H=-\lambda\varepsilon_V/2(\lambda+\mu) = -\nu\ \varepsilon_V$ $\quad (\nu = \lambda/2(\lambda+\mu))$

$\boldsymbol{\sigma}(\mathbf{i}_3)= (2\lambda\varepsilon_H+(\lambda+2\mu)\varepsilon_V))\ \mathbf{i}_3 \quad => E\ \varepsilon_V = F/S$ $\quad (E = \mu(3\lambda+2\mu)/(\lambda+\mu))$

$$\boldsymbol{\sigma} = F/S\ \mathbf{i}_3\otimes\mathbf{i}_3$$

$u_3 = \varepsilon_V\ x_3$ (CL en $x_3=0$) $=> u_3(x_3=L) = FL/ES$

Cas 2 : $\boldsymbol{\sigma}(\mathbf{i}_3)=F/S\ \mathbf{i}_3$, $\varepsilon_H=0$

$$\boldsymbol{\sigma} = \varepsilon_V\ (\lambda\ (\mathbf{i}_1\otimes\mathbf{i}_1+\mathbf{i}_2\otimes\mathbf{i}_2)\ + (\lambda+2\mu)\ \mathbf{i}_3\otimes\mathbf{i}_3\ )$$

$\boldsymbol{\sigma}(\mathbf{i}_3)=F/S\ \mathbf{i}_3 => \varepsilon_V = F/S(\lambda+2\mu) = F(1-2\nu)(1+\nu)/SE(1-\nu)$

$(\lambda=E\nu/(1+\nu)(1-2\nu),\ \mu=E/2(1+\nu))$

$u_3 = \varepsilon_V\ x_3$ (CL en $x_3=0$) $=> u_3(x_3=L) = (FL/SE)\ (1-2\nu)(1+\nu)/(1-\nu) \sim 0.4\ FL/SE$

Cas 3 : $\varepsilon_{H1}=0$, $\boldsymbol{\sigma}(\mathbf{i}_2)=0$

$$\boldsymbol{\sigma} = \lambda(\varepsilon_{H2}+\varepsilon_V)\ \mathbf{i}_1\otimes\mathbf{i}_1 + ((\lambda+2\mu)\varepsilon_{H2}+\lambda\varepsilon_V)\ \mathbf{i}_2\otimes\mathbf{i}_2\ + ((\lambda+2\mu)\varepsilon_V+\lambda\varepsilon_{H2})\ \mathbf{i}_3\otimes\mathbf{i}_3$$

$\boldsymbol{\sigma}(\mathbf{i}_2)= ((\lambda+2\mu)\varepsilon_{H2}+\lambda\varepsilon_V)\ \mathbf{i}_2 \ => \varepsilon_{H2}=-\lambda\varepsilon_V/(\lambda+2\mu) = -(\nu/(1-\nu))\ \varepsilon_V$

$\boldsymbol{\sigma}\ (\mathbf{i}_3)= (\lambda+2\mu)\varepsilon_V+\lambda\varepsilon_{H2})\ \mathbf{i}_3 => (\lambda+2\mu)\varepsilon_V+\lambda\varepsilon_{H2} = F/S => \ \varepsilon_V = F(1-\nu^2)/SE$

$u_3 = \varepsilon_V\ x_3$ (CL en $x_3=0$) $=> u_3(x_3=L) = (FL/SE)\ (1-\nu^2) \sim 0.9\ FL/SE$

Conclusion : FL/ES, 0.74 FL/SE, 0.9 FL/SE
aucune différence si ν=0, énorme différence si ν->0.5
Donc rôle important de ν dès que la pièce n'est pas libre latéralement

Figure 2.5 Solution of Problem 2.10.

2.11 Homework 1 - Problem 3

Consider an initial state of stress at a point, given by: $\sigma_{xx}/\sigma_{zz} = 0.5$, $\sigma_{yy}/\sigma_{zz} = 0.75$, $\sigma_{xz}/\sigma_{zz} = 0.25$, $\sigma_{xy}/\sigma_{zz} = 0$, $\sigma_{yz}/\sigma_{zz} = 0$, $\sigma_{zz} = 1$.

1. Calculate the invariants of the stress tensor: I_1, I_2 and I_3.

2. Calculate the magnitude of the principal stresses σ_1, σ_2 and σ_3. Calculate the angles of orientation of the principal planes.

3. The state of stress at the point is changed by application of the following stress increments: $\Delta\sigma_{zz} = 0$, $\Delta\sigma_{xx}/\sigma_{zz} = -0.25$, $\Delta\sigma_{xz}/\sigma_{zz} = 0.25$. Draw the initial and

final Mohr cirlces. What are the final magnitudes of the principal stresses and the orientations of the principal planes?

Solution:

1. Calculations can easily be handled with MATLAB:

$$I_1 = Tr\left(\boldsymbol{\sigma}\right) = 2.25, \quad I_2 = \frac{1}{2}\left[\left(Tr\left(\boldsymbol{\sigma}\right)\right)^2 - Tr\left(\boldsymbol{\sigma}^2\right)\right] = 1.5625, \quad I_3 = \det\left(\boldsymbol{\sigma}\right) = 0.3281$$

2. Solving the characteristic equation with MATLAB, we get:

$$\sigma_1 = 1.1036, \quad \sigma_2 = \sigma_{yy} = 0.75, \quad \sigma_3 = 0.3964$$

It was expected that σ_{yy} would be a principal stress, since $\sigma_{xy} = \sigma_{yz} = 0$. As a result, $\boldsymbol{e}_y$ is a principal direction. The other tow principal directions are in the plane ($\boldsymbol{e}_x$,$\boldsymbol{e}_x$). We can find the orientation of the principal axes by calculating first the angle θ_p between the normal to the plane subjected to the major principal stress, as follows:

$$\tan(2\theta_p) = \frac{2\sigma_{xz}}{\sigma_{zz} - \sigma_{xx}} = 1 \Rightarrow \theta_p = 22.5^o + k * 90^o,\ k \in \mathbb{Z}$$

One principal plane has a normal oriented by an angle of 22.5^o to the x-axis, and another principal plane is oriented to an angle 112.5^o to the x-axis. The unit vectors that are normal to the principal planes are expressed in the Cartesian base as follows:

$$\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} \cos(22.5^o) \\ 0 \\ \sin(22.5^o) \end{Bmatrix}, \quad \begin{Bmatrix} \sin(22.5^o) \\ 0 \\ \cos(22.5^o) \end{Bmatrix}$$

3. The new state of stress is:

$$[\boldsymbol{\sigma'}] = [\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}] = \begin{bmatrix} 0.25 & 0 & 0.5 \\ 0 & 0.75 & 0 \\ 0.5 & 0 & 1 \end{bmatrix}$$

σ'_{yy} is a principal stress. The other two principal stresses are found by solving the characteristic equation in MATLAB:

$$\sigma'_1 = 1.25, \quad \sigma'_2 = \sigma'_{yy} = 0.75, \quad \sigma'_3 = 0$$

One principal plane is normal to $\boldsymbol{e}_y$. The normal vector of the other two planes is contained in the plane ($\boldsymbol{e}_x$,$\boldsymbol{e}_z$). The orientation of these two normal vectors compared to the x-axis is found in the same way as in question 2:

$$\tan\left(2\theta'_p\right) = \frac{2\sigma'_{xz}}{\sigma'_{zz} - \sigma'_{xx}} = \frac{4}{3} \Rightarrow \theta'_p = 26.56^o + k * 90^o,\ k \in \mathbb{Z}$$

The unit vectors normal to the principal planes are the following:

$$\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} \cos(26.56^o) \\ 0 \\ \sin(26.56^o) \end{Bmatrix}, \quad \begin{Bmatrix} \sin(26.56^o) \\ 0 \\ \cos(26.56^o) \end{Bmatrix}$$

2.12 Homework 1 - Problem 4

Consider a specimen that has the shape of a disc with a diameter d. The disc is subjected to two forces P that are diametrically opposite, as shown in Figure 2.6. We consider the following stress field:

$$\frac{\boldsymbol{\sigma}}{a} = \left(\frac{\cos\theta_1}{r_1}\right) \boldsymbol{e}_{r1} \otimes \boldsymbol{e}_{r1} + \left(\frac{\cos\theta_2}{r_2}\right) \boldsymbol{e}_{r1} \otimes \boldsymbol{e}_{r1} - \frac{1}{d}\boldsymbol{I}$$

in which a is a constant that will be defined later, $\boldsymbol{I}$ is the second-order identity tensor and r_1, r_2, θ_1, θ_2, $\boldsymbol{e}_{r1}$ and $\boldsymbol{e}_{r2}$ are defined in Figure 2.6.

1. Show that $\boldsymbol{div}\left[(\cos\theta/r)\boldsymbol{e}_r \otimes \boldsymbol{e}_r\right] = \boldsymbol{0}$, in which r, θ and $\boldsymbol{e}_r$ are the usual coordinates and coordinate directions used in a cylindrical coordinate system.
2. Deduce from the previous question that the stress field is in equilibrium, i.e. that $\boldsymbol{div}\boldsymbol{\sigma} = \boldsymbol{0}$. Neglect the forces of gravity and inertia.
3. Show that the traction force $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$ is zero on the free surface $r = d/2$. Consider points of the free surface that are far from the point of application of the forces P.
4. Calculate the stress tensor on the axis $x_1 = 0$. Explain why the specimen has a low resistance in traction on this axis.

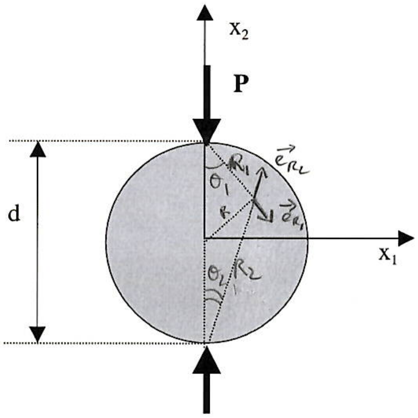


Figure 2.6 Schematic of the Brazilian test studied in Problem 4.

1. **Proof of : $\boldsymbol{div}\,[(\cos\theta/r)\boldsymbol{e}_r\otimes\boldsymbol{e}_r]=\boldsymbol{0}$:**

$$\boldsymbol{div}\,[(\cos\theta/r)\boldsymbol{e}_r\otimes\boldsymbol{e}_r] \;=\; \boldsymbol{div}\,[(\cos\theta/r)]\cdot(\boldsymbol{e}_r\otimes\boldsymbol{e}_r)\;+(\cos\theta/r)\boldsymbol{div}\,[\boldsymbol{e}_r\otimes\boldsymbol{e}_r] \tag{2.2}$$

in which: $\boldsymbol{e}_r=\cos\theta\boldsymbol{e}_x+\sin\theta\boldsymbol{e}_y$ (noting x for x_1 and y for x_2 for compactness). We have $\boldsymbol{e}_\theta=-\sin\theta\boldsymbol{e}_x+\cos\theta\boldsymbol{e}_y$, so that:

$$\frac{\partial\boldsymbol{e}_r}{\partial\theta}\;=\;\boldsymbol{e}_\theta$$

We thus have:

$$\begin{aligned}\boldsymbol{\nabla}\,[\boldsymbol{e}_r]\;&=\;0\times\;\boldsymbol{e}_r\otimes\boldsymbol{e}_r\;+\frac{1}{r}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_\theta\\ \boldsymbol{\nabla}\,[\boldsymbol{e}_r\otimes\boldsymbol{e}_r]\;&=\;\frac{1}{r}\boldsymbol{e}_r\otimes\boldsymbol{e}_\theta\otimes\boldsymbol{e}_\theta\;+\frac{1}{r}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_\theta\otimes\boldsymbol{e}_r\end{aligned}$$

And so:

$$\boldsymbol{div}\,[\boldsymbol{e}_r\otimes\boldsymbol{e}_r]\;=\;\frac{1}{r}\boldsymbol{e}_r \tag{2.3}$$

On the other hand:

$$\boldsymbol{\nabla}\,[(\cos\theta/r)]\;=\;-\frac{\cos\theta}{r^2}\boldsymbol{e}_r\;-\;\frac{\sin\theta}{r^2}\boldsymbol{e}_\theta \tag{2.4}$$

From equations 2.2, 2.3 and 2.4, we have:

$$\boldsymbol{div}\,[(\cos\theta/r)\boldsymbol{e}_r\otimes\boldsymbol{e}_r]\;=\;-\frac{\cos\theta}{r^2}\boldsymbol{e}_r\;+\;\frac{\cos\theta}{r}\frac{1}{r}\boldsymbol{e}_r\;=\;0 \tag{2.5}$$

So we showed that $\boldsymbol{div}\,[(\cos\theta/r)\boldsymbol{e}_r\otimes\boldsymbol{e}_r]=\boldsymbol{0}$.

2. **Proof of equilibrium:** Noting x for x_1 and y for x_2 for compactness, we have:

$$\begin{aligned}\boldsymbol{e}_{r_1}\;&=-\cos\theta_1\,\boldsymbol{e}_y\;+\sin\theta_1\,\boldsymbol{e}_x\\ \boldsymbol{e}_{\theta_1}\;&=\sin\theta_1\,\boldsymbol{e}_y\;+\cos\theta_1\,\boldsymbol{e}_x\\ \boldsymbol{e}_{r_2}\;&=\cos\theta_2\,\boldsymbol{e}_y\;+\sin\theta_2\,\boldsymbol{e}_x\\ \boldsymbol{e}_{\theta_2}\;&=-\sin\theta_2\,\boldsymbol{e}_y\;+\cos\theta_2\,\boldsymbol{e}_x\end{aligned}$$

Equation 2.5 is valid for any cylindircal coordinate system, and so:

$$\begin{aligned}\boldsymbol{div}_{r_1,\theta_1}\,[(\cos\theta_1/r_1)\boldsymbol{e}_{r_1}\otimes\boldsymbol{e}_{r_1}]\;&=\;\boldsymbol{0}\\ \boldsymbol{div}_{r_2,\theta_2}\,[(\cos\theta_2/r_2)\boldsymbol{e}_{r_2}\otimes\boldsymbol{e}_{r_2}]\;&=\;\boldsymbol{0}\end{aligned}$$

Note that the two equations above are derived in a different coordinate system than equation 2.5. To perform the coordinate change and convert a derivation in (r_1,θ_1) or (r_2,θ_2) into a derivation in $(r,\theta,$ we introduce the following Jacobian matrices:

$$[J_i]\;=\;\begin{bmatrix}\frac{\partial r_i}{\partial r} & r_i\frac{\partial\theta_i}{\partial r}\\ \frac{\partial r_i}{r\partial\theta} & \frac{\partial\theta_i}{\partial\theta}\end{bmatrix},\quad i=1,2$$

We have:

$$\boldsymbol{div}_{r,\theta}\,[(\cos\theta_1/r_1)\boldsymbol{e}_{r_1}\otimes\boldsymbol{e}_{r_1}] \;=[J]_1\cdot\boldsymbol{div}_{r_1,\theta_1}\,[(\cos\theta_1/r_1)\boldsymbol{e}_{r_1}\otimes\boldsymbol{e}_{r_1}] \;=\;0$$

$$\boldsymbol{div}_{r,\theta}\,[(\cos\theta_2/r_2)\boldsymbol{e}_{r_2}\otimes\boldsymbol{e}_{r_2}] \;=[J]_2\cdot\boldsymbol{div}_{r_2,\theta_2}\,[(\cos\theta_2/r_2)\boldsymbol{e}_{r_2}\otimes\boldsymbol{e}_{r_2}] \;=\;0$$

Since: $\boldsymbol{I}/d$ is a constant tensor, we deduce from the two previous equations that:

$$\boldsymbol{div}_{r,\theta}\boldsymbol{\sigma}\;=\;0$$

which shows that the system is in equilibrium.

3. When $r = d/2$, $\cos\theta_1/r_1 = \cos\theta_2/r_2 = 1/d$. We have $\boldsymbol{n} = \boldsymbol{e}_r$ and so the surface traction is:

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \frac{a}{d}(\boldsymbol{e}_{r1}\cdot\boldsymbol{e}_r)\boldsymbol{e}_{r1} + \frac{a}{d}(\boldsymbol{e}_{r2}\cdot\boldsymbol{e}_r)\boldsymbol{e}_{r2} - \frac{a}{d}\boldsymbol{e}_r$$

We have $\boldsymbol{e}_{r_1}\cdot\boldsymbol{e}_{r_2} = 0$ and so $\boldsymbol{e}_{r1}\cdot\boldsymbol{e}_r)\boldsymbol{e}_{r1} + \boldsymbol{e}_{r2}\cdot\boldsymbol{e}_r)\boldsymbol{e}_{r2} = \boldsymbol{e}_r$. So we have:

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \frac{a}{d}\boldsymbol{e}_r - \frac{a}{d}\boldsymbol{e}_r\;=\;\boldsymbol{0}$$

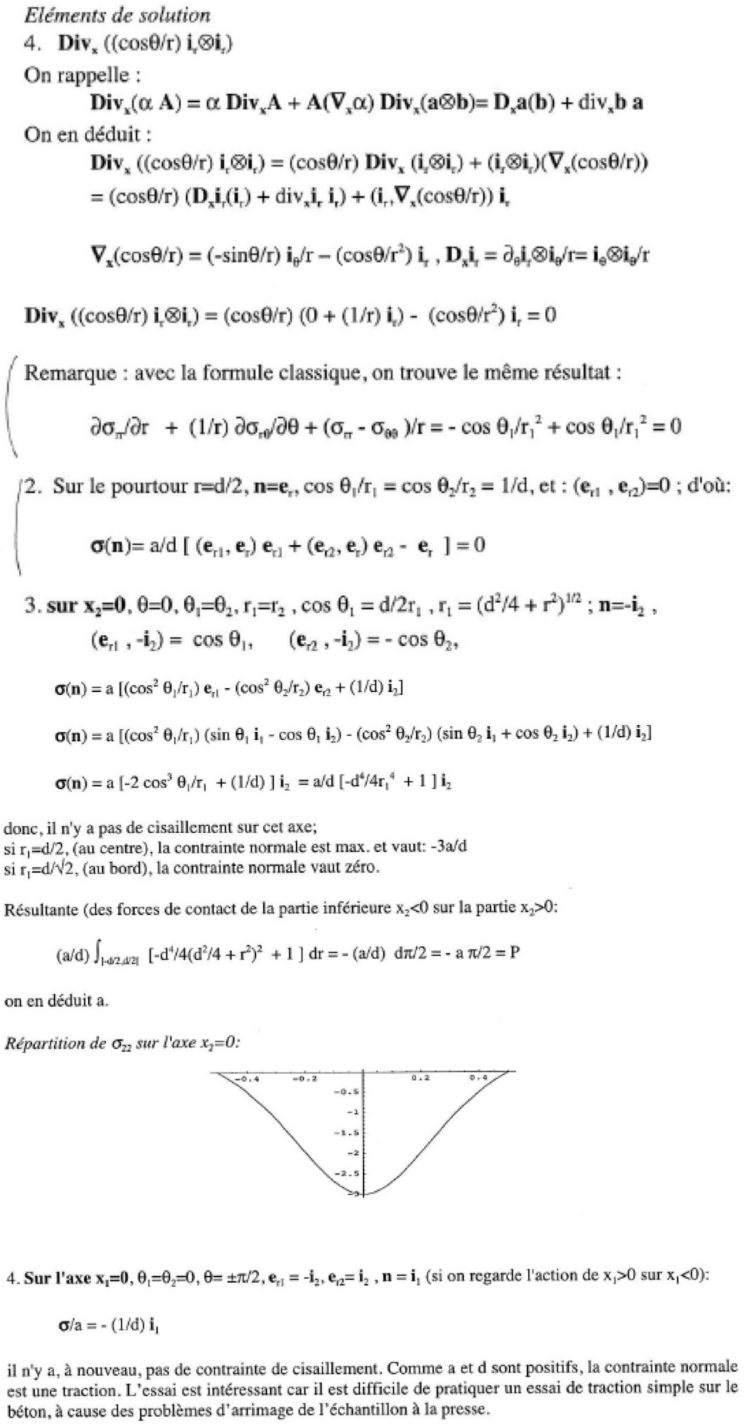
So the radial traction force is zero on the circumference of the specimen.

4. On the vertical axis ($x_1 = 0$), we have $\theta_1 = \theta_2 = 0$, $\theta = \pm\pi/2$, $\boldsymbol{e}_{r_1} = -\boldsymbol{e}_y$ and $\boldsymbol{e}_{r_2} = \boldsymbol{e}_y$ (using $x = x_1$, $y = x_2$). The stress is:

$$\frac{\boldsymbol{\sigma}}{a} = \frac{1}{r_1}\boldsymbol{e}_y\otimes\boldsymbol{e}_y + \frac{1}{r_2}\boldsymbol{e}_y\otimes\boldsymbol{e}_y - \frac{1}{d}\boldsymbol{e}_x\otimes\boldsymbol{e}_x - \frac{1}{d}\boldsymbol{e}_y\otimes\boldsymbol{e}_y$$

Take a vetical cut in the specimen and analyze the traction force that applies on the right side. We have $\boldsymbol{n} = -\boldsymbol{e}_x$ and $\frac{\boldsymbol{\sigma}}{a} = \boldsymbol{e}_x/d$. Similarly, the traction force that applies on the left side, where $\boldsymbol{n} = \boldsymbol{e}_x$, is $\frac{\boldsymbol{\sigma}}{a} = -\boldsymbol{e}_x/d$. So the specimen is subjected to a horizontal force pointing the the left on the left side, and to a horizontal force pointing to the right on the right side. That means that along the axis $x_1 = 0$, the specimen is subjected to opposite horzontal forces that subject the specimen to traction. That explains why the specimen has a low resistance to traction along the axis $x_1 = 0$.

See Figure 2.7 for more details.



Eléments de solution

4. $\mathbf{Div}_x\ ((\cos\theta/r)\ \mathbf{i}_r\otimes\mathbf{i}_r)$

On rappelle :

$$\mathbf{Div}_x(\alpha\ \mathbf{A}) = \alpha\ \mathbf{Div}_x\mathbf{A} + \mathbf{A}(\nabla_x\alpha)\ \mathbf{Div}_x(\mathbf{a}\otimes\mathbf{b}) = \mathbf{D}_x\mathbf{a}(\mathbf{b}) + \mathrm{div}_x\mathbf{b}\ \mathbf{a}$$

On en déduit :

$$\mathbf{Div}_x\ ((\cos\theta/r)\ \mathbf{i}_r\otimes\mathbf{i}_r) = (\cos\theta/r)\ \mathbf{Div}_x\ (\mathbf{i}_r\otimes\mathbf{i}_r) + (\mathbf{i}_r\otimes\mathbf{i}_r)(\nabla_x(\cos\theta/r))$$

$$= (\cos\theta/r)\ (\mathbf{D}_x\mathbf{i}_r(\mathbf{i}_r) + \mathrm{div}_x\mathbf{i}_r\ \mathbf{i}_r) + (\mathbf{i}_r, \nabla_x(\cos\theta/r))\ \mathbf{i}_r$$

$$\nabla_x(\cos\theta/r) = (-\sin\theta/r)\ \mathbf{i}_\theta/r - (\cos\theta/r^2)\ \mathbf{i}_r\ ,\ \mathbf{D}_x\mathbf{i}_r = \partial_\theta\mathbf{i}_r\otimes\mathbf{i}_\theta/r = \mathbf{i}_\theta\otimes\mathbf{i}_\theta/r$$

$$\mathbf{Div}_x\ ((\cos\theta/r)\ \mathbf{i}_r\otimes\mathbf{i}_r) = (\cos\theta/r)\ (0 + (1/r)\ \mathbf{i}_r) - (\cos\theta/r^2)\ \mathbf{i}_r = 0$$

Remarque : avec la formule classique, on trouve le même résultat :

$$\partial\sigma_{rr}/\partial r + (1/r)\ \partial\sigma_{r\theta}/\partial\theta + (\sigma_{rr} - \sigma_{\theta\theta})/r = -\cos\theta_1/r_1^{\ 2} + \cos\theta_1/r_1^{\ 2} = 0$$

2. Sur le pourtour $r=d/2$, $\mathbf{n}=\mathbf{e}_r$, $\cos\theta_1/r_1 = \cos\theta_2/r_2 = 1/d$, et : $(\mathbf{e}_{r1}\ , \mathbf{e}_{r2})=0$; d'où:

$$\boldsymbol{\sigma}(\mathbf{n}) = a/d\ [\ (\mathbf{e}_{r1}, \mathbf{e}_r)\ \mathbf{e}_{r1} + (\mathbf{e}_{r2}, \mathbf{e}_r)\ \mathbf{e}_{r2} - \mathbf{e}_r\ ] = 0$$

3. **sur** $\mathbf{x_2=0}$, $\theta=0$, $\theta_1=\theta_2$, $r_1=r_2$, $\cos\theta_1 = d/2r_1$, $r_1 = (d^2/4 + r^2)^{1/2}$; $\mathbf{n}=-\mathbf{i}_2$,
$(\mathbf{e}_{r1}\ , -\mathbf{i}_2) = \cos\theta_1$, $(\mathbf{e}_{r2}\ , -\mathbf{i}_2) = -\cos\theta_2$,

$$\boldsymbol{\sigma}(\mathbf{n}) = a\ [(\cos^2\theta_1/r_1)\ \mathbf{e}_{r1} - (\cos^2\theta_2/r_2)\ \mathbf{e}_{r2} + (1/d)\ \mathbf{i}_2]$$

$$\boldsymbol{\sigma}(\mathbf{n}) = a\ [(\cos^2\theta_1/r_1)\ (\sin\theta_1\ \mathbf{i}_1 - \cos\theta_1\ \mathbf{i}_2) - (\cos^2\theta_2/r_2)\ (\sin\theta_2\ \mathbf{i}_1 + \cos\theta_2\ \mathbf{i}_2) + (1/d)\ \mathbf{i}_2]$$

$$\boldsymbol{\sigma}(\mathbf{n}) = a\ [-2\cos^3\theta_1/r_1 + (1/d)\ ]\ \mathbf{i}_2 = a/d\ [-d^4/4r_1^{\ 4} + 1\ ]\ \mathbf{i}_2$$

donc, il n'y a pas de cisaillement sur cet axe;
si $r_1=d/2$, (au centre), la contrainte normale est max. et vaut: $-3a/d$
si $r_1=d/\sqrt{2}$, (au bord), la contrainte normale vaut zéro.

Résultante (des forces de contact de la partie inférieure $x_2<0$ sur la partie $x_2>0$:

$$(a/d) \int_{]-d/2,d/2[} [-d^4/4(d^2/4 + r^2)^2 + 1\ ]\ dr = -(a/d)\ d\pi/2 = -a\ \pi/2 = P$$

on en déduit a.

Répartition de σ_{22} sur l'axe $x_2=0$:

4. **Sur l'axe** $\mathbf{x_1=0}$, $\theta_1=\theta_2=0$, $\theta=\pm\pi/2$, $\mathbf{e}_{r1} = -\mathbf{i}_2$, $\mathbf{e}_{r2}=\mathbf{i}_2$, $\mathbf{n} = \mathbf{i}_1$ (si on regarde l'action de $x_1>0$ sur $x_1<0$):

$$\boldsymbol{\sigma}/a = -(1/d)\ \mathbf{i}_1$$

il n'y a, à nouveau, pas de contrainte de cisaillement. Comme a et d sont positifs, la contrainte normale est une traction. L'essai est intéressant car il est difficile de pratiquer un essai de traction simple sur le béton, à cause des problèmes d'arrimage de l'échantillon à la presse.

Figure 2.7 Solution of Problem 2.12.

2.13 Homework 1 - Problem 5

Taking the axis x_3 normal to the sheet of the paper, draw the final configuration of the displacement fields ($\underline{u}$) given below, in which a is a small positive quantity. Calculate the deformation tensor that contributes to these displacements:

1. $u_1 = a\,x_1$, $u_2 = a\,x_2$, $u_3 = 0$
2. $u_1 = a\,x_2$, $u_2 = a\,x_1$, $u_3 = 0$
3. $u_1 = a\,x_2$, $u_2 = a\,x_2$, $u_3 = 0$
4. $u_1 = a\,x_2$, $u2 = -a\,x_1$, $u_3 = 0$

Solution: There is no displacement in direction 3, and none of the other displacements depends on x_3, so that all the derivatives $\partial u_i/\partial x_3$ are zero. As a result, each of the configurations presented are in plane strain, with the strains remaining in the plane (x_1,x_2). Additionally, $a << 1$, so we can make the assumption of small deformation. The final configurations of the four displacement fields are shown in Figure 2.8. The deformation tensor is 2 x 2 because we are in plane strain (i.e., the components 33, 23 and 13 are zero) and it is calculated by using the definition of the linearized deformation tensor since we are in small deformation. Results are as follows:

1. Gradient of the displacement field:

   $$\left[\underline{\underline{\nabla}}\left(\underline{u}\right)\right] = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

   Deformation tensor:

   $$\left[\underline{\underline{\epsilon}}\right] = \frac{1}{2}\left(\underline{\underline{\nabla}}\left(\underline{u}\right) + {}^{T}\,\underline{\underline{\nabla}}\left(\underline{u}\right)\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

   This is a volumetric deformation.

2. Gradient of the displacement field:

   $$\left[\underline{\underline{\nabla}}\left(\underline{u}\right)\right] = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

   Deformation tensor:

   $$\left[\underline{\underline{\epsilon}}\right] = \frac{1}{2}\left(\underline{\underline{\nabla}}\left(\underline{u}\right) + {}^{T}\,\underline{\underline{\nabla}}\left(\underline{u}\right)\right) = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

   This is a shear deformation.

3. Gradient of the displacement field:

   $$\left[\underline{\underline{\nabla}}\left(\underline{u}\right)\right] = \begin{bmatrix} 0 & a \\ 0 & a \end{bmatrix}$$

Deformation tensor:

$$[\underline{\epsilon}] = \frac{1}{2}\left(\underline{\underline{\nabla}}\,(\underline{u}) + ^T\underline{\underline{\nabla}}\,(\underline{u})\right) = \begin{bmatrix} 0 & a/2 \\ a/2 & a \end{bmatrix}$$

4. Gradient of the displacement field:

$$\left[\underline{\underline{\nabla}}\,(\underline{u})\right] = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

Deformation tensor:

$$[\underline{\epsilon}] = \frac{1}{2}\left(\underline{\underline{\nabla}}\,(\underline{u}) + ^T\underline{\underline{\nabla}}\,(\underline{u})\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This is a rotation, which is a rigid body motion. Therefore, there is no deformation.

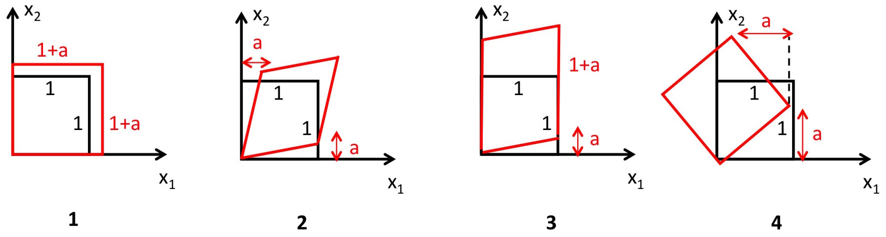


Figure 2.8 Illustration of the deformations produced by the displacements in Problem 2.13.

2.14 Homework 1 - Problem 6

A constant volume axial compression test is conducted on a soil sample which has initial dimensions x_1, x_2, $x_2 = x_3$. Assuming that the specimen has lubricated ends and that it deforms with plane sides, express the strain state in the sample using the Lagrangian and Eulerian finite strain tensors for a vertical displacement, $u_1 = k\,x_1$. Show how the major principal strains compare with the principal strains under the assumption of small deformation, as functions of k.

Solution: The specimen is lubricated at top and bottom, its faces remain plane, and the deformations in direction 2 and 3 are expected to be the same. Therefore it is anticipated that the specimen will undergo no distortion (or shear), and that it will undergo the same elongation in directions 2 and 3. Additionally, there is a negative elongation in direction 1, due to the loading. The displacement field is thus of the form:

$$\boldsymbol{u} = k\,x_1\boldsymbol{e}_1 + \alpha\,x_2\boldsymbol{e}_2 + \alpha\,x_3\boldsymbol{e}_3$$

The experiment is conducted under constant volume, so $x_1 x_2 x_3 = x'_1 x'_2 x'_3$. Since $x'_1 = (1+k)x_1$, then we have $x'_2 = x_2/\sqrt{(1+k)}$ and $x'_2 = x_3/\sqrt{(1+k)}$ and therefore:

$$\boldsymbol{u} = k\,x_1 \boldsymbol{e}_1 + \left(\frac{1}{\sqrt{1+k}} - 1\right) x_2 \boldsymbol{e}_2 + \left(\frac{1}{\sqrt{1+k}} - 1\right) x_3 \boldsymbol{e}_3$$

The Lagrangian finite strain tensor $\boldsymbol{L}$ is defined as follows:

$$L_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right]$$

and we calculate:

$$[\boldsymbol{L}] = \frac{1}{2}\begin{bmatrix} 2k+k^2 & 0 & 0 \\ 0 & -\frac{k}{1+k} & 0 \\ 0 & 0 & -\frac{k}{1+k} \end{bmatrix}$$

By definition of the displacement vector, $\boldsymbol{u} = \boldsymbol{x'} - \boldsymbol{x}$, in which $\boldsymbol{x'}$ is the the new position vector, which can be expressed explicitly as:

$$\boldsymbol{x'} = (1+k)\,x_1 \boldsymbol{e}_1 + (1+\alpha)\,x_2 \boldsymbol{e}_2 + (1+\alpha)\,x_3 \boldsymbol{e}_3$$

We can now express the displacement vector in terms of the new position vector:

$$\boldsymbol{u} = \frac{k}{1+k}\,x'_1 \boldsymbol{e}_1 + \frac{\alpha}{1+\alpha}\,x'_2 \boldsymbol{e}_2 + \frac{\alpha}{1+\alpha}\,x'_3 \boldsymbol{e}_3$$

$$\boldsymbol{u} = \frac{k}{1+k}\,x'_1 \boldsymbol{e}_1 + \left(1 - \sqrt{1+k}\right)\,x'_2 \boldsymbol{e}_2 + \left(1 - \sqrt{1+k}\right)\,x'_3 \boldsymbol{e}_3$$

We can now calculate the Eulerian finite strain tensor, which is defined as:

$$E_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x'_j} + \frac{\partial u_j}{\partial x'_i} - \frac{\partial u_k}{\partial x'_i}\frac{\partial u_k}{\partial x'_j}\right]$$

We find:

$$[\boldsymbol{E}] = \frac{1}{2}\begin{bmatrix} \frac{2k+k^2}{(1+k)^2} & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & -k \end{bmatrix}$$

In small deformation, the linearized deformation tensor, defined as the symmetric part of the gradient of displacement in the initial configuration, is:

$$[\boldsymbol{\epsilon}] = \begin{bmatrix} k & 0 & 0 \\ 0 & \left(\frac{1}{\sqrt{1+k}} - 1\right) & 0 \\ 0 & 0 & \left(\frac{1}{\sqrt{1+k}} - 1\right) \end{bmatrix}$$

The major principal strains are $\epsilon_{11} = k$ in small deformation, $L_{11} = k + k^2/2$ in Lagrangian finite strain and $E_{11} = \frac{k+k^2/2}{(1+k)^2}$ in Eulerian finite strain.

2.15 Homework 2 - Problem 1

Let us consider a cylinder of length L and circular cross-section with radius a. We assume that the state of stress in the cylinder is given as $\boldsymbol{\sigma} = \alpha\, r\, (\boldsymbol{e}_\theta \otimes \boldsymbol{e}_z + \boldsymbol{e}_z \otimes \boldsymbol{e}_\theta)$. Gravity and inertia forces are neglected.

1. Show that the state of stress satisfies the equations of equilibrium and the boundary condition on the lateral face (free of stress).
2. Calculate the state of stress on the top and bottom faces ($z = 0$ and $z = L$). Calculate the resulting moment about the axis of the cylinder. From there, calculate α as a function of the loading and as a function of I_p, the polar moment of inertia of the section.
3. Propose an experimental set-up that can create this state of stress. Discuss any challenge that you anticipate in this set-up.
4. Calculate the first two invariants of the stress tensor and of the deviatoric tensor.
5. Suppose that the constitutive material breaks when the norm of the deviatoric stress exceeds a certain limit k. Calculate the maximum couple that can be exerted on the cylinder. Dimension the cylindrical transmission axis of a truck for a couple $C = 3,000\, N.m$, with $k = 120\, MPa$ (steel).
6. Now suppose that the cylindrical transmission axis is made of a composite material with long fibers. In what direction should the fibers be oriented to optimize the design of the cylindrical transmission axis?

Solution:

1. Since gravity and inertia forces are neglected, the equation of equilibrium is satsified if and only if:

$$\boldsymbol{div\sigma} = \boldsymbol{0}$$

We have:

$$\boldsymbol{div\sigma} = \frac{\partial \sigma_{ij}}{\partial x_j}\boldsymbol{e}_i = \frac{\partial \sigma_{rj}}{\partial x_j}\boldsymbol{e}_r + \frac{\partial \sigma_{\theta j}}{\partial x_j}\boldsymbol{e}_\theta + \frac{\partial \sigma_{zj}}{\partial x_j}\boldsymbol{e}_z$$

Since $\sigma_{rr} = 0$ and $\sigma_{\theta r} = \sigma_{\theta\theta} = \sigma_{zr} = \sigma_{zz} = 0$

$$\boldsymbol{div\sigma} = \frac{\partial \sigma_{\theta z}}{\partial z}\boldsymbol{e}_\theta + \frac{\partial \sigma_{z\theta}}{\partial \theta}\boldsymbol{e}_z$$

Since $\sigma_{\theta z} = \sigma_{z\theta}$ does not depend on θ or z:

$$\boldsymbol{div\sigma} = \boldsymbol{0}$$

which shows that the proposed state of stress verifies the conditions of equilibrium. On the lateral face, $r = a$ and the normal is $\boldsymbol{e}_r$. With the proposed state of stress, the tractions on the lateral face are thus:

$$\boldsymbol{t} = \boldsymbol{\sigma}(r = a) \cdot \boldsymbol{e}_r = \alpha\, a\, (\boldsymbol{e}_\theta \otimes \boldsymbol{e}_z + \boldsymbol{e}_z \otimes \boldsymbol{e}_\theta) \cdot \boldsymbol{e}_r = \boldsymbol{0}$$

Hence, the proposed state of stress ensures that the lateral faces are free of stress.

2. On the top face, $z = L$ and $\boldsymbol{n} = +\boldsymbol{e}_z$. The state of stress is thus:

$$\sigma_{rz} = \boldsymbol{e}_r \cdot \boldsymbol{\sigma}(z = L) \cdot \boldsymbol{e}_z = 0, \quad \sigma_{\theta z} = \boldsymbol{e}_\theta \cdot \boldsymbol{\sigma}(z = L) \cdot \boldsymbol{e}_z = +\alpha\, r, \quad \sigma_{zz} = \boldsymbol{e}_z \cdot \boldsymbol{\sigma}(z = L) \cdot \boldsymbol{e}_z = 0$$

Similarly on the borrom face, $z = 0$ and $\boldsymbol{n} = -\boldsymbol{e}_z$, and the state of stress is:

$$\sigma_{rz} = 0, \quad \sigma_{\theta z} = -\alpha\, r, \quad \sigma_{zz} = 0$$

The resulting moment about the axis of the cylinder is:

$$\mathcal{M}_z = \int_{S_z} (r\, \boldsymbol{e}_r) \times (\boldsymbol{\sigma} \cdot \boldsymbol{e}_z)\, dS = \int_{S_z} (r\, \boldsymbol{e}_r) \times (\alpha\, r \boldsymbol{e}_\theta)\, dS$$

Using cylindrical coordinates:

$$\mathcal{M}_z = \alpha \left(\int_{r=0}^{a} r^3 dr \int_{\theta=0}^{2\pi} d\theta \right) \boldsymbol{e}_z = \frac{\pi \alpha a^4}{2}\, \boldsymbol{e}_z$$

Noting that:

$$I_p = \frac{\pi a^4}{2}$$

We have:

$$\alpha = \frac{\mathcal{M}_z \cdot \boldsymbol{e}_z}{I_p}$$

3. The proposed state of stress could be created by fixing the bottom face and subjecting the top face to a couple equal to $\mathcal{M}_z$, while leaving the lateral faces free of stress. Some challenges: (i) Ensuring propoer grip at the top and bottom; (ii) Avoiding concentration of stresses for those grips.

4. Stress invariants:

$$I_1 = Tr\,(\boldsymbol{\sigma}) = 0$$

$$I_2 = \frac{1}{2}\left[Tr\,(\boldsymbol{\sigma})^2 - Tr\,\left(\boldsymbol{\sigma}^2\right) \right] = -\frac{1}{2} Tr\,\left(\boldsymbol{\sigma}^2\right)$$

with:

$$\boldsymbol{\sigma}^2 = \alpha^2\, r^2\, (\boldsymbol{e}_\theta \otimes \boldsymbol{e}_z + \boldsymbol{e}_z \otimes \boldsymbol{e}_\theta) \cdot (\boldsymbol{e}_\theta \otimes \boldsymbol{e}_z + \boldsymbol{e}_z \otimes \boldsymbol{e}_\theta) = \alpha^2\, r^2\, (\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_z \otimes \boldsymbol{e}_z)$$

So that:

$$I_2 = -\alpha^2\, r^2$$

The stress tensor is equal to the deviatoric stress tensor, so that $I_1 = J_1 = 0$ and $I_2 = -J_2 = -\alpha^2\, r^2$.

5. Here we recall that the norm of a matrix is:

$$|\boldsymbol{A}| = Tr\,\left(\boldsymbol{A} \cdot \boldsymbol{A}^T\right)^{1/2}$$

As a result, the norm of the deviatoric stress is:

$$|\boldsymbol{s}| = Tr\,\left(\boldsymbol{s}^2\right)^{1/2} = \sqrt{2\, J_2} = \left(2\alpha^2\, r^2\right)^{1/2} = \sqrt{2}\alpha\, r$$

The maximum deviatoric stress is reached at $r = a$:

$$|\boldsymbol{s}^{max}| = \sqrt{2}\alpha\, a$$

If the material breaks when $|\boldsymbol{s}^{max}|$ reaches the value k, then the maximum possible value of α is:

$$\alpha^{max} = \frac{k}{a\sqrt{2}}$$

and the maximum couple that can be exerted on the specimen is:

$$\mathcal{C}^{max} = \alpha^{max}\, I_p = \alpha^{max}\,\frac{\pi a^4}{2} = \frac{k\pi a^3}{2\sqrt{2}}$$

For the dimensions given in the problem, we find that the minimum radius of the cylinder should be: $a_{min} = 2.82\,\mathrm{cm}$.

6. It is advised to orient the fibers such that the direction of maximum shear in the specimen corresponds to the direction of maximum strength of the fibers. Fibers typically resist the most to tension in the direction of their axis. The maximum shear stress in the specimen is oriented at an angle of $\pm 45^o$ to the specimen axis. Hence, it is recomended to orient the fibers at $\pm 45^o$ to the cylinder axis.

2.16 Homework 2 - Problem 2

For standard triaxial tests (axisymmetric geometry) on soils, a good representation of stress states in the soil is obtained by introducing the stresses $\sigma_{oct} = \sigma_{kk}/3$ and $q = \sigma_1 - \sigma_3$, which are related to the invariants of the stress tensor I_1 and J_2, respectively. The stress-strain behavior can be considered by introducing two measures of strains, ϵ_{vol} and ϵ_s, which are energetically compatible with theses stresses such that:

$$\delta W = \sigma_{ij}\delta\epsilon_{ij} = \sigma_{oct}\delta\epsilon_{vol} + q\delta\epsilon_s$$

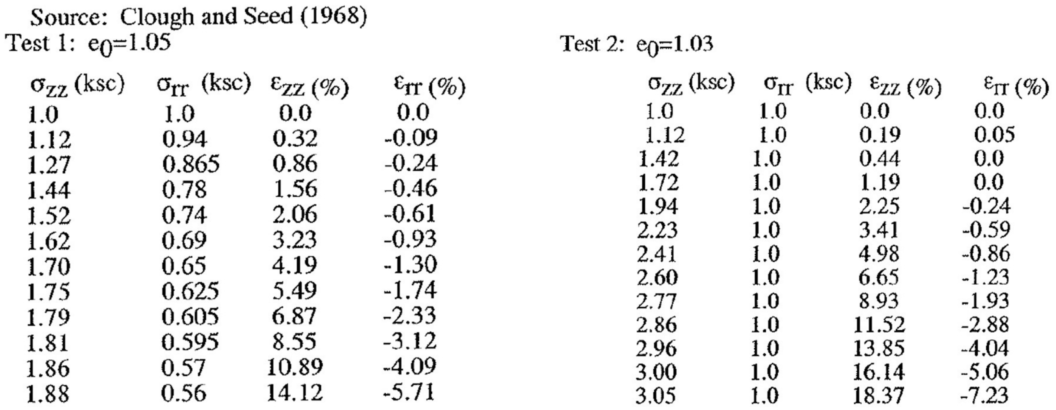
1. Find an expression for ϵ_s to satisfy this relationship. Show that ϵ_{vol} and ϵ_s can be expressed in terms of the invariants of the strain tensor.

2. Figure 2.9 lists the results of two drained triaxial tests on medium dense sand. Plot the results (i.e. plot q vs. σ_{oct}, q vs. ϵ_s, ϵ_{vol} vs. ϵ_s and σ_{oct} vs. ϵ_{vol}). How well does the linear, isotropic elastic model describe the behavior of sands measured in these tests? (Assume values for the elastic moduli K, G and compare with the measurements).

Solution:

1. We first note that due to the symmetry of the test, the stress and strain tensors take the following forms:

$$\begin{cases} \boldsymbol{\sigma} = \sigma_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_3 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \sigma_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \\ \boldsymbol{\epsilon} = \epsilon_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \epsilon_3 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \epsilon_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \end{cases}$$



Source: Clough and Seed (1968)

Test 1: e_0=1.05

σ_{zz} (ksc)	σ_{rr} (ksc)	ε_{zz} (%)	ε_{rr} (%)
1.0	1.0	0.0	0.0
1.12	0.94	0.32	-0.09
1.27	0.865	0.86	-0.24
1.44	0.78	1.56	-0.46
1.52	0.74	2.06	-0.61
1.62	0.69	3.23	-0.93
1.70	0.65	4.19	-1.30
1.75	0.625	5.49	-1.74
1.79	0.605	6.87	-2.33
1.81	0.595	8.55	-3.12
1.86	0.57	10.89	-4.09
1.88	0.56	14.12	-5.71

Test 2: e_0=1.03

σ_{zz} (ksc)	σ_{rr} (ksc)	ε_{zz} (%)	ε_{rr} (%)
1.0	1.0	0.0	0.0
1.12	1.0	0.19	0.05
1.42	1.0	0.44	0.0
1.72	1.0	1.19	0.0
1.94	1.0	2.25	-0.24
2.23	1.0	3.41	-0.59
2.41	1.0	4.98	-0.86
2.60	1.0	6.65	-1.23
2.77	1.0	8.93	-1.93
2.86	1.0	11.52	-2.88
2.96	1.0	13.85	-4.04
3.00	1.0	16.14	-5.06
3.05	1.0	18.37	-7.23

Figure 2.9 Triaxial test data.

in which the loading is done in the first direction. We have:

$$\begin{cases} \delta W = \sigma_{ij}\delta\epsilon_{ij} = \sigma_1\delta\epsilon_1 + 2\sigma_3\delta\epsilon_3 \\ \delta W = \frac{1}{3}\left(\sigma_1 + 2\sigma_3\right)\left(\delta\epsilon_1 + 2\delta\epsilon_3\right) + \left(\sigma_1 - \sigma_3\right)\delta\epsilon_s \end{cases}$$

To find the expression of ϵ_s, we proceed by identification, and we find:

$$\epsilon_s = \frac{2}{3}\left(\epsilon_1 - \epsilon_3\right)$$

Let us recall that the second invariant of the deviatoric stress tensor can be calculated as:

$$J_2 = \frac{1}{2}Tr(\boldsymbol{s}^2) = \frac{1}{2}\left[\frac{4}{9}\left(\sigma_1 - \sigma_3\right)^2 + \frac{1}{9}\left(\sigma_3 - \sigma_1\right)^2 + \frac{1}{9}\left(\sigma_3 - \sigma_1\right)^2\right]$$

After simplification:

$$J_2 = \frac{1}{3}\left(\sigma_1 - \sigma_3\right)^2$$

And in the same way, the second invariant of the deviatoric strain tensor is:

$$J_2^\epsilon = \frac{1}{3}\left(\epsilon_1 - \epsilon_3\right)^2$$

As a result, we have:

$$\epsilon_s = \frac{2}{3}\sqrt{3\,J_2^\epsilon}$$

Additionally, we can readily see that:

$$\epsilon_{vol} = Tr\left(\boldsymbol{\epsilon}\right) = I_1^\epsilon$$

2. This question requires plotting the experimental results, fitting them with a linear elastic law, and calculating the error made by this fit.

2.17 Homework 2 - Problem 3

When we interpret the results of a triaxial axisymmetric test, we assume a uniform state of stress in the sample (as described in Problem 2), which requires that the porous stones (platens) are frictionless.

1. Draw a sketch of the triaxial specimen (with height H and diameter D) and indicate the boundary conditions which are implied by this assumption. If the piston is rigid and the sample is made of linear, isotropic (elastic) material, determine the stresses and strains in the sample in terms of the confining pressure, σ_3, and the displacement of the upper piston, δ.
2. In a real experiment, there is no slippage between the specimen and the porous stones. How does this affect the solution of the problem? Can you solve this problem when $\sigma_3 = 0$ and $\nu = 0$?

Solution:

1. Boundary conditions (direction 1 is the loading direction):

    - Top surface, $z = H$, normal $+\boldsymbol{e}_1$: $u_1 = \delta$ (controlled displacement), $\sigma_{21} = \sigma_{31} = 0$ (no friction);
    - Bottom surface, $z = 0$, normal $-\boldsymbol{e}_1$: $u_1 = 0$ (fixed displacement), central node: $u_2 = u_3 = 0$ (fixed) and rest of the boundary: $\sigma_{21} = \sigma_{31} = 0$ (no friction);
    - Lateral surface, $r = D/2$, normal $\boldsymbol{e}_r$: $\sigma_{22} = \sigma_{33} = \sigma_{rr} = \sigma_3$ (uniform confining stress) and $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$ (no tangential traction on the lateral face).

    In the absence of shear stresses, the stresses σ_1 and σ_3 are the principal stresses and the strains ϵ_1 and ϵ_3 are the principal strains. Due to the symmetry of the test, the stress and strain tensors take the following forms:

    $$\boldsymbol{\sigma} = \sigma_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_3 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \sigma_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

    $$\boldsymbol{\epsilon} = \epsilon_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \epsilon_3 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \epsilon_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

    in which σ_3 is known and in which ϵ_1 is imposed by the boundary conditions:

    $$\epsilon_1 = \delta/H$$

    Next, σ_1 and ϵ_3 can be found by using Hooke's law, as follows:

    $$\begin{cases} \sigma_1 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_1 + 2\nu\epsilon_3\right] \\ \epsilon_3 = \frac{\sigma_3}{E} - \nu\frac{(\sigma_1+\sigma_3)}{E} \end{cases}$$

    We get:

    $$\epsilon_3 = \frac{(1-\nu)\sigma_3}{E} - \frac{\nu}{E} \times \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\delta}{H} + 2\nu\epsilon_3\right]$$

from which we get:

$$\epsilon_3 = \frac{(1+\nu)(1-2\nu)}{E}\sigma_3 - \frac{\nu(1+\nu)}{(1-\nu)} \times \frac{\delta}{H}$$

and lastly:

$$\sigma_1 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\delta}{H} - \frac{2\nu^2(1+\nu)}{(1-\nu)} \times \frac{\delta}{H} + 2\nu\frac{(1+\nu)(1-2\nu)}{E}\sigma_3\right]$$

$$\sigma_1 = 2\nu\sigma_3 + \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu) - \frac{2\nu^2(1+\nu)}{(1-\nu)}\right]\frac{\delta}{H}$$

2. In a real problem, the boundary conditions are:

   - Top surface, $z = H$, normal $+\boldsymbol{e}_1$: $u_1 = \delta$ (controlled displacement), $u_2 = u_3 = 0$ (no slippage);
   - Bottom surface, $z = 0$, normal $-\boldsymbol{e}_1$: $u_1 = 0$ (fixed displacement), $u_2 = u_3 = 0$ (no slippage);
   - Lateral surface, $r = D/2$, normal $\boldsymbol{e}_r$: $\sigma_{22} = \sigma_{33} = \sigma_{rr} = \sigma_3$ (uniform confining stress) and $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$ (no tangential traction on the lateral face).

   In a real test, shear stresses σ_{12} and σ_{13} can develop at the the top and bottom surfaces of the specimen. The state of stress is not uniform, since σ_{12} and σ_{13} are zero at the lateral surface for $0 < z < H$. Maintaining the axis-smmetry assumption, the stress and strain tensors are now expressed as:

$$\boldsymbol{\sigma} = \sigma_{11}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_3\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \sigma_3\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \ + \sigma_{13}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \ + \sigma_{13}\boldsymbol{e}_1 \otimes \boldsymbol{e}_3$$

$$\boldsymbol{\epsilon} = \epsilon_1\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \epsilon_{33}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \epsilon_{33}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \ + \epsilon_{13}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \ + \sigma_{13}\boldsymbol{e}_1 \otimes \boldsymbol{e}_3$$

   with $\epsilon_1 = \delta/H$ and σ_3 given. The unknowns are σ_{11}, σ_{13}, ϵ_{33} and ϵ_{13}, which now depend on the position in the specimen. It is immossible to solve for the stress and strain fields in the specimen unless additional assumptions are made. For instance, if $\nu = 0$, then the specimen will not undergo any lateral strains due to the compression in direction 1 and by the same token, there will not be any friction applied on the top and bottom platens. As a result: $\sigma_{13} = \epsilon_{13} = 0$ and $\epsilon_{33} = \sigma_3/E$. If, in addition, $\sigma_3 = 0$, we get $\epsilon_{33} = 0$, and also: $\sigma_1 = E\epsilon_1 = E\,\delta/H$. So we can solve the problem if we assume $\nu = 0$ and $\sigma_3 = 0$.

2.18 Exam 1 - Problem 1

Let us consider a cylindrical specimen of length L and initial section S_0. We seek to design a mechanical test in order to reproduce a uniform state of stress $\boldsymbol{\sigma} = \alpha\boldsymbol{e}_3 \otimes \boldsymbol{e}_3$ in the specimen, by only applying tractions at the boundary of the cylinder. α is a scalar constant and $\boldsymbol{e}_3$ is a unit vector parallel to the axis of the cylinder. Inertia forces are neglected.

1. Calculate the forces that need to be applied to the specimen. Propose an experimental set-up to apply those forces. Comment on the way the specimen will be anchored. Calculate the value of α as a function of the applied forces.

2. The peak force at failure, noted F_f, is measured during the experiment.What is the error made on the calculation of the ultimate stress (i.e. the stress at failure) if the stress is calculated by using the initial cross-section of the specimen (S_0) instead of the current cross section (noted S_t). Calcualte the error for the special case $S_t = 0.8\,S_0$.

3. Suppose that the stress field induced by gravity is of the form $\boldsymbol{\sigma}_g = -\rho\, g\, x_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$, in which ρ is the specific mass of the material (expressed in kg/m^3) and g is the gravity acceleration. Show that the equilibrium equations are satisfied. At what condition on the value of α can the gravity stress be neglected? Now suppose that the material that makes the specimen is used to design elevator cables. If the maximum stress that can be applied to the elevator cable is 200 MPa, what is the minimum number of elevators necessary to reach the bottom of a diamond mine located at a depth of 4 km? Assume $\rho = 7.7 \times 10^3$ kg/m^3.

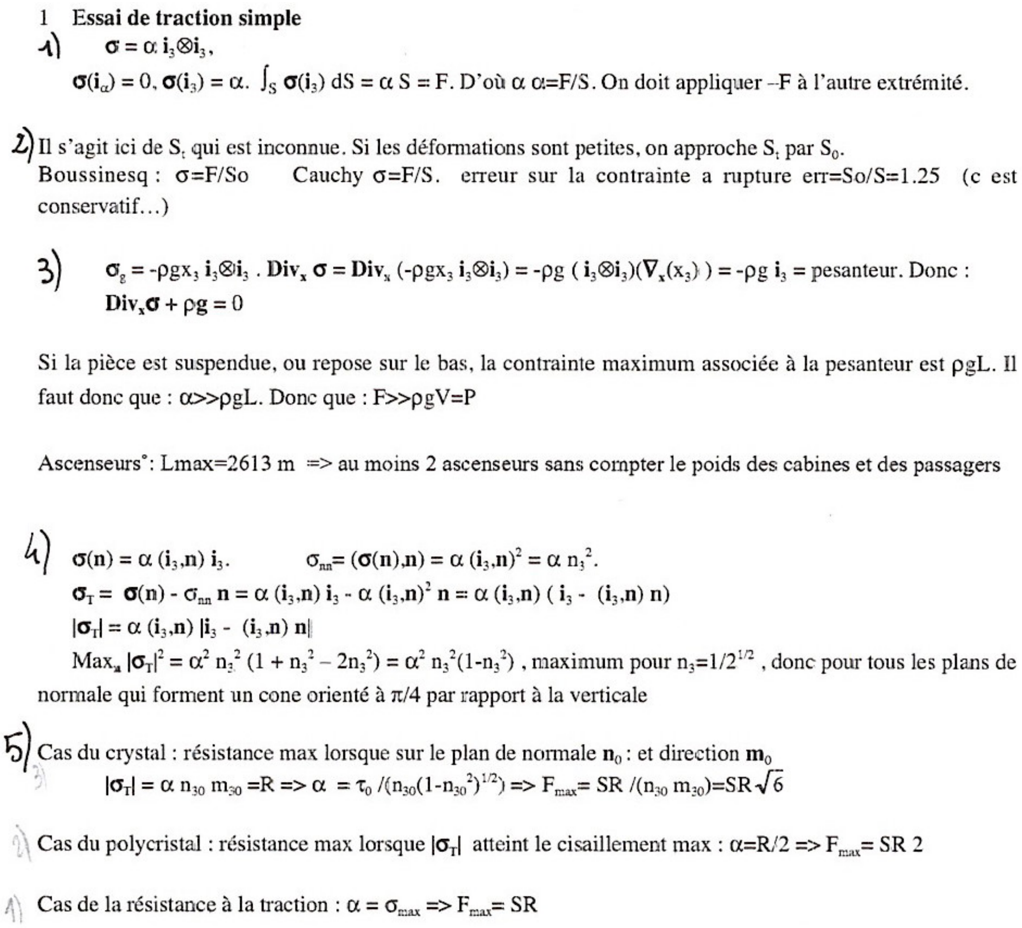
4. In the following, gravity forces are neglected. Express the stress components on a plane of normal $\boldsymbol{n}$ in the specimen. What is the orientation of the plane where the shear stress is maximum?

5. Three simple traction tests are performed with three different materials:

    - a material that breaks when the normal stress exceeds a certain resistance R;
    - a poly-crystalline material that breaks when the maximum shear stress exceeds the threshold R;
    - a mono-crystalline material that breaks when the shear stress in the direction $\boldsymbol{m_0} = \frac{1}{\sqrt{2}}(-1, 0, +1)$ on a plane of normal $\boldsymbol{n_0} = \frac{1}{\sqrt{3}}(1, 1, 1)$ exceeds the threshold R.

    Calculate the maximum force that can be applied to each of the three specimens. Assume small deformations.

Solution: see Figure 2.10



1 **Essai de traction simple**

1) $\boldsymbol{\sigma} = \alpha\, \mathbf{i}_3 \otimes \mathbf{i}_3$,

$\boldsymbol{\sigma}(\mathbf{i}_\alpha) = 0$, $\boldsymbol{\sigma}(\mathbf{i}_3) = \alpha$. $\int_S \boldsymbol{\sigma}(\mathbf{i}_3)\, dS = \alpha\, S = F$. D'où α α=F/S. On doit appliquer –F à l'autre extrémité.

2) Il s'agit ici de S_t qui est inconnue. Si les déformations sont petites, on approche S_t par S_0.
Boussinesq : σ=F/So Cauchy σ=F/S. erreur sur la contrainte a rupture err=So/S=1.25 (c est conservatif…)

3) $\boldsymbol{\sigma}_g = -\rho g x_3\, \mathbf{i}_3 \otimes \mathbf{i}_3$. $\mathbf{Div_x}\, \boldsymbol{\sigma} = \mathbf{Div}_x\, (-\rho g x_3\, \mathbf{i}_3 \otimes \mathbf{i}_3) = -\rho g\, (\mathbf{i}_3 \otimes \mathbf{i}_3)(\boldsymbol{\nabla}_\mathbf{x}(x_3)) = -\rho g\, \mathbf{i}_3$ = pesanteur. Donc :
$\mathbf{Div_x} \boldsymbol{\sigma} + \rho \mathbf{g} = 0$

Si la pièce est suspendue, ou repose sur le bas, la contrainte maximum associée à la pesanteur est ρgL. Il faut donc que : α>>ρgL. Donc que : F>>ρgV=P

Ascenseurs°: Lmax=2613 m => au moins 2 ascenseurs sans compter le poids des cabines et des passagers

4) $\boldsymbol{\sigma}(\mathbf{n}) = \alpha\, (\mathbf{i}_3,\mathbf{n})\, \mathbf{i}_3$. $\sigma_{nn} = (\boldsymbol{\sigma}(\mathbf{n}),\mathbf{n}) = \alpha\, (\mathbf{i}_3,\mathbf{n})^2 = \alpha\, n_3^2$.
$\boldsymbol{\sigma}_T = \boldsymbol{\sigma}(\mathbf{n}) - \sigma_{nn}\, \mathbf{n} = \alpha\, (\mathbf{i}_3,\mathbf{n})\, \mathbf{i}_3 - \alpha\, (\mathbf{i}_3,\mathbf{n})^2\, \mathbf{n} = \alpha\, (\mathbf{i}_3,\mathbf{n})\, (\mathbf{i}_3 - (\mathbf{i}_3,\mathbf{n})\, \mathbf{n})$
$|\boldsymbol{\sigma}_T| = \alpha\, (\mathbf{i}_3,\mathbf{n})\, |\mathbf{i}_3 - (\mathbf{i}_3,\mathbf{n})\, \mathbf{n}|$
$\mathrm{Max}_\mathbf{a}\, |\boldsymbol{\sigma}_T|^2 = \alpha^2 n_3^2\, (1 + n_3^2 - 2n_3^2) = \alpha^2 n_3^2 (1-n_3^2)$, maximum pour $n_3=1/2^{1/2}$, donc pour tous les plans de normale qui forment un cone orienté à $\pi/4$ par rapport à la verticale

5) Cas du crystal : résistance max lorsque sur le plan de normale $\mathbf{n}_0$: et direction $\mathbf{m}_0$
3) $|\boldsymbol{\sigma}_T| = \alpha\, n_{30}\, m_{30}$ =R => $\alpha = \tau_0 /(n_{30}(1-n_{30}^2)^{1/2})$ => F_{max}= SR /($n_{30}\, m_{30}$)=SR$\sqrt{6}$

2) Cas du polycristal : résistance max lorsque $|\boldsymbol{\sigma}_T|$ atteint le cisaillement max : α=R/2 => F_{max}= SR 2

1) Cas de la résistance à la traction : $\alpha = \sigma_{max}$ => F_{max}= SR

Figure 2.10 Solution of Exam 1, Problem 1.

CHAPTER 3

ANALYTICAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS IN LINEAR ELASTICITY

PROBLEMS

3.1 Prove the Navier's equations of motion in Cartesian coordinates:

$$\lambda\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}\right)+G\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho f_x=0$$

$$\lambda\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\partial z}\right)+G\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho f_y=0$$

$$\lambda\left(\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial z^2}\right)+G\left(\frac{\partial^2 u}{\partial z\partial y}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho f_z=0$$

Solution: We start with the equilibrium equation:

$$\boldsymbol{div}\left(\boldsymbol{\sigma}\right)+\rho\boldsymbol{f}=\boldsymbol{0}$$

A projection along the x-direction provides:

$$\frac{\partial\sigma_{xj}}{\partial x_j}+\rho f_x=0$$

Now, we recall the constitutive equation, in index notation:

$$\sigma_{ij} = \lambda Tr\left(\boldsymbol{\epsilon}\right)\delta_{ij} + 2G\epsilon_{ij}$$

Combining the two last equations above, we get:

$$\lambda \frac{\partial Tr\left(\boldsymbol{\epsilon}\right)}{\partial x_j}\delta_{x,j} + 2G\frac{\partial \epsilon_{xj}}{\partial x_j} + \rho f_x = 0$$

$$\lambda \frac{\partial}{\partial x}\left(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}\right) + 2G\left(\frac{\partial \epsilon_{xx}}{\partial x} + \frac{\partial \epsilon_{xy}}{\partial y} + \frac{\partial \epsilon_{xz}}{\partial z}\right) + \rho f_x = 0$$

We now recall the strain-displacement relationships:

$$\epsilon_{xx} = \frac{\partial u}{\partial x},\ \epsilon_{yy} = \frac{\partial v}{\partial y},\ \epsilon_{zz} = \frac{\partial w}{\partial z},\ \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ \epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),\ \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

The combination of the two last equations provides:

$$\lambda\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right) + G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho f_x = 0$$

which is the first of the three Navier's equations. The two other equations are obtained in the same way, by projecting the equilibrium equation in directions y and z, respectively.

3.2 We consider a pressurized circular cavity of radius a subjected to a uniform state of stress at a distance b from the center, as shown in Figure 3.1. We focus on a 2D elasticity problem (plane strain or plane stress). Find the state of stress around the cavity, between a and b, by solving the biharmonic equation.

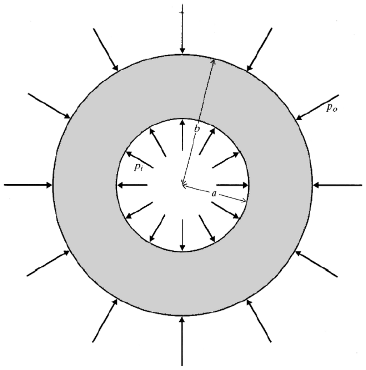


Figure 3.1 Circular pressurized cavity subject to isotropic stress in the far field. Picture taken from (Brady & Brown, 2004).

Solution: The equilibrium equation in terms of Airy's stress function is:

$$\nabla^4 U = 0$$

In cylindrical coordinates:

$$\boldsymbol{\nabla} U(r,\theta,z) = \frac{\partial U}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial U}{\partial z}\boldsymbol{e}_z$$

This problem is axis-symmetric. Suppose it is plane stress (the solution for the plane strain case is left for the reader). Then the equation above reduces to:

$$\boldsymbol{\nabla} U(r) = \frac{dU}{dr}\boldsymbol{e}_r$$

Using the formulae given in chapter 1 for the gradient of a vector in cylindrical coordinates, we find:

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla} U(r)\right) = \frac{d^2U}{dr^2}\boldsymbol{e}_r\otimes\boldsymbol{e}_r + \frac{1}{r}\frac{dU}{dr}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_\theta \tag{3.1}$$

so that:

$$\boldsymbol{\nabla}^2 U = Tr\left(\boldsymbol{\nabla}\left(\boldsymbol{\nabla} U\right)\right) = \frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr}$$

From there, we get:

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla}^2 U\right) = \left(\frac{d^3U}{dr^3} + \frac{1}{r}\frac{d^2U}{dr^2} - \frac{1}{r^2}\frac{dU}{dr}\right)\boldsymbol{e}_r$$

$$\begin{aligned}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\left(\boldsymbol{\nabla}^2 U\right)\right) = \quad & \left(\frac{d^4U}{dr^4} + \frac{1}{r}\frac{d^3U}{dr^3} - \frac{1}{r^2}\frac{d^2U}{dr^2} - \frac{1}{r^2}\frac{d^2U}{dr^2} + \frac{2}{r^3}\frac{dU}{dr}\right)\boldsymbol{e}_r\otimes\boldsymbol{e}_r \\ & \frac{1}{r}\left(\frac{d^3U}{dr^3} + \frac{1}{r}\frac{d^2U}{dr^2} - \frac{1}{r^2}\frac{dU}{dr}\right)\boldsymbol{e}_\theta\otimes\boldsymbol{e}_\theta\end{aligned}$$

And we finally get:

$$\boldsymbol{\nabla}^4 U = \frac{d^4U}{dr^4} + \frac{2}{r}\frac{d^3U}{dr^3} - \frac{1}{r^2}\frac{d^2U}{dr^2} + \frac{1}{r^3}\frac{dU}{dr} \tag{3.2}$$

The solution to this equation is not trivial. We listed two of them in the notebook:

- The solution by Timoshenko and Goodier (1970):

  $$U(r) = A\,ln(r) + Br^2\,ln(r) + Cr^2 + D$$

- The solution by Barber (2010):

  $$U(r\theta) = A\,ln(r) + Cr^2 + E\theta$$

  in which A, B, C, D and E are constants. The reader can check that both solutions satisfy equation 3.2.

Using the relationships between the stress components and Airy's stress function, and using equation 3.1:

$$\sigma_{rr} = \boldsymbol{\nabla}_{\theta\theta}\left(\boldsymbol{\nabla} U\right) = \frac{1}{r}\frac{dU}{dr}, \quad \sigma_{\theta\theta} = \boldsymbol{\nabla}_{rr}\left(\boldsymbol{\nabla} U\right) = \frac{d^2U}{dr^2}, \quad \sigma_{r\theta} = \boldsymbol{\nabla}_{\theta r}\left(\boldsymbol{\nabla} U\right) = 0$$

From which we get:

- For the solution of Timoshenko and Goodier (1970):

$$\sigma_{rr} = \frac{A}{r^2} + 2B\,ln(r) + B + 2C, \quad \sigma_{\theta\theta} = -\frac{A}{r^2} + 2B\,ln(r) + 3B + 2C, \quad \sigma_{r\theta} = 0$$

- For the solution of Barber (2010):

$$\sigma_{rr} = \frac{A}{r^2} + 2C, \quad \sigma_{\theta\theta} = -\frac{A}{r^2} + 2C, \quad \sigma_{r\theta} = 0$$

We now use the boundary conditions to find the unknown constants: $\sigma_{rr}(r = a) = p_i$, $\sigma_{rr}(r = b) = p_0$. For instance, using the solution of Barber:

$$A = \frac{(p_i - p_0)a^2b^2}{(b^2 - a^2)}, \quad 2C = \frac{p_0b^2 - p_ia^2}{(b^2 - a^2)}$$

At this point, the state of stress is known. The reader can show that the same state of stress is found if the solution of Timoshenko and Goodier is used.

3.3 Show that in Cartesian coordinates, the stress solution can be expressed in terms of two complex potentials $\phi(z)$ and $\psi(z) = \chi'(z)$, as follows:

$$\sigma_{xx} + \sigma_{yy} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 2\left[\phi'(z) + \overline{\phi'(z)}\right] = 2\mathcal{R}\left[\phi'(z)\right]$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} - 2i\frac{\partial^2 U}{\partial x \partial y} = 2\left[\overline{z}\phi''(z) + \psi'(z)\right]$$

Solution: To solve this problem, one has to be able to calculate the second derivatives of Airy's stress function, by using the derivation rules that apply to complex functions. To illustrate the procedure, we provide the details of the calculation of the first order derivatives of Airy's stress function. The second-order derivations are left to the reader. The final expressions of the fist and second - order derivatives of Airy's stress function are provided in the notebook for reference. We recall that, in virtue of Cauchy-Rieman's equation, for a complex function $\zeta(z) = \xi(z) + i\eta(z)$:

$$\zeta'(z) = \frac{\partial \xi}{\partial x} - i\frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial y} + i\frac{\partial \eta}{\partial x} = \frac{\partial \xi}{\partial x} + i\frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial y} - i\frac{\partial \xi}{\partial y}$$

Then, noting that $z = x + iy$ and $\overline{z} = x - iy$, we have:

$$2\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}\left(\overline{z}\phi(z) + z\overline{\phi}(z) + \chi(z) + \overline{\chi}(z)\right)$$

$$2\frac{\partial U}{\partial x} = \phi(z) + \overline{z}\phi'(z) + \overline{\phi}(z) + z\overline{\phi}'(z) + \chi'(z) + \overline{\chi}'(z)$$

Similarly:

$$2\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left(\overline{z}\phi(z) + z\overline{\phi}(z) + \chi(z) + \overline{\chi}(z)\right)$$

$$2\frac{\partial U}{\partial y} = -i\phi(z) + i\overline{z}\phi'(z) + i\overline{\phi}(z) - iz\overline{\phi}'(z) + i\chi'(z) - i\overline{\chi}'(z)$$

3.4 At a depth of 750 m, a 10-m diameter circular tunnel is driven in rock having a unit weight of 26 kN/m^3 and uniaxial compressive and tensile strengths of 80 MPa and 3 MPa, respectively. Will the strength of the rock on the tunnel boundary be exceeded if: (i) K=0.3? (ii) K=2?

Solution: This is a problem of free circular cavity subjected to a biaxial state of stress in the far field. We can solve it by using Kirsch's equations. At the cavity wall, $\sigma_{rr} = \sigma_{r\theta} = 0$ (free cavity), and we have:

$$\sigma_{\theta\theta}(r = a) = \sigma_v \left(1 + K + 2(1 - K)\cos 2\theta\right)$$

We have:

$$\sigma_v = -\gamma h = 26 \times 10^3 \times 750 = -19.5\text{MPa}$$

(i) If $K = 0.3$:

$$\sigma_{\theta\theta}(r = a) = -\gamma h\,(1.3 + 1.4\cos 2\theta)$$

The maximum tension (counted positive here) occurs when $\cos 2\theta = -1$, and $\sigma_{\theta\theta}(r = a) = +0.1\gamma h = 1.95\text{MPa}$. The maximum tension is below the tensile strength, so there is no risk of tensile failure at the cavity wall. The maximum compression occurs when $\cos 2\theta = +1$, and $\sigma_{\theta\theta}(r = a) = -2.7\gamma h = -52.7\text{MPa}$. The maximum compression is below the compressive strength, so there is no risk of compressive failure at the cavity wall.

(ii) If $K = 2$:

$$\sigma_{\theta\theta}(r = a) = -\gamma h\,(3 - 2\cos 2\theta)$$

The maximum value of $\sigma_{\theta\theta}(r = a)$ occurs when $\cos 2\theta = +1$. The stress is still negative for those angles, which means that for $K = 2$, the cavity is entirely in compression. Therefore there is no risk of tensile failure at the cavity wall. The maximum compression occurs when $\cos 2\theta = -1$, and $\sigma_{\theta\theta}(r = a) = -5\gamma h = -97.5\text{MPa}$, which exceeds the compressive strength. So there is a risk of compressive failure at the cavity wall.

3.5 A gold-bearing quartz vein, 2 m thick and dipping 90^o, is to be exploited by a small-cut-and-fill stoping operation. The mining is to take place at a depth of 800 m, and the average unit weight of the granite host rock above this level is 29 kN/m^3. The strike of the vein is parallel to the intermediate stress, and the major principal stress is horizontal with a magnitude of 37 MPa. The uniaxial compressive strength of the vein material is 218 MPa (in absolute value), and the tensile strength of the host rock is 5 MPa (in absolute value). What is the maximum permissible stope height before failure occurs?

Solution: Here, the cross section of the cavity is oriented with its width horizontal, its height vertical, and the axis of the cavity is orthogonal to the plane of the paper sheet (parallel to the intermediate stress direction). The width W of the cavity is 2 meters (to exploit full the vein), and the height H of the cavity is to be determined to avoid failure. To solve the problem, we use the formulae of the course that give the extremum values of the orthoradial stress $\sigma_{\theta\theta}$, at A (sidewall) and and at B (crown):

$$\sigma_A = p\left(1 - K + 2\frac{W}{H}\right), \quad \sigma_B = p\left(K - 1 + 2K\frac{H}{W}\right)$$

In this problem, $p = -\gamma h = 29 \times 10^3 \times 800 = -23.2$MPa, and $K = \sigma_H/p = 37/23.2 = 1.59$. Since the horizontal stress is larger than the vertical stress, it is expected that there will be compression at the crown (B) and tension at the sidewall (A). So to avoid failure at the wall, one must ensure: $\sigma_A \leq 5$MPa and $\sigma_B \geq 218$MPa. From there, we get:

$$-23.2\left(-0.59 + \frac{4}{H}\right) \leq 5$$

$$-23.2\left(0.59 + 3.18\frac{H}{2}\right) \geq -218$$

We get:

$$\frac{4}{H} \geq 0.59 - \frac{5}{23.2}$$

$$3.18\frac{H}{2} \leq \frac{218}{23.2} + 0.59$$

and lastly:

$$H \leq 5.5\text{m}$$

$$H \leq 10.7\text{m}$$

So overall, the height of the cavity should not exceed 5.5 meters.

3.6 In Figure 3.2, the uniaxial rock compressive strength is 50 MPa and the corresponding crack initiation stress is $\sigma_c = 16$ MPa. Calculate the extent of the failure zone in tension and compression.

Solution: This is a problem of free circular cavity subjected to a biaxial state of stress in the far field. We can solve it by using Kirsch's equations. At the cavity wall, $\sigma_{rr} = \sigma_{r\theta} = 0$ (free cavity), and we have:

$$\sigma_{\theta\theta}(r = a) = p\left(1 + K + 2(1 - K)\cos 2\theta\right)$$

with $p = 7.5$ MPa (compression counted positive) and $K = 1/3$:

$$\sigma_{\theta\theta}(r = a) = \frac{4p}{3}\left(1 + \cos 2\theta\right)$$

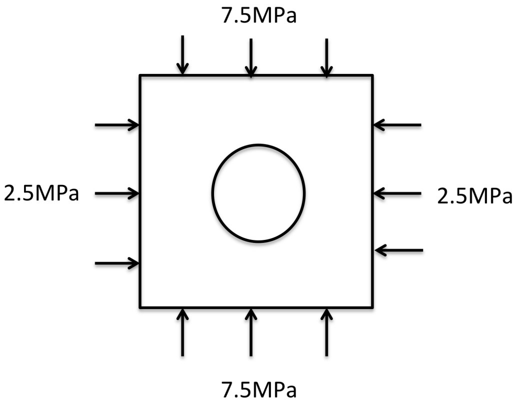


Figure 3.2 Cavity studied in Problem 3.6.

Failure occurs at the wall in compression if $\sigma_{\theta\theta}(r = a) \geq 16$MPa, i.e. $\cos 2\theta \geq 0.6$, i.e. for $-26^o \leq \theta \geq 26^o$. Failure occurs at the wall in tension if $\sigma_{\theta\theta}(r = a) \leq 0$MPa, i.e. $1+\cos 2\theta \leq 0$. This only happens at the crown ($\theta = 90^o$) and at the foot ($\theta = -90^o$).

3.7 Provided the boundary conditions in Figure 3.3, and knowing that the following purely frictional strength criterion holds: $\sigma_1 = d\,\sigma_3 + C_0$, $C_0 = 0$, calculate: (i) the extent of the damaged zone (r_e), (ii) the pressure in the damaged zone (p_1).

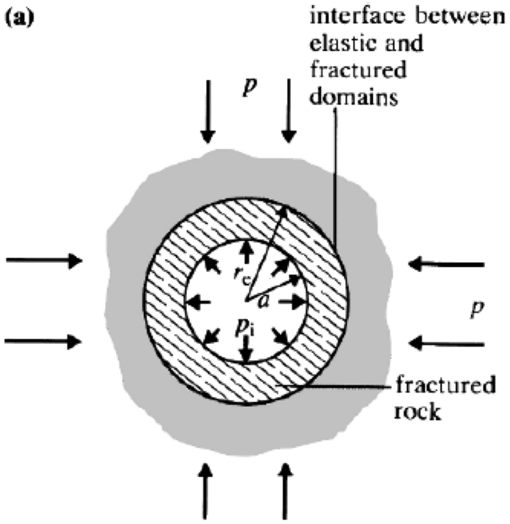


Figure 3.3 Cavity studied in Problem 3.7. (Brady & Brown, 2004)

Solution: It is not possible to use a known analytical solution in the damaged zone, since the behavior of the rock mass is not linear elastic. So we start by establishing the equation of equilibrium in the damaged zone:

$$\boldsymbol{div}\,(\boldsymbol{\sigma}) = \mathbf{0}$$

We calculate the divergence of the stress tensor in cylindrical coordinates:

$$\begin{aligned}\boldsymbol{div}\,(\boldsymbol{\sigma}) = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \quad & \left(\frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \frac{\partial}{\partial z}\boldsymbol{e}_z\right)\cdot\left(\sigma_{rr}\boldsymbol{e}_r\otimes\boldsymbol{e}_r + \sigma_{r\theta}\boldsymbol{e}_r\otimes\boldsymbol{e}_\theta + \sigma_{rz}\boldsymbol{e}_r\otimes\boldsymbol{e}_z\right)\\ & + \left(\frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \frac{\partial}{\partial z}\boldsymbol{e}_z\right)\cdot\left(\sigma_{\theta r}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_r + \sigma_{r\theta}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_\theta + \sigma_{\theta z}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_z\right)\\ & + \left(\frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \frac{\partial}{\partial z}\boldsymbol{e}_z\right)\cdot\left(\sigma_{zr}\boldsymbol{e}_z\otimes\boldsymbol{e}_r + \sigma_{z\theta}\boldsymbol{e}_z\otimes\boldsymbol{e}_\theta + \sigma_{zz}\boldsymbol{e}_z\otimes\boldsymbol{e}_z\right)\end{aligned}$$

The full derivation of this expression requires deriving the position vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ by θ. We have: $\boldsymbol{e}_r = \cos\theta\boldsymbol{e}_x + \sin\theta\boldsymbol{e}_y$ and $\boldsymbol{e}_\theta = -\sin\theta\boldsymbol{e}_x + \cos\theta\boldsymbol{e}_y$ so that $d\boldsymbol{e}_r/d\theta = \boldsymbol{e}_\theta$ and $d\boldsymbol{e}_\theta/d\theta = -\boldsymbol{e}_r$. One can show that the divergence of the stress tensor in cylindrical coordinates is:

$$\begin{aligned}\boldsymbol{div}\,(\boldsymbol{\sigma}) = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \quad & \left(\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} + \frac{\partial\sigma_{zr}}{\partial z} - \frac{\sigma_{\theta\theta}}{r}\right)\boldsymbol{e}_r\\ & + \left(\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{r\theta}}{\partial r} + 2\frac{\sigma_{r\theta}}{r} + \frac{\partial\sigma_{z\theta}}{\partial z}\right)\boldsymbol{e}_\theta\\ & + \left(\frac{\partial\sigma_{zz}}{\partial z} + \frac{\partial\sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta}\right)\boldsymbol{e}_z\end{aligned}$$

The present problem is plane stress and axis-symmetric, so:

$$\begin{aligned}\boldsymbol{div}\,(\boldsymbol{\sigma}) = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \quad & \left(\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{r} - \frac{\sigma_{\theta\theta}}{r}\right)\boldsymbol{e}_r\\ & + \left(\frac{\partial\sigma_{r\theta}}{\partial r} + 2\frac{\sigma_{r\theta}}{r}\right)\boldsymbol{e}_\theta\end{aligned}$$

The projection of the divergence on the radial axis provides the following equilibrium equation (often encountered in axis-symmetric problems of cavity expansion):

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

The problem is isotropic so there is no shear stress. Therefore $\sigma_1 = \sigma_{\theta\theta}$ (compressive hoop stress) and $\sigma_3 = \sigma_{rr}$. In the damaged zone, the frictional strength criterion is expressed as: $\sigma_1 = d\,\sigma_3$, therefore $\sigma_{\theta\theta} = d\,\sigma_{rr}$. The introduction of that relation in the equation of equilibrium provides:

$$\frac{d\sigma_{rr}}{dr} + (1-d)\frac{\sigma_{rr}}{r} = 0$$

Or:

$$\frac{d\sigma_{rr}}{\sigma_{rr}} = (d-1)\frac{dr}{r}$$

Integrating the above equation between $r = a$ (cavity wall) and $r \leq r_e$ (where $r = r_e$ is the boundary between the damaged zone and the elastic zone, we get:

$$\ln\left(\frac{\sigma_{rr}}{p_i}\right) = (d-1)\ln\left(\frac{r}{a}\right)$$

And finally:

$$\sigma_{rr} = p_i\left(\frac{r}{a}\right)^{(d-1)}$$

At the boundary between the elastic zone and the damaged zone: $\sigma_{rr} = p_1$ (by definition) and $r = r_e$. Therefore:

$$p_1 = p_i\left(\frac{r_e}{a}\right)^{(d-1)}$$

and we get:

$$r_e = a\left(\frac{p_1}{p_i}\right)^{1/(d-1)} \tag{3.3}$$

So if we find p_1, Equation 3.3 provides r_e and we solved the problem. To find p_1, we analyze the state of stress in the elastic zone. The elastic zone can be viewed as a pressurized circular cavity of internal radius r_e, internal pressure p_1, and far-field pressure p. We use the analytical solution seen in class for this case:

$$\begin{aligned}
\sigma_{rr} &= p + \left(\frac{r_e}{r}\right)^2 (p_1 - p) \\
\sigma_{\theta\theta} &= p - \left(\frac{r_e}{r}\right)^2 (p_1 - p) \\
\sigma_{r\theta} &= 0
\end{aligned}$$

At $r = r_e$: the boundary conditions impose: $\sigma_{rr} = p_1$ and we find $\sigma_{\theta\theta} = 2p - p_1$. In the elastic zone, the material has some cohesion and the strength criterion is therefore $\sigma_1 = d\,\sigma_3 + C_0$, i.e. $\sigma_{\theta\theta} = d\,\sigma_{rr} + C_0$. As a result, we have:

$$2p - p_1 = d\,p_1 + C_0$$

and finally:

$$p_1 = \frac{2p - C_0}{d+1}$$

Then r_e is found from Equation 3.3:

$$r_e = a\left(\frac{2p - C_0}{(d+1)p_i}\right)^{1/(d-1)}$$

3.8 In Figure 3.4: Does the plane of weakness affect the elastic stress distribution? Under which conditions does the rock mass slip along the plane of weakness?

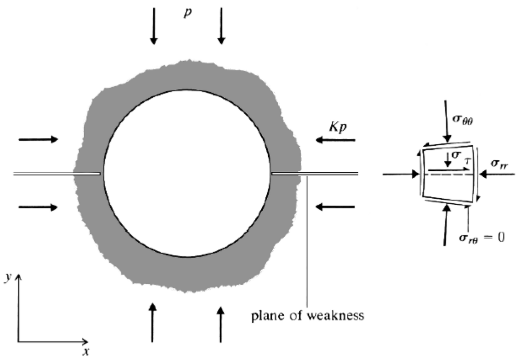


Figure 3.4 Cavity studied in Problem 3.8. (Brady & Brown, 2004)

Solution: This is a problem of free circular cavity subjected to a biaxial state of stress in the far field. We can solve it by using Kirsch's equations. The shear stress that is applied on the plane of weakness is equal to $\sigma_{r\theta}$ at $\theta = 0$. According to Kirsch's equations, $\sigma_{r\theta}(\theta = 0) = 0$, which means that no shear stress is applied on the plane of weakness, and therefore, there is no slip on the plane of weakness. According to Kirsch's equations, the hoop stress $\sigma_{\theta\theta}$ on the plane of weakness at the cavity wall is:

$$\sigma_{\theta\theta}(r = a, \theta = 90^o) = p(1 + K) + 2p(1 - K) = p(3 - K)$$

If $K \leq 3$, $\sigma_{\theta\theta}$ is compressive (compression counted positive) and the plane of weakness cannot open; the presence of the plane of weakness does not change the elastic state of stress. If $K \geq 3$, $\sigma_{\theta\theta}$ is tensile and the plane of weakness could open, and disturb the elastic state of stress.

3.9 In Figure 3.5: Does the plane of weakness affect the elastic stress distribution? Under which conditions does the rock mass slip along the plane of weakness?

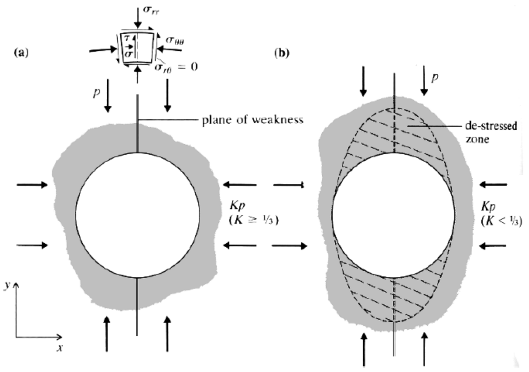


Figure 3.5 Cavity studied in Problem 3.9. (Brady & Brown, 2004)

Solution: This is a problem of free circular cavity subjected to a biaxial state of stress in the far field. We can solve it by using Kirsch's equations. The shear stress that is applied on the plane of weakness is equal to $\sigma_{r\theta}$ at $\theta = 90^o$. According to Kirsch's equations, $\sigma_{r\theta}(\theta = 0) = 0$, which means that no shear stress is applied on the plane of weakness, and therefore, there is no slip on the plane of weakness. The plane of weakness could open if $\sigma_{\theta\theta}$ was positive (tension). At the cavity wall, $\sigma_{rr} = 0$ and, according to Kirsch's equations:

$$\sigma_{\theta\theta}(r = a, \theta = 90^o) = p(1 + K) - 2p(1 - K) = p(3K - 1)$$

If $K \geq 1/3$, $\sigma_{\theta\theta}$ is compressive and the plane of weakness cannot open; the presence of the plane of weakness does not change the elastic state of stress. If $K \leq 1/3$, there is a risk that the fracture (plane of weakness) opens, thus disturbing the elastic state of stress.

3.10 In Figure 3.6: Does the plane of weakness affect the elastic stress distribution? Under which conditions does the rock mass slip along the plane of weakness?

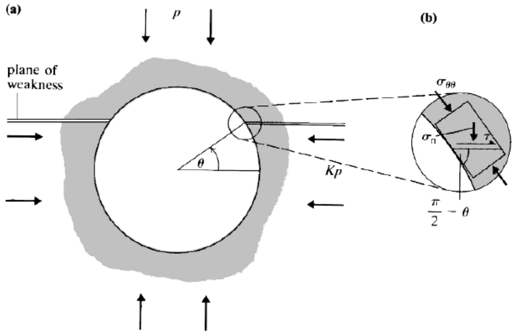


Figure 3.6 Cavity studied in Problem 3.10. (Brady & Brown, 2004)

Solution: There is slippage along the plane of weakness if $\tau \geq \tan\phi\,\sigma_n$. The transformation of stress provides:

$$\tau = \sigma_{\theta\theta}\sin\theta\cos\theta$$

$$\sigma_n = \sigma_{\theta\theta}(\cos\theta)^2$$

So there is slippage if:

$$\sin\theta \geq \tan\phi\cos\theta$$

In other words:

$$\tan\theta \geq \tan\phi$$

And we conclude that for angles that are under 90^o, that implies that there is slippage if:

$$\theta \geq \phi$$

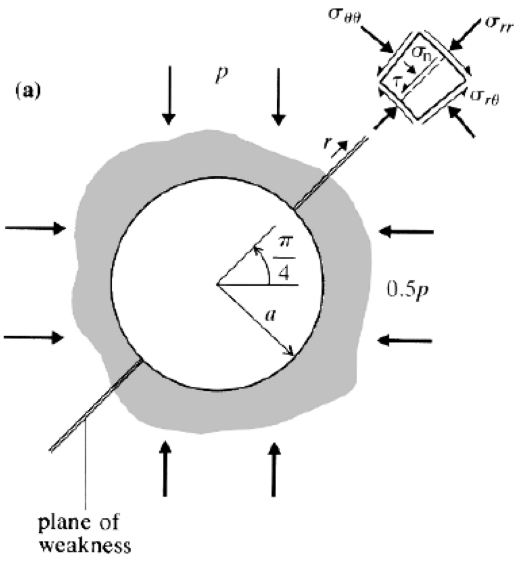


Figure 3.7 Cavity studied in Problem 3.11. (Brady & Brown, 2004)

3.11 In Figure 3.7: Does the plane of weakness affect the elastic stress distribution? Under which conditions does the rock mass slip along the plane of weakness?

Solution: This is a problem of free circular cavity subjected to a biaxial state of stress in the far field. We can solve it by using Kirsch's equations. According to Kirsch's equations, for $\theta = 45^o$ and $K = 0.5$:

$$\begin{aligned} \sigma_n &= \sigma_{\theta\theta} = \frac{3p}{4}\left(1 + \frac{a^2}{r^2}\right) \\ \tau &= \sigma_{r\theta} = \frac{p}{4}\left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right) \end{aligned}$$

From here, it is possible to plot τ/σ_n and to check when this ratio exceeds $\tan\phi$ (when $\tau/\sigma_n \geq \tan\phi$, there is slippage). The plot shows that τ/σ_n reaches a maximum for $r/a = 2.5$, with $\tau = 0.357\sigma_n$, which corresponds to $\phi = 19.6^o$. Most rock materials have a larger friction angle, so the risk of slippage along the plane of weakness is low. One can see that σ_n is always compressive so there is no risk of plane weakness opening. As a conclusion, the plane of weakness is unlikely to change the state of elastic stress.

3.12 In Figure 3.8: Does the plane of weakness affect the elastic stress distribution? Under which conditions does the rock mass slip along the plane of weakness?

Solution: This is a problem of free circular cavity subjected to a biaxial state of stress in the far field. We can solve it by using Kirsch's equations. According to Kirsch's equations,

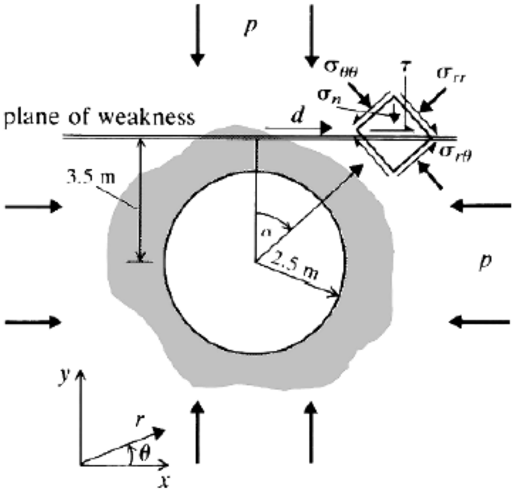


Figure 3.8 Cavity studied in Problem 3.12. (Brady & Brown, 2004)

for $K = 1$:

$$\begin{aligned}
\sigma_{rr} &= p\left(1 - \frac{a^2}{r^2}\right) \\
\sigma_{\theta\theta} &= p\left(1 + \frac{a^2}{r^2}\right) \\
\sigma_{r\theta} &= 0
\end{aligned}$$

We can find the state of stress on the plane of weakness by performing a transformation of stress:

$$\begin{aligned}
\sigma_n &= \frac{1}{2}\left(\sigma_{rr} + \sigma_{\theta\theta}\right) + \frac{1}{2}\left(\sigma_{rr} - \sigma_{\theta\theta}\right)\cos 2\theta \\
\tau &= -\frac{1}{2}\left(\sigma_{rr} - \sigma_{\theta\theta}\right)\sin 2\theta
\end{aligned}$$

By combining the two sets of equations above, we get that:

$$\begin{aligned}
\sigma_n &= p + p\left(\frac{a^2}{r^2}\right)\cos 2\theta \\
\tau &= -p\left(\frac{a^2}{r^2}\right)\sin 2\theta
\end{aligned}$$

One can readily see that σ_n is always compressive, so there is no risk of fracture opening. Slip could happen along the plane of weakness if $\tau/\sigma_n \geq \tan\phi$. To determine whether there is slippage or not, one has to plot τ/σ_n, find the maximum of τ/σ_n and check the value of the friction angle required to trigger slippage (see previous problem).

3.13 Considering Figure 3.9: How can we improve the design in order to avoid compressive failure at the sidewalls?

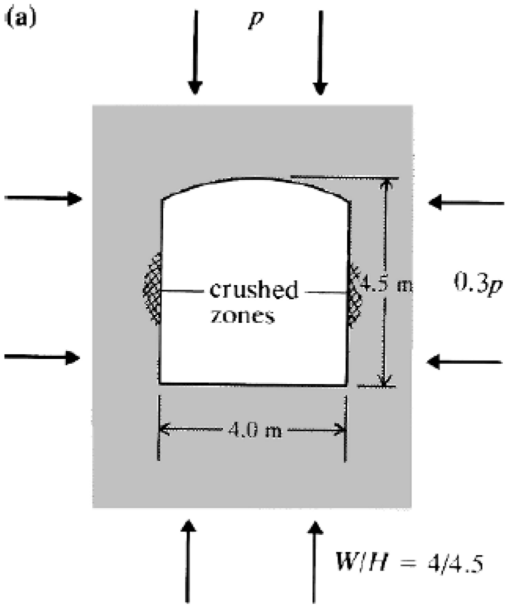


Figure 3.9 Cavity studied in Problem 3.13. (Brady & Brown, 2004)

Solution: We recall the stress at the wall of an ellipse: at A (sidewall) and and at B (crown):

$$\sigma_A = p\left(1 - K + 2\frac{W}{H}\right), \quad \sigma_B = p\left(K - 1 + 2K\frac{H}{W}\right)$$

For a field stress ratio K of 0.3, an inscribed ellipse indicates approximate sidewall stresses of 2.5p, using equation 3.13. If the observed performance of the opening involved crushing of the sidewalls, its redesign should aim to reduce stresses in these areas. Inspection of equation 3.13 indicates this can be achieved by reducing the excavation width/height ratio. For example, if the width/height ratio is reduced to 0.5, the peak sidewall stress is calculated to be 1.7p. While the practicality of mining an opening to this shape is not certain, the general principle is clear, that the maximum boundary stress can be reduced if the opening dimension is increased in the direction of the major principal stress. For this case, a practical solution could be achieved by mining an opening with a low width/height ratio, and leaving a bed of mullock in the base of the excavation.

3.14 Considering Figure 3.10: What are the stresses at the sidewall and at the crown?

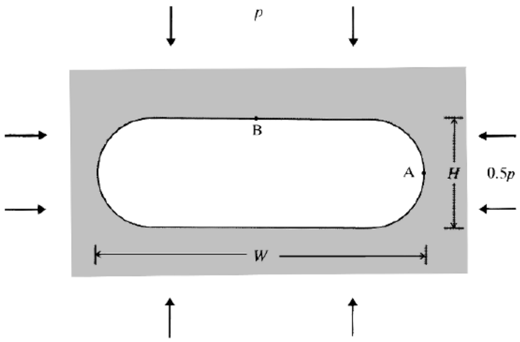


Figure 3.10 Cavity studied in Problem 3.14. (Brady & Brown, 2004)

Solution: The width/height ratio for the opening is three, and the radius of curvature for the side wall is H/2. For a ratio of 0.5 of the horizontal and vertical field principal stresses, the sidewall boundary stress is given, by substitution in Equation 3.13 (see previous problem):

$$\sigma_A = p\left(1 - 0.5 + \sqrt{\frac{2 \times 3H}{H/2}}\right) = 3.96p$$

An independent boundary element analysis of this problem yields a sidewall boundary stress of 3.60p, which is sufficiently close for practical design purposes. Although the radius of curvature, for the ovaloid, is infinite at point B in the centre of the crown of the excavation, it is useful to consider the state of stress at the centre of the crown of an ellipse inscribed in the ovaloid. This predicts a value of B, according to Equation 3.13, of -0.17p, while the boundary element analysis for the ovaloid produces a value at B of -0.15p. This suggests that excavation aspect ratio (say, W/H), as well as boundary curvature, can be used to develop a reasonably accurate picture of the state of stress around an opening.

3.15 Considering Figure 3.11: What are is the stress at edge A?

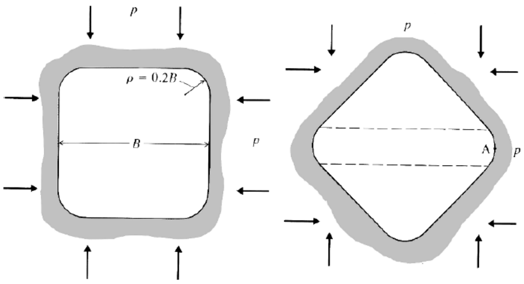


Figure 3.11 Cavity studied in Problem 3.15. (Brady & Brown, 2004)

Solution: The square hole has rounded corners, each with radius of curvature $\rho = 0.2B$.

For a hydrostatic stress field, the problem shown in the left figure is mechanically equivalent to that shown in the right figure. The inscribed ovaloid has a width of $2B[2^{1/2} - 0.4(2^{1/2} - 1)]$, from the geometry. The boundary stress at the rounded corner is estimated from Equation 3.13, as:

$$\sigma_A = p\left(1 - 1 + \sqrt{\frac{2B(2^{1/2} - 0.4(2^{1/2} - 1))}{0.2B}}\right) = 3.53p$$

The corresponding boundary element solution is 3.14p. The effect of boundary curvature on boundary stress appears to be a particular consequence of St Venant's Principle, in that the boundary state of stress is dominated by the local geometry, provided the excavation surface contour is relatively smooth.

3.16 Considering Figure 3.12: Comment on the boundary stresses at the crown, at the sidewall, and at edges A, B, C and D.

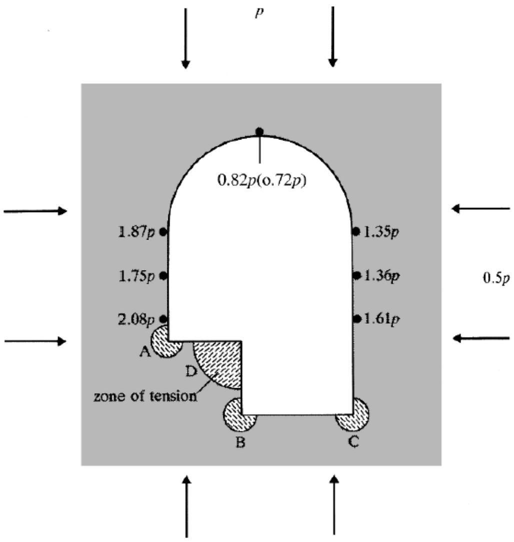


Figure 3.12 Cavity studied in Problem 3.16. (Brady & Brown, 2004)

Solution: Using the general notions developed above, the opening geometry (width/height ratio = 2/3) and pre-mining stress ratio (K = 0.5), the following information concerning boundary stresses can be deduced:

- The zones A, B, C are likely to be highly stressed, since the boundary curvature at these locations is high. Local cracking is to be expected in these zones, but this would compromise neither the integrity of the excavation nor the validity of the stress analysis.
- The bench area D is likely to be at a low state of stress, due to the notionally negative curvature of the prominence forming the bench.

- The boundary stress at the centre of the crown would be approximately 0.72p, estimated from Equation 3.13 (The boundary element solution is 0.82p.)
- An estimate of the sidewall boundary stress, obtained by considering an inscribed ellipse and applying Equation 3.13, yields A=1.83p. For the sifewall locations in the left wall, boundary element analysis gives values of 1.87p, 1.75p and 2.08p. For the locations in the right wall, the A values are 1.35p, 1.36p and 1.61p. The average of these six values is 1.67p.

Boundary element analysis also confirms the two first conclusions above. The demonstration, in an elastic analysis, of a zone of tensile stress, such as in the bench of the current excavation design, has significant engineering implications. Since a rock mass must be assumed to have zero tensile strength, stress redistribution must occur in the vicinity of the bench. This implies the development of a de-stressed zone in the bench and some loss of control over the behaviour of rock in this region. The important point is that a rock mass in compression may behave as a stable continuum. In a de-stressed state, small imposed or gravitational loads can cause large displacements of component rock units.

3.17 Consider Figure 3.13. We give:

- Solution for the line load:

$$\sigma_{xx} = \frac{2P}{\pi}\frac{x^2\,z}{R^4}, \quad \sigma_{zz} = \frac{2P}{\pi}\frac{z^3}{R^4}, \quad \sigma_{xz} = \frac{2P}{\pi}\frac{z^2\,x}{R^4}$$

- Solution for the footing:

$$\sigma_{zz} = \frac{Q}{\pi}\left(a + \sin a \cos(a + 2\delta)\right)$$

1. Using the solution for the stresses (σ_{xx}, σ_{zz}, σ_{xz}) under a line load of intensity P (force/unit length) acting normal to the surface, obtain the stresses due to a strip footing of width $2a$ with an applied surface traction $Q(x)$. Write specific solutions for the case when $Q = Q_0$ (i.e. for surface constant loading).
2. Show that the loci of points with σ_1 = constant (or σ_3 = constant) describe a circle.
3. What is the locus of $q = 0.5(\sigma_1 - \sigma_3)$=constant? The value of q represents the maximum shear stress acting at a point. Show that $q_{max} = Q/\pi$.

Solution:

1. The first question can be answered by integrating the given line loads over the width of the strip:

$$\sigma_{xx}(x,z) = \frac{2z}{\pi}\int_{x'=-a}^{+a} \frac{x'^2\,Q(x')}{(z^2+(x-x')^2}dx'$$

$$\sigma_{zz}(x,z) = \frac{2z^3}{\pi}\int_{x'=-a}^{+a} \frac{Q(x')}{(z^2+(x-x')^2}dx'$$

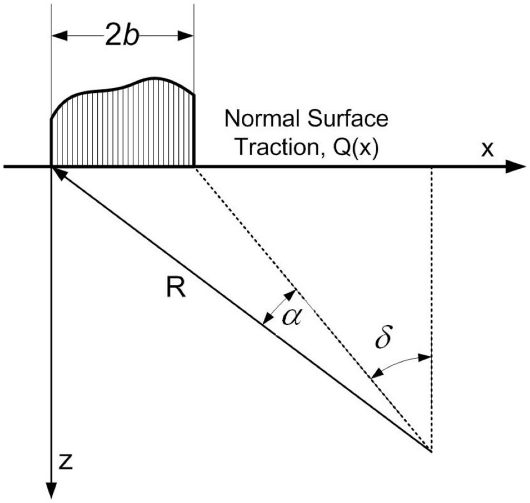


Figure 3.13 Distributed load studied in Problem 3.17. (Assimaki, 2014)

$$\sigma_{xz}(x,z) = \frac{2z^2}{\pi}\int_{x'=-a}^{+a}\frac{x'\,Q(x')}{(z^2+(x-x')^2}dx'$$

For a uniform surface traction:

$$\sigma_{xx}(x,z) = \frac{2Q_0 z}{\pi}\int_{x'=-a}^{+a}\frac{x'^2}{(z^2+(x-x')^2}dx'$$

$$\sigma_{zz}(x,z) = \frac{2Q_0 z^3}{\pi}\int_{x'=-a}^{+a}\frac{1}{(z^2+(x-x')^2}dx'$$

$$\sigma_{xz}(x,z) = \frac{2Q_0 z^2}{\pi}\int_{x'=-a}^{+a}\frac{x'}{(z^2+(x-x')^2}dx'$$

The end result of these calculations is provided in the notebook:

$$\sigma_{zz} = \frac{Q_0}{\pi}\left[(\theta_1-\theta_2)-\sin(\theta_1-\theta_2)\cos(\theta_1+\theta_2)\right]$$

$$\sigma_{xx} = \frac{Q_0}{\pi}\left[(\theta_1-\theta_2)+\sin(\theta_1-\theta_2)\cos(\theta_1+\theta_2)\right]$$

$$\sigma_{xz} = \frac{Q_0}{\pi}\left[\sin(\theta_1-\theta_2)\sin(\theta_1+\theta_2)\right]$$

in which $\theta_1 = 90^o - \delta$ and $\theta_2 = 90^o - (\delta+\alpha)$.

2. We first calculate the principal stresses:

$$\sigma_1 = \sigma_{av} + \mathcal{R}, \quad \sigma_3 = \sigma_{av} - \mathcal{R}$$

with:

$$\sigma_{av} = \frac{1}{2}\left(\sigma_{xx}+\sigma_{zz}\right), \quad \mathcal{R}^2 = \left(\frac{\sigma_{zz}-\sigma_{xx}}{2}\right)^2 + \left(\sigma_{xz}\right)^2$$

We have:

$$\sigma_{av} = \frac{Q_0}{\pi}(\theta_1 - \theta_2), \quad \mathcal{R} = \frac{Q_0}{\pi}\sin(\theta_1 - \theta_2)$$

σ_1 is constant if:

$$\sigma_1 = \frac{Q_0}{\pi}\left((\theta_1 - \theta_2) + \sin(\theta_1 - \theta_2)\right) = \text{cst}$$

By deriving the above expression by $\theta_1 - \theta_2$, we obtain the following requirement:

$$1 + \cos(\theta_1 - \theta_2) = 0$$

which implies:

$$\theta_1 = \theta_2 + 180^o + k \times 360^o$$

Working with σ_3, we similarly obtain:

$$\sigma_3 = \frac{Q_0}{\pi}\left((\theta_1 - \theta_2) - \sin(\theta_1 - \theta_2)\right) = \text{cst}$$

$$1 - \cos(\theta_1 - \theta_2) = 0$$

$$\theta_1 = \theta_2 + k \times 360^o$$

We have $\theta_1 = \theta_2$ or $\theta_1 = \theta_2 + 180^o$, which both indicate that the point of observation lies on a circle [check why].

3. We can calculate the value of q from the calculation of σ_1 and σ_3:

$$q = 0.5(\sigma_1 - \sigma_3) = \mathcal{R} = \frac{Q_0}{\pi}\sin(\theta_1 - \theta_2)$$

We see that the maximum value of the maximum shear stress is:

$$q_{max} = \frac{Q_0}{\pi}$$

The locus of the points characterized by $q = cst$ is found by deriving the expression of q by $\theta_1 - \theta_2$:

$$\cos(\theta_1 - \theta_2) = 0$$

which implies:

$$\theta_1 = \theta_2 + 90^o + 2 \times 180^o$$

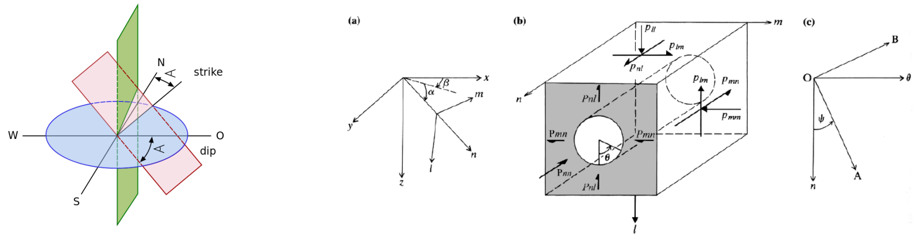
The locus is the vertical axis z [check].

3.18 Homework 2 - Problem 4

A strain cell was used to determine the state of strain in the walls of a borehole. The borehole was oriented $300^o/70^o$, in which 300^o is the angle of strike, and 70^o is the angle of dip (Figure 3.14). Using the angular coordinates and orientations defined in Figure 3.14, the measured states of strain in the wall of the hole, for various angles θ and ψ, are reported in Table 3.1. The Young's modulus of the rock was 40 GPa, and the Poisson's ratio, 0.25.

Table 3.1 Measures taken by the strain cell at the wall of a borehole (expressed in microstrains).

	$\psi = 0^o$	$\psi = 45^o$	$\psi = 90^o$	$\psi = 135^o$
$\theta = 0^o$	N/A	213.67	934.41	821.11
$\theta = 120^o$	96.36	N/A	349.15	131.45
$\theta = 240^o$	96.36	N/A	560.76	116.15



(Wikipedia, 08/2014)

(Brady & Brown, 2004)

Figure 3.14 Conventions adopted to define orientations in strain cell tests.

1. Set up the set of nine equations relating measured strain and gauge location.
   Note that, for $\psi = 0^o$, identical equations are obtained, independent of θ.

2. Solve the system of equations for the field stress components p_{ll}, p_{mm}, p_{nn}, p_{lm}, p_{ln}, p_{mn}.
   It is recommended to use a computational tool (such as MATLAB). Note that you will have to select six independent equations from the set of eight equations established in question 2.1.

3. Determine the field principal stresses.

4. Determine the orientations of the field principal stresses relative to the mine global axes (x-north, y-east, z-down).

Solution:

1. First let us recall (or establish) Leeman's analytical solution for the stress distribution around circular cavities subject to the 3D field stress $\mathbf{p}$. Using the notations of Figure

3.14:

$$\begin{aligned}
\sigma_{rr} &= \sigma_{r\theta} = \sigma_{rn} = 0 \\
\sigma_{\theta\theta} &= p_{ll}(1 - 2cos(2\theta)) + p_{mm}(1 + 2cos(2\theta)) - 4p_{lm}sin(2\theta) \\
\sigma_{nn} &= p_{nn} + 2\nu\left(-p_{ll}cos(2\theta) + p_{mm}cos(2\theta) - 2p_{lm}sin(2\theta)\right) \\
\sigma_{\theta n} &= 2p_{mn}cos\theta - 2p_{nl}sin\theta
\end{aligned}$$

Changing coordinate systems to express the state of plane stress, the normal components of the boundary stress, in the directions OA, OB are given by:

$$\begin{aligned}
\sigma_A &= \frac{\sigma_{nn} + \sigma_{\theta\theta}}{2} + \frac{\sigma_{nn} - \sigma_\theta}{2}cos(2\psi) + \sigma_{n\theta}sin(2\psi) \\
\sigma_B &= \frac{\sigma_{nn} + \sigma_{\theta\theta}}{2} - \frac{\sigma_{nn} - \sigma_\theta}{2}cos(2\psi) - \sigma_{n\theta}sin(2\psi)
\end{aligned}$$

Six independent stress measures in six different directions (OA) provide six independent equations:

$$\sigma_{Ai} = \frac{\sigma_{nn} + \sigma_{\theta\theta}}{2} + \frac{\sigma_{nn} - \sigma_\theta}{2}cos(2\psi) + \sigma_{n\theta}sin(2\psi), \qquad i = 1, ..., 6$$

Which depend on six unknowns p_{ll}, p_{mm}, p_{nn}, p_{lm}, p_{ln}, p_{mn}. The 3D field stress $\mathbf{p}$ is determined by solving a system of equations which is written in the form:

$$[M]\{p\} = \{\sigma_A\}$$

Assuming a state of plane stress in the plane of the borehole cross section:

$$\begin{aligned}
\epsilon_A &= \frac{\sigma_A}{E} - \frac{\nu}{E}\sigma_B \\
\epsilon_B &= \frac{\sigma_B}{E} - \frac{\nu}{E}\sigma_A
\end{aligned}$$

Six independent strain measures in six different directions (OA) provide six independent equations, which depend on six unknowns p_{ll}, p_{mm}, p_{nn}, p_{lm}, p_{ln}, p_{mn}. The 3D field stress $\mathbf{p}$ is determined by solving a system of equations which is written in the form:

$$[M]\{p\} = \{\epsilon_A\}$$

The system of 9 equations is set up by introducing the values of the measured strains in the following equations, for the positions θ and ψ reported in the table:

$$\begin{aligned}
E\,\epsilon_A = \quad & p_{ll}\left[\frac{1}{2}\left((1-\nu)-(1+\nu)\cos 2\psi\right)-(1-\nu^2)(1-\cos 2\psi)\cos 2\theta\right] \\
& +p_{mm}\left[\frac{1}{2}\left((1-\nu)-(1+\nu)\cos 2\psi\right)+(1-\nu^2)(1-\cos 2\psi)\cos 2\theta\right] \\
& +p_{nn}\times\frac{1}{2}\left[(1-\nu)+(1+\nu)\cos 2\psi\right] \\
& -p_{lm}\times 2(1-\nu^2)(1-\cos 2\psi)\sin 2\theta \\
& +p_{mn}\times 2(1+\nu)\sin 2\psi\cos\theta \\
& -p_{nl}\times 2(1+\nu)\sin 2\psi\sin\theta
\end{aligned}$$

2. The 9 equations obtained are not independent. A subsystem of 6 independent equations is solved for the unknown field stress components (with a computational tool such as MATLAB):

$$\begin{aligned}
p_{ll} &= 10.22\,MPa \\
p_{mm} &= 17.09\,MPa \\
p_{nn} &= 10.68\,MPa \\
p_{lm} &= -1.30\,MPa \\
p_{mn} &= -4.86\,MPa \\
p_{nl} &= 1.12\,MPa
\end{aligned}$$

3. Principal stresses are obtained by using the governing equations of Mohr's circles (3 Mohr's circles in 3D):

$$\begin{aligned}
p_1 &= 20\,MPa \\
p_2 &= 10\,MPa \\
p_3 &= 8\,MPa
\end{aligned}$$

4. The directions of principal stresses relative to the mine global axes are determined by establishing geometric relations with Mohr's circles:

p_1 is oriented $030^o/30^o$ (strike/dip)
p_2 is oriented $135^o/24^o$ (strike/dip)
p_3 is oriented $257^o/50^o$ (strike/dip)

3.19 Homework 2 - Problem 5

Figure 3.15 represents a cross section through a long opening. The magnitudes of the plane components of the field stresses are p_{xx} = 13.75MPa, p_{yy} = 19.25 MPa, p_{xy} = 4.76 MPa, expressed relative to the reference axes shown.

1. Calculate the maximum and minimum boundary stresses in the excavation perimeter, defining the locations of the relevant points.
2. If the strength of the rock mass is defined by a maximum shear strength criterion, and the shear strength is 20MPa, estimate the extent of boundary failure, in terms of the angular range over the perimeter.
3. Comment on the significance of this result for any mining operations in the opening.

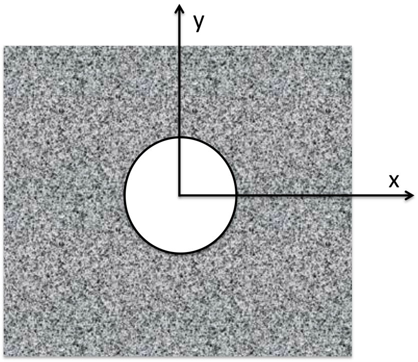


Figure 3.15 Orientation of the axes of cross section through a long opening.

Solution:

1. According to Leeman's equations (equation 3.4), the boundary stresses in the plane of the tunnel cross section are:

$$\begin{aligned}\sigma_{rr} &= \sigma_{r\theta} = 0\\ \sigma_{\theta\theta} &= p_{xx}(1-2cos(2\theta-\pi)) + p_{yy}(1+2cos(2\theta+\pi)) - 4p_{xy}sin(2\theta+\pi)\end{aligned}$$

Injecting the values of the field stress in the expression of the hoop stress at the cavity boundary:

$$\sigma_{\theta\theta} = 13.75\,(1+2cos(2\theta)) + 19.25\,(1-2cos(2\theta)) + 19.04\,sin(2\theta)$$

The maximum value is $\sigma_{\theta\theta} = 55\,MPa$, for $\theta = 60^o$ and $\theta = 240^o$ (relative to the vertical axis pointing downwards, Figure **??**), i.e. for points located 30^o below the horizontal on the right hand-side, and 30^o above the horizontal on the left hand-side. The minimum value is $\sigma_{\theta\theta} = 11\,MPa$, for for $\theta = 150^o$ and $\theta = 330^o$ (relative to the vertical axis pointing downwards, Figure **??**), i.e. for points located 60^o above the horizontal on the right hand-side, and 60^o below the horizontal on the left hand-side.

2. A coordinate change provides the expression of shear stress in a material element at point P located at the cavity wall:

$$\sigma_{r1\,\theta 1} \;=\; -\frac{\sigma_{rr}-\sigma_{\theta\theta}}{2}sin(2\alpha)\;+\sigma_{r\theta}cos(2\alpha)\;=\;\frac{\sigma_{\theta\theta}}{2}sin(2\alpha)$$

In which α is the angle between axes r and $r1$. The shear strength criterion provides the following requirement:

$$\sigma_{\theta\theta}\,sin(2\alpha)\;\leq\;2\,\tau_c$$

With τ_c = 20 MPa. At any point at the cavity wall (i.e. for any θ), shear failure first occurs for $\alpha = 45^o$, i.e. along planes oriented at 45^o from the radial axis, for which $sin(2\alpha) = 1$. Failure occurs for elements in which the hoop stress can exceed shear strength. Using equation 3.4:

$$f(\theta)\;=\;13.75\,(1+2cos(2\theta))\;+19.25\,(1-2cos(2\theta))\;+19.04\,sin(2\theta)\;\geq\;\tau_c$$

A plot of $f(\theta)$ against θ indicates that shear failure can occur for $25.5^o \leq \theta \leq 95.5^o$ and for $204.5^o \leq \theta \leq 275.5^o$, i.e. at $+/-\;35.5^o$ from the points of the cavity wall where the hoop stress is maximum.

3. The maximum principal field stress is 22 MPa, oriented at 30^o below the horizontal, and the minimum principal field stress is 11 MPa, at 60^o above the horizontal. Bray's solution of stress distribution around an elliptical cavity provides the following expressions for the stress at the sidewall (σ_A) and at the crown (σ_B):

$$\begin{aligned}\sigma_A &= p(1-K+2q)\\ \sigma_B &= p(K-1+\frac{2K}{q})\end{aligned} \tag{3.4}$$

In which $q = W/H$ is the aspect ratio of the ellipse. The present cavity can be viewed as an elliptical cavity subject to shear failure at the crown in a plane oriented according to the principal directions of the field stress, as shown in Figure 3.16. To avoid failure, the stress at the crown must be decreased. According to equations 3.4, this can be achieved by increasing q, i.e., by enlarging the size of the opening in the direction perpendicular to the maximum principal field stress direction, as shown in Figure 3.16. The additional volume at the bottom of the opening can be filled with rock to maintain the depth location of the infrastructure in the tunnel.

3.20 Exam 1 - Problem 2

A vertical shaft of elliptical section is being considered in a region of high horizontal stress. Measurements along the shaft route indicate the preshaft principal stresses in psi (compression positive) are given by a vertical stress $\sigma_v = 1.1\,h$ and horizontal stresses $\sigma_h = 100 + 1.5\,h$, $\sigma_H = 500 + 2.2\,h$, in which h is the depth in feet. Figure 3.17 gives the

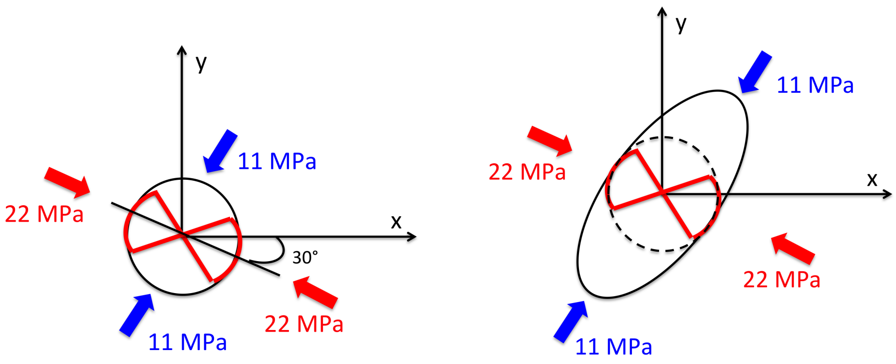
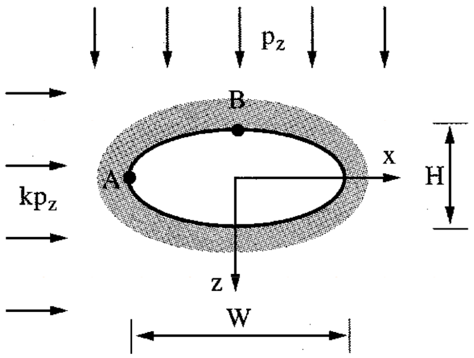


Figure 3.16 Adaptation of the mine shape to avoid shear failure.



$$\sigma_A = p(1 - k + 2q) = p\left(1 - k + \sqrt{\frac{2W}{\rho_A}}\right),$$

$$\sigma_B = p\left(k - 1 + \frac{2k}{q}\right) = p\left(k - 1 + k\sqrt{\frac{2H}{\rho_B}}\right)$$

where, for an ellipse, the radii of curvature are

$$\rho_A = \frac{H^2}{2W} \text{ and } \rho_B = \frac{W^2}{2H}$$

Hudson & Harrison (1997)

Figure 3.17 Peak stress concentrations at the wall of an elliptical cavity.

expressions of the peak stress concentrations at the wall of an elliptical section. Determine the optimum orientation and aspect ratio of the section.

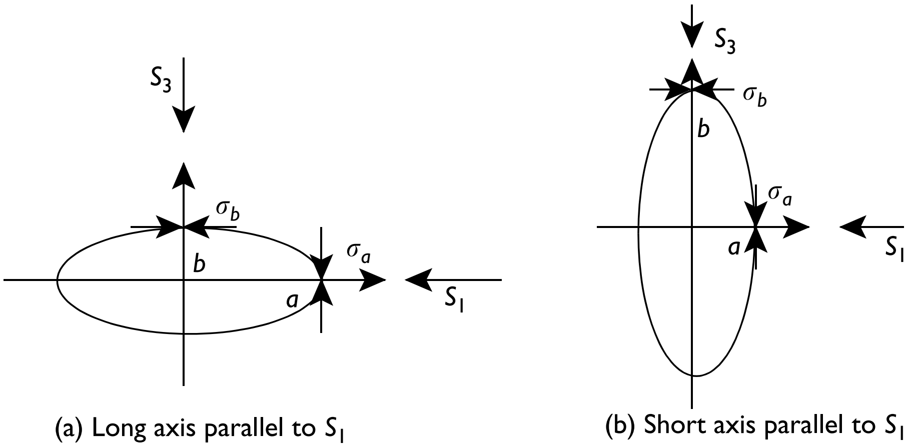
Solution: see the two following pages (from Pariseau, 2007) can now be obtained by reducing the larger of K_a or K_b. Ideally, one has neither $K_a > K_b$ nor $K_b > K_a$. Hence, the optimum occurs when $K_a = K_b$, that is, when $k = 1$. Thus, a circular shape, a special case of an ellipse, is optimum when the preshaft stress state is hydrostatic. This situation occurs for vertical shafts in ground where the stress state is caused by gravity alone.

Example 3.5 A vertical shaft of elliptical section is being considered in a region of high horizontal stress. Measurements along the shaft route indicate the preshaft principal stresses in psi (compression positive) are given by $S_{\mathrm{v}} = 1.1h, S_{\mathrm{h}} = 100 + 1.5h, S_{\mathrm{H}} = 500 + 2.2h$ (h is depth in feet). Determine the optimum orientation and aspect ratio of the section.

Solution: The peak stress concentrations at the wall of an elliptical section occur at the ends of the geometric axes. The peak stresses are given by the formulas

$$\sigma_a = K_a S_1 = -S_1 + (1 + 2/k)S_3 = S_1[(1 + 2/k)M - 1]$$
$$\sigma_b = K_b S_1 = -S_3 + (1 + 2k)S_1 = S_1[(1 + 2k) - M]$$

where the meaning of terms is shown in the sketch, $k = b/a$ and $M = S_3/S_1$.



(a) Long axis parallel to S_1 (b) Short axis parallel to S_1

Sketch illustrating the meaning of ellipse formula terms

An optimum orientation would be one that reduces stress concentration to a minimum, so the stress concentration at b is no greater or less than at a and tension is absent. Thus,

$$K_a = K_b \geq 0$$
$$\therefore$$
$$(1 + 2/k)M - 1 = (1 + 2k) - M$$
$$\therefore$$
$$k = M = \frac{100 + 1.5h}{500 + 2.2h}$$

so the aspect ratio varies with depth, but is equal to the preexcavation stress ratio at any particular depth. At the surface $k = 0.200$ and at great depth $k = 1.5/2.2 = 0.682$.

The best orientation is with the long axis a parallel to S_H. In this orientation, with $k = \mathrm{M}$, $K_a = K_b = 1 + M$ and indeed the wall of the ellipse is in a uniform compression. At the surface, the stress concentration is 1.2; at great depth the stress concentration is 1.682.

With the short axis parallel to $S_H, K_a' = 2M^2 + M$ and $K_b' = 1 - M + 2/M$, which is greater than $1 + M$ and therefore less favorable at any depth.

If a circular section were considered, the stress concentrations would be $-1 + 3M$ and $3 - M$ or -0.4 and 2.8 at the surface; 1.046 and 2.318 at great depth. The elliptical section shows a considerable advantage over the circular section in this stress field. However, other considerations such as difficulty in excavation to the proper shape would need to be considered before final selection of the shaft shape.

In vertical section, the shaft wall stress is equal to the preshaft stress and stress concentration is one. This value is less than stress concentration in plan view of the best orientation and aspect ratio (1+M) for $M = k > 0$. Hence, the plan view section governs design.

Example 3.6 Suppose a vertical shaft of elliptical section 14 ft × 21 ft is being considered for sinking to a depth of 4,350 ft. The horizontal stress in the north–south direction is estimated to be twice the vertical stress caused by rock mass weight, while the east–west stress is estimated to be equal to the "overburden" stress. Unconfined compressive strength of rock is 22,000 psi, as measured in the laboratory; tensile strength is 2,200 psi. Determine shaft wall safety factors as functions of depth with the section oriented in a way that minimizes compressive stress concentration.

***Solution*:** Shaft wall safety factors with respect to compression and tension, by definition, are

$$\mathrm{FS_c} = \frac{C_\mathrm{o}}{\sigma_\mathrm{c}}, \quad \mathrm{FS_t} = \frac{T_\mathrm{o}}{\sigma_\mathrm{t}}$$

In case of an ellipse, the peak stresses at the ends of the semi-axes are:

$$\begin{aligned}
\sigma_a &= K_a S_1 = -S_1 + (1 + 2/k)S_3 = S_1[(1 + 2/k)M - 1] \\
\sigma_b &= K_b S_1 = -S_3 + (1 + 2k)S_1 = S_1[(1 + 2k) - M] \\
k &= 14/21 = 2/3 \leq 1, \quad M = S_{\mathrm{EW}}/S_{\mathrm{NS}} = 1/2, \\
\sigma_a &= K_a S_1 = -S_1 + (1 + 2/k)S_3 = S_1[(1 + 2/(2/3))(1/2) - 1] = S_1(1.00) \\
\sigma_b &= K_b S_1 = -S_3 + (1 + 2k)S_1 = S_1[(1 + 2(2/3)) - 1/2] = S_1(1.83)
\end{aligned}$$

The maximum and minimum stress concentration factors are both positive, so no tension is present. With respect to compressive stress, the shaft wall safety factor as a function of depth is

$$\begin{aligned}
\mathrm{FS_c} &= \frac{C_\mathrm{o}}{1.83 S_1} \\
&= \frac{22{,}000}{1.83(2)(158/144)h} \\
\mathrm{FS_c} &= \frac{5{,}478}{h}
\end{aligned}$$

CHAPTER 4

FINITE ELEMENT METHOD IN LINEAR ELASTICITY

PROBLEMS

4.1 Write the variational formulation of the following problem:

$$\frac{d^2}{dx^2}\left[EI\frac{d^2u(x)}{dx^2}\right] - q_0 \,=\, 0, \quad 0 < x < L$$

$$u(0) = \left[\frac{du(x)}{dx}\right]_{x=0} = 0, \quad EI\left[\frac{d^2u(x)}{dx^2}\right]_{x=L} = -M_0, \quad EI\left[\frac{d^3u(x)}{dx^3}\right]_{x=L} = F_0$$

Solution: Weighted integral statement:

$$\forall \delta w \sim u, \qquad \int_0^L w(x)\frac{d^2}{dx^2}\left[EI\frac{d^2u(x)}{dx^2}\right]dx \,-\, \int_0^L w(x)q_0dx \,=\, 0$$

First integration by parts (assuming E, I constants):

$$\forall \delta w \sim u, \qquad \left[w(x)EI\frac{d^3u(x)}{dx^3}\right]_0^L - \int_0^L \frac{dw}{dx}EI\frac{d^3u(x)}{dx^3}dx \,-\, \int_0^L w(x)q_0dx \,=\, 0$$

Second integration by parts:

$$\forall \delta w \sim u, \qquad \left[w(x)EI\frac{d^3u(x)}{dx^3}\right]_0^L - \left[\frac{dw}{dx}EI\frac{d^2u(x)}{dx^2}\right]_0^L + \int_0^L \frac{d^2w}{dx^2}EI\frac{d^2u(x)}{dx^2}dx - \int_0^L w(x)q_0dx \,=\, 0$$

Primary variables: $u(x)$ (deflection) and du/dx (deflection angle). Secondary variables: $EI\frac{d^3u(x)}{dx^3}$ (shear force) and $EI\frac{d^2u(x)}{dx^2}$ (bending moment). Applying boundary conditions:

$$\forall \delta w \sim u, \quad w(L)F_0 + \frac{dw}{dx}(L)M_0 + \int_0^L \frac{d^2w}{dx^2}EI\frac{d^2u(x)}{dx^2}dx - \int_0^L w(x)q_0dx = 0$$

4.2 Find the expression of the interpolation functions of order 1 is by using the interpolation property. Consider elements with 2 and 3 nodes.

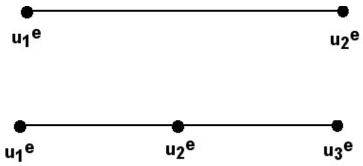


Figure 4.1 Finite Elements considered in Problem 4.2.

Solution: Element with two nodes:

$$\Psi_1(x_1) = 1, \quad \Psi_1(x_2) = 0, \quad \Psi_2(x_1) = 0, \quad \Psi_2(x_2) = 1$$

Using linear polynomials:

$$\Psi_1(x) = \frac{x_2 - x}{x_2 - x_1}, \quad \Psi_2(x) = \frac{x_1 - x}{x_1 - x_2}$$

Element with three nodes:

$$\begin{aligned} \Psi_1(x_1) &= 1, \quad \Psi_1(x_2) = 0, \quad \Psi_1(x_3) = 0 \\ \Psi_2(x_1) &= 0, \quad \Psi_2(x_2) = 1, \quad \Psi_2(x_3) = 0 \\ \Psi_3(x_1) &= 0, \quad \Psi_3(x_2) = 0, \quad \Psi_3(x_3) = 1 \end{aligned}$$

Using quadratic polynomials:

$$\Psi_1(x) = \frac{(x_2 - x)(x_3 - x)}{(x_2 - x_1)(x_3 - x_1)}, \quad \Psi_2(x) = \frac{(x_1 - x)(x_3 - x)}{(x_1 - x_2)(x_3 - x_2)}, \quad \Psi_3(x) = \frac{(x_1 - x)(x_2 - x)}{(x_1 - x_3)(x_2 - x_3)}$$

4.3 Consider the following boundary value problem:

$$-\frac{d^2u(x)}{dx^2} - u(x) + x^2 = 0, \quad 0 < x < 1$$

$$u(0) = u(1) = 0$$

Provide the FEM equations to solve for the unknown primary and secondary variables when the domain is discretized with: (a) four linear elements; (b) two quadratic elements (Figure 4.2).

Figure 4.2 Meshes considered in Problem 4.3.

Solution: Weak formulation of the problem on one element $[x_a, x_b]$:

$$\forall w \sim \delta u, \quad -\left[w(x)\frac{du(x)}{dx}\right]_{x_a}^{x_b} + \int_{x_a}^{x_b} \frac{dw}{dx}\frac{du}{dx}dx - \int_{x_a}^{x_b} w(x)u(x)dx + \int_{x_a}^{x_b} w(x)x^2dx = 0$$

Using Ritz method with N nodes per element:

$$\forall i = 1...N, \quad -\left[\Psi_i(x)\frac{du(x)}{dx}\right]_{x_a}^{x_b} + \sum_{j=1}^{N}\left[\int_{x_a}^{x_b} \frac{d\Psi_i}{dx}\frac{d\Psi_j}{dx}dx\right]u_j - \sum_{j=1}^{N}\left[\int_{x_a}^{x_b} \Psi_i(x)\Psi_j(x)dx\right]u_j + \int_{x_a}^{x_b} \Psi_i(x)x^2dx = 0$$

In a matrix form:

$$\forall i, j = 1...N, \quad K_{ij}u_j + M_{ij}u_j = F_i + Q_i$$

with:

$$K_{ij} = \int_{x_a}^{x_b} \frac{d\Psi_i}{dx}\frac{d\Psi_j}{dx}dx$$

$$M_{ij} = -\int_{x_a}^{x_b} \Psi_i(x)\Psi_j(x)dx$$

$$F_i = -\int_{x_a}^{x_b} \Psi_i(x)x^2dx$$

$$Q_i = \left[\Psi_i(x)\frac{du(x)}{dx}\right]_{x_a}^{x_b}$$

For the mesh with four linear elements, the length of each element is 1/4 and for each element:

$$[K] = 4\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[M] = \frac{-1}{24}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

For the mesh with two quadratic elements, the length of each element is 1/2 and for each element:

$$[K] = \frac{2}{3}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$$[M] = \frac{-1}{60}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

Assembly for 4 linear elements (note: we only show the top triangle of the stiffness matrix, since the stiffness matrix is symmetric; the same assembly process is used to find the global mass matrix; the assembly process is the same for $\{F\}$ and $\{Q\}$):

$$[\overline{K}] = \begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 & 0 \\ & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & 0 \\ & & K_{22}^2 + K_{11}^3 & K_{12}^3 & 0 \\ & & & K_{22}^3 + K_{11}^4 & K_{12}^4 \\ & & & & K_{22}^4 \end{bmatrix}$$

$$\{\overline{F}\} = \begin{Bmatrix} F_1^1 \\ F_2^1 + F_1^2 \\ F_2^2 + F_1^3 \\ F_2^3 + F_1^4 \\ F_2^4 \end{Bmatrix}$$

Assembly for 2 quadratic elements (note: we only show the top triangle of the stiffness matrix, since the stiffness matrix is symmetric; the same assembly process is used to find the global mass matrix; the assembly process is the same for $\{F\}$ and $\{Q\}$):

$$[\overline{K}] = \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & 0 & 0 \\ & K_{22}^1 & K_{23}^1 & 0 & 0 \\ & & K_{33}^1 + K_{11}^2 & K_{12}^2 & K_{13}^2 \\ & & & K_{22}^2 & K_{23}^2 \\ & & & & K_{33}^2 \end{bmatrix}$$

$$\{\overline{F}\} = \left\{ \begin{array}{l} F_1^1 \\ F_2^1 \\ F_3^1 + F_1^2 \\ F_2^2 \\ F_3^2 \end{array} \right\}$$

Since the primary variable is known at nodes 1 and 5, the global system of FEM equations can be condensed as follows (regardless of the mesh considered):

$$\left(\begin{bmatrix} \overline{K}_{22} & \overline{K}_{23} & 0 \\ & \overline{K}_{33} & \overline{K}_{34} \\ & & \overline{K}_{44} \end{bmatrix} + \begin{bmatrix} \overline{M}_{22} & \overline{M}_{23} & 0 \\ & \overline{M}_{33} & \overline{M}_{34} \\ & & \overline{M}_{44} \end{bmatrix} \right) \left\{ \begin{array}{l} \overline{F}_2 \\ \overline{F}_3 \\ \overline{F}_4 \end{array} \right\}$$

in which we used $u_1 = u_5 = 0$ and $Q_2 = Q_3 = Q_4 = 0$. From there, it is possible to solve for u_2, u_3 and u_4. Usually, the secondary variable is post-processed. In the following, we explain the post-processing process for both meshes. For the four linear elements:

$$\overline{Q}_1 = Q_1^1 = \Psi_1^1(x_1) \frac{d}{dx} \left(u_1 \Psi_1^1(x) + u_2 \Psi_2^1(x) \right)_{x=x_1}$$

$$\overline{Q}_1 = u_2 \frac{d\Psi_2(x)}{dx}|_{x=x_1} = u_2 \frac{d}{dx} \left(\frac{x}{(1/4)} \right)_{x=0}$$

$$\overline{Q}_1 = 4u_2$$

In the same way:

$$\overline{Q}_5 = 4u_4$$

For the two quadratic elements:

$$\overline{Q}_1 = Q_1^1 = \Psi_1^1(x_1) \frac{d}{dx} \left(u_1 \Psi_1^1(x) + u_2 \Psi_2^1(x) + u_3 \Psi_3^1(x) \right)_{x=x_1}$$

$$\overline{Q}_1 = u_2 \frac{d}{dx} \left(\frac{x(1/2 - x)}{(1/16)} \right)_{x=0} + u_3 \frac{d}{dx} \left(\frac{x(x - 1/4)}{(1/8)} \right)_{x=0}$$

$$\overline{Q}_1 = 8u_2 - 2u_3$$

In the same way:

$$\overline{Q}_5 = 8u_4 - 2u_3$$

4.4 Solve the problem of heat transfer through the composite wall shown in Figure 4.3 by using Ritz method, with four linear elements.

$$-\frac{\partial}{\partial x} \left(kA \frac{\partial T}{\partial x} \right) + \beta P (T - T_\infty) \; = \; Aq_0$$

Solution: There is no heat source in the problem. Advection takes place on the lect and

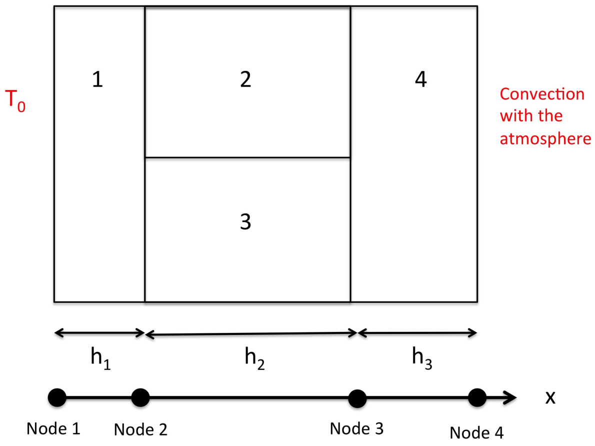


Figure 4.3 Heat transfer through a composite wall.

right hand sides of the domain, but not on the top or bottom, i.e., there is no advection along the periphery of the walls (advection can be accounted for in the boundary conditions). As a result, the governing equation reduces to:

$$-\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) = 0$$

The Ritz Finite Element equations are the following (for each element):

$$K_{ij}T_j = Q_i$$

with:

$$K_{ij} = \int_{x_a}^{x_b} kA\frac{d\Psi_i}{dx}\frac{d\Psi_j}{dx}dx$$

$$Q_i = \left[\Psi_i(x)kA\frac{dT}{dx}\right]_{x_a}^{x_b}$$

For linear elements:

$$[K] = \frac{kA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where l is the length of the element. The assembled FE equations are (note: we do not write the coefficients below the diagonal, which can be obtained by symmetry):

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 \\ & K_{22}^1 + K_{11}^2 + K_{11}^3 & K_{12}^2 + K_{12}^3 & 0 \\ & & K_{22}^2 + K_{22}^3 & K_{12}^4 \\ & & & K_{22}^4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 + Q_1^3 \\ Q_2^2 + Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

The prescribed nodal conditions are: $T_1 = T_0$, $Q_2 = Q_3 = 0$ and $Q_4 = -\beta A(T_4 - T_\infty)$ (advection with atmosphere). Using these nodal conditions, the condensed system of Finite

Element equations is:

$$\begin{bmatrix} K^1_{22}+K^2_{11}+K^3_{11} & K^2_{12}+K^3_{12} & 0 \\ & K^2_{22}+K^3_{22} & K^4_{12} \\ & & K^4_{22}-\beta A \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} -\overline{K}_{21}T_0 \\ -\overline{K}_{31}T_0 \\ \beta A T_\infty - \overline{K}_{41}T_0 \end{Bmatrix}$$

in which $[\overline{K}]$ is the global stiffness matrix (before condensation). Using the elementary stiffness matrices, we get:

$$\begin{bmatrix} A_1k_1/l_1+A_2k_2/l_2+A_3k_3/l_3 & -A_2k_2/l_2-A_3k_3/l_3 & 0 \\ & A_2k_2/l_2+A_3k_3/l_3 & -A_4k_4/l_4 \\ & & A_4k_4/l_4-\beta A_4 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} A_1k_1T_0/l_1 \\ 0 \\ \beta A_4 T_\infty \end{Bmatrix}$$

From there, it is possible to solve for the nodal values of the temperature (T_2, T_3 and T_4), and then, for the nodal value of the heat flux Q_1.

4.5 Solve the problem of heat transfer through the cylindrical canister shown in Figure 4.4 by using Ritz method, with two linear elements.

$$-\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = q_0(r)$$

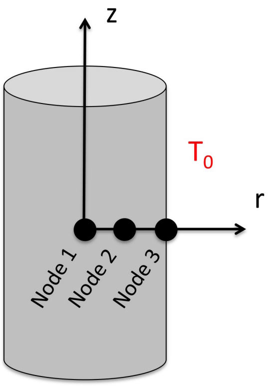


Figure 4.4 Heat transfer through a cylindrical canister.

Solution:

1. Discretization
   Linear interpolation functions: for an element defined on the segment $[r_a, r_b]$

$$\Psi^e_1(r) = \frac{-(r-r_b)}{h_e}, \quad \Psi^e_2(r) = \frac{(r-r_a)}{h_e}, \quad h_e = r_b - r_a$$

2. Elementary equations
   Using Ritz Method:

$$[K]^e\{T\}^e = \{F\}^e + \{Q\}^e$$

$$K_{ij}^e = 2\pi \int_{r_a}^{r_b} k \frac{d\Psi_i^e}{dr}\frac{d\Psi_j^e}{dr} r dr$$

$$F_i^e = 2\pi \int_{r_a}^{r_b} \Psi_i^e(r) q_0(r) r dr$$

$$Q_i^e = \pm 2\pi\, kr \frac{dT}{dr}(r = r_i^e)$$

   The integrals depends on the bounds. The stiffness matrix and the force vector have thus different expressions for the two elements. In a matrix form:

$$[K^{(1)}] = k\pi \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K^{(2)}] = k\pi \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

   Assuming that the heat source function is a constant: $q_0(r) = q_0$:

$$\{F^{(1)}\} = \frac{\pi\, q_0 {R_0}^2}{12}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$\{F^{(2)}\} = \frac{\pi\, q_0 {R_0}^2}{12}\begin{Bmatrix} 4 \\ 5 \end{Bmatrix}$$

3. Assembly

$$k\pi \begin{bmatrix} 1 & -1 & 0 \\ -1 & 4 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \frac{\pi\, q_0 {R_0}^2}{12}\begin{Bmatrix} 1 \\ 6 \\ 5 \end{Bmatrix} + \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} + Q_1^{(2)} \\ Q_2^{(2)} \end{Bmatrix}$$

4. Nodal conditions

$$Q_1 = 0, \quad Q_2 = 0, \quad T_3 = T_0$$

5. Condensation and resolution

$$k\pi \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{\pi\, q_0 {R_0}^2}{12}\begin{Bmatrix} 1 \\ 6 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 3k\, r\, T_0 \end{Bmatrix}$$

   From here, solve for T_1 and T_2.

6. Post-processing Q_3 can be obtained by a direct method, using the stiffness matrix, as follows:

$$Q_3 = -3\pi k T_2 + 3\pi k T_0 - \frac{5\pi q_0 {R_0}^2}{12}$$

7. Approximate solution:

$$T(r) \simeq \begin{cases} T_1\Psi_1^{(1)}(r) \ + T_2\Psi_2^{(1)}(r), & r \in]0, R_0/2[\\ T_2\Psi_1^{(2)}(r) \ + T_3\Psi_2^{(2)}(r), & r \in]R_0/2, R_0[\end{cases}$$

$$T(r) \simeq \begin{cases} T_1\frac{R_0/2-r}{R_0/2} \ + T_2\frac{r}{R_0/2}, & r \in]0, R_0/2[\\ T_2\frac{R_0-r}{R_0/2} \ + T_0\frac{r-R_0/2}{R_0/2}, & r \in]R_0/2, R_0[\end{cases}$$

4.6 Solve the 1D Newtonian fluid flow problem with Ritz method, by using two linear elements, and for the two following sets of boundary conditions: (a) $v_x(-L) = v_x(L) = 0$; (b) $v_x(-L) = 0$, $v_x(L) = v_0$.

Solution: The governing equation is:

$$\mu\frac{d^2v_x(y)}{dy^2} = \frac{dP}{dx}$$

The Ritz Finite Element equations (for one element) are:

$$K_{ij}v_j = F_i + Q_i$$

in which:

$$K_{ij} = \int y_a{}^{y_b}\mu\frac{d\Psi_i}{dy}\frac{d\Psi_j}{dy}dy$$

$$F_i = -\int y_a{}^{y_b}\Psi_i(y)\frac{dP}{dx}dy$$

$$Q_i = \left[\Psi(y)\mu\frac{dv_x}{dy}\right]_{y_a}^{y_b}$$

For linear elements:

$$[K] = \frac{\mu}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For the special where $\frac{dP}{dx} = f_0$ (constant):

$$\{F\} = \frac{f_0 l}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

where l is the length of the element. After assembling the elementary equations (assuming that all elements have same length l):

$$\frac{\mu}{l}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \frac{f_0 l}{2}\begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

For the first set of boundary conditions: $v_1 = v_3 = 0$ and $Q_2 = 0$, the system of equations can be condensed as:

$$\frac{2\mu}{l} v_2 = f_0 l$$

In other words, the solution for the primary variable is:

$$v_2 = \frac{f_0 l^2}{2\mu}$$

And the FEM approximation of the solution on $[-l, +l]$ is:

$$\begin{aligned} v_x(y) &= \frac{f_0 l^2}{2\mu} \frac{y+l}{l}, \quad -l \leq y \leq 0 \\ v_x(y) &= \frac{f_0 l^2}{2\mu} \frac{y}{l}, \quad 0 \leq y \leq l \end{aligned}$$

Using the FEM equations to solve for the secondary variable:

$$\begin{aligned} Q_1 &+ \frac{f_0 l}{2} = -\frac{\mu}{l} v_2 = -\frac{f_0 l}{2} \\ Q_3 &+ \frac{f_0 l}{2} = -\frac{\mu}{l} v_2 = -\frac{f_0 l}{2} \end{aligned}$$

so that: $Q_1 = Q_3 = -f_0 l$.
For the second set of boundary conditions: $v_1 = 0$, $Q_2 = 0$ and $v_3 = v_0$ and the system of equations can be condensed as:

$$\frac{2\mu}{l} v_2 = f_0 l + \frac{\mu}{l} v_0$$

In other words, the solution for the primary variable is:

$$v_2 = \frac{f_0 l^2}{2\mu} + \frac{v_0}{2}$$

And the FEM approximation of the solution on $[-l, +l]$ is:

$$\begin{aligned} v_x(y) &= \left(\frac{f_0 l^2}{2\mu} + \frac{v_0}{2} \right) \frac{y+l}{l}, \quad -l \leq y \leq 0 \\ v_x(y) &= \left(\frac{f_0 l^2}{2\mu} + \frac{v_0}{2} \right) \frac{y}{l}, \quad 0 \leq y \leq l \end{aligned}$$

Using the FEM equations to solve for the secondary variables:

$$\begin{aligned} Q_1 &+ \frac{f_0 l}{2} = -\frac{\mu}{l} v_2 = -\frac{f_0 l}{2} - \frac{\mu v_0}{2l} \\ Q_3 &+ \frac{f_0 l}{2} = -\frac{\mu}{l} v_2 + \frac{\mu}{l} v_0 = -\frac{f_0 l}{2} + \frac{\mu v_0}{2l} \end{aligned}$$

so that:

$$Q_1 = -f_0 l - \frac{\mu v_0}{2l}$$

$$Q_3 = -f_0 l + \frac{\mu v_0}{2l}$$

4.7 Solve the problem of bar elongation shown in Figure 4.5 by using Ritz method, with two linear elements.

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f(x)$$

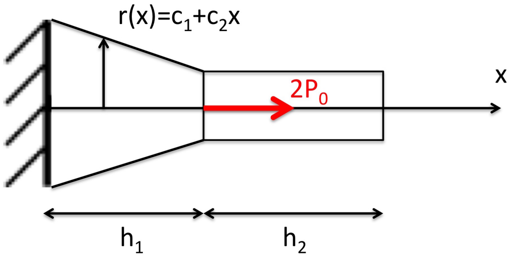


Figure 4.5 Bar element with a non-uniform cross section.

Solution: Here, there is no distributed horizontal force, i.e. $f(x) = 0$, but there is a concentrated load $2P_0$ applied at $x = h_1$. The concentrated load will be accounted for in theboundary conditions. The governing equation is thus:

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$$

The area of the cross section of the bar for $0 \leq x \leq h_1$ is:

$$A_1(x) = \pi(c_1 + c_2 x)^2$$

The area of the cross section of the bar for $h_1 \leq x \leq h_1 + h_2$ is:

$$A_2(x) = A_2 = \pi(c_1 + c_2 h_1)^2$$

We use two linear elements (one for $0 \leq x \leq h_1$ and one for $h_1 \leq x \leq h_1 + h_2$). The Ritz Finite Element equations are:

$$K_{ij} u_j = Q_i$$

For the first element:

$$K^1_{ij} = \pi E \int_0^{h_1} (c_1 + c_2 x)^2 \frac{d\Psi_i}{dx} \frac{d\Psi_j}{dx} dx$$

For the second element:

$$K_{ij}^2 = EA_2 \int_{h_1}^{h_2} \frac{d\Psi_i}{dx}\frac{d\Psi_j}{dx}dx$$

We calculate the average cross section on the first element:

$$\overline{A}_1 = \frac{1}{h_1}\int_0^{h_1} \pi(c_1 + c_2 x)^2 dx$$

And we evaluate the stiffness coefficients of the first element as:

$$K_{ij}^1 = E\overline{A}_1 \int_0^{h_1} \frac{d\Psi_i}{dx}\frac{d\Psi_j}{dx}dx$$

Using the expression of the stiffnes matrix of a linear 1D element, the assembled system of FEM equations becomes (only writing the terms of the stiffness matrix above the diagonal):

$$\begin{bmatrix} \overline{A}_1 E/h_1 & -\overline{A}_1 E/h_1 & 0 \\ & \overline{A}_1 E/h_1 + A_2 E/h_2 & -A_2 E/h_1 \\ & & A_2 E/h_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

The prescribed nodal conditions are: $u_1 = 0$, $Q_2 = 2P_0$ and $Q_3 = 0$. Therefore, we can condense the system of equations as:

$$\begin{bmatrix} \overline{A}_1 E/h_1 + A_2 E/h_2 & -A_2 E/h_1 \\ -A_2 E/h_1 & A_2 E/h_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 2\,P_0 \\ 0 \end{Bmatrix}$$

From there, it is possible to solve for u_2 and u_3, and then for the reaction Q_1 (either by using hte FEM equations, or by post-processing).

4.8 Solve the problem of beam deflection shown in Figure 4.6 by using two Euler-Bernouilli elements.

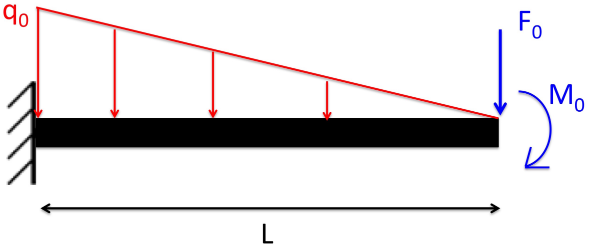


Figure 4.6 Deflection problem with Euler-Bernouilli beam elements.

Solution: The governing equation is:

$$EI\frac{d^4 w(x)}{dx^4} = q_0 - \frac{q_0}{L}x$$

in which $w(x)$ is the deflection. The Ritz Finite Element equations are (for one cubic Euler-Bernouilli element):

$$\forall i = 1...4, \qquad \sum_{j=1}^{4}\left(EI\int_{x_a}^{x_b}\frac{d^2\Psi_i(x)}{dx^2}\frac{d^2\Psi_j(x)}{dx^2}dx\right)W_j =$$

$$\int_{x_a}^{x_b}\Psi_i(x)\left(q_0 - \frac{q_0}{L}x\right)dx - \left[\Psi_i x EI\frac{d^3w(x)}{dx^3}\right]_{x_a}^{x_b} + \left[\frac{d\Psi_i x}{dx}EI\frac{d^2w(x)}{dx^2}\right]_{x_a}^{x_b}$$

in which W_j designates the jth degree of freedom (for each node, the deflection is listed first, and the deflection angle, second). The cubic Hermite polynomial interpolation functions are:

$$\begin{aligned}
\Psi_1(x) &= 1 - 3\left(\frac{x-x_a}{x_b-x_a}\right)^2 + 2\left(\frac{x-x_a}{x_b-x_a}\right)^3 \\
\Psi_2(x) &= -(x-x_a)\left(1-\frac{x-x_a}{x_b-x_a}\right)^2 \\
\Psi_3(x) &= 3\left(\frac{x-x_a}{x_b-x_a}\right)^2 - 2\left(\frac{x-x_a}{x_b-x_a}\right)^3 \\
\Psi_4(x) &= -(x-x_a)\left(\frac{x-x_a}{x_b-x_a}\right)\left(\frac{x-x_a}{x_b-x_a}-1\right)
\end{aligned}$$

in which degrees of freedom 1, 2, 3 and 4 are respectively: the deflection at node 1, the deflection angle at node 1, the deflection at node 2 and the deflection angle at node 2. Noting that the length of each element is L/2, the elementary stiffness matrix is:

$$[K] = \frac{16EI}{L^3}\begin{bmatrix} 6 & -3L/2 & -6 & -3L/2 \\ -3L/2 & L^2/2 & 3L/2 & L^2/4 \\ -6 & 3L/2 & 6 & 3L/2 \\ -3L/2 & L^2/4 & 3L/2 & L^2/2 \end{bmatrix}$$

Assembling the Finite Element equations, we obtain the following global Finite Element equations (in which only the coefficients of hte stiffness matrix that are above the diagonal are noted):

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & K_{14}^1 & 0 & 0 \\ & K_{22}^1 & K_{23}^1 & K_{24}^1 & 0 & 0 \\ & & K_{33}^1 + K_{11}^2 & K_{34}^1 + K_{12}^2 & K_{13}^2 & K_{14}^2 \\ & & & K_{44}^1 + K_{22}^2 & K_{23}^2 & K_{24}^2 \\ & & & & K_{33}^2 & K_{34}^2 \\ & & & & & K_{44}^2 \end{bmatrix}\begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \end{Bmatrix} = \begin{Bmatrix} F_1^1 \\ F_2^1 \\ F_3^1 + F_1^2 \\ F_4^1 + F_2^2 \\ F_3^2 \\ F_4^2 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

The nodal conditions are the following: $W_1 = W_2 = 0$ (fixed support), $Q_3 = Q_4 = 0$ (no concentrated loads), $Q_5 = F_0$, $Q_6 = -M_0$. The system of equations can be condensed as follows:

$$\frac{16EI}{L^2}\begin{bmatrix} 12 & 0 & -6 & -3L/2 \\ 0 & L^2 & 3L/2 & L^2/4 \\ -6 & 3L/2 & 6 & 3L/2 \\ -3L/2 & L^2/4 & 3L/2 & L^2/2 \end{bmatrix}\begin{Bmatrix} W_3 \\ W_4 \\ W_5 \\ W_6 \end{Bmatrix} = \begin{Bmatrix} F_3^1 + F_1^2 \\ F_4^1 + F_2^2 \\ F_3^2 \\ F_4^2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ F_0 \\ -M_0 \end{Bmatrix}$$

From there, it is possible to solve for W_3, W_4, W_5 and W_6. The secondary variables Q_1 and Q_2 can be post-processed as follows:

$$\begin{aligned} Q_1 &= Q_1^1 = EI\frac{d^3}{dx^3}\left(W_1\Psi_1(x) + W_2\Psi_2(x) + W_3\Psi_3(x) + W_4\Psi_4(x)\right)_{x=0} \\ Q_1 &= EI\left(-\frac{96}{L^3}W_3 - \frac{24}{L^2}W_4\right) \end{aligned}$$

In the same way:

$$Q_2 = \frac{24EI}{L^2}W_3 + \frac{4EI}{L}W_4$$

The approximate solution for the deflection is expressed as follows:
For $0 \le x \le L/2$:

$$w(x) = W_3\left[3\left(\frac{x}{L/2}\right)^2 - 2\left(\frac{x}{L/2}\right)^3\right] + W_4\left[-x\left(\frac{x}{L/2}\right)\left(\frac{x}{L/2} - 1\right)\right]$$

For $L/2 \le x \le L$:

$$\begin{aligned} w(x) = \quad & W_3\left[1 - 3\left(\frac{x - L/2}{L/2}\right)^2 + 2\left(\frac{x - L/2}{L/2}\right)^3\right] + W_4\left[-(x - L/2)\left(1 - \frac{x - L/2}{L/2}\right)^2\right] \\ & W_5\left[3\left(\frac{x - L/2}{L/2}\right)^2 - 2\left(\frac{x - L/2}{L/2}\right)^3\right] + W_6\left[-(x - L/2)\left(\frac{x - L/2}{L/2}\right)\left(\frac{x - L/2}{L/2} - 1\right)\right] \end{aligned}$$

4.9 Calculate the coefficients of the elementary stiffness matrix and force vector of a linear trianular element, if the numbering convention in Figure **??** is changed such that: node 1 is (a,0); node 2 is (0,b); node 3 is (0,0).

Solution: The solution is obtained by performin permutations of the coefficients of the stiffness matrix given in the problem:

$$[K] = \frac{1}{2ab}\left(a_{11}\begin{bmatrix} b^2 & 0 & -b^2 \\ 0 & 0 & 0 \\ -b^2 & 0 & b^2 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & 0 \\ 0 & a^2 & -a^2 \\ 0 & -a^2 & a^2 \end{bmatrix}\right)$$

4.10 Calculate the three boundary integrals Q_1, Q_2 and Q_3 for the linear triangular element shown in Figure 4.7.

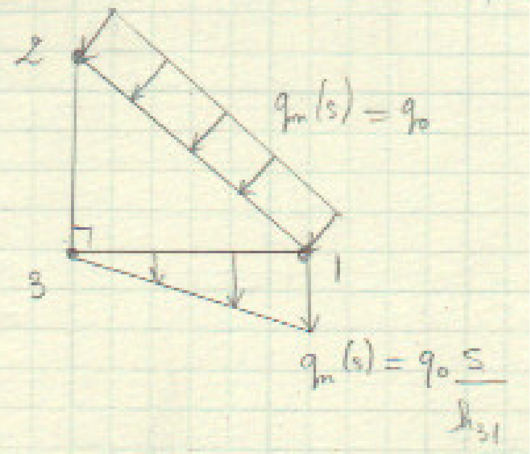


Figure 4.7 Triangular element subject to boundary loads.

Solution: We calculate the boundary integrals, one by one. For the first node:

$$Q_1 = \oint \Psi_1(s)q_n(s)ds = \int_0^{h_{12}} \Psi_1(s)q_n(s)ds + \int_0^{h_{23}} \Psi_1(s)q_n(s)ds + \int_0^{h_{31}} \Psi_1(s)q_n(s)ds$$

Since $\Psi_1(s) = 0$ on side 2-3:

$$Q_1 = \int_0^{h_{12}} \Psi_1(s)q_0 ds + \int_0^{h_{31}} \Psi_1(s)q_0 \frac{s}{h_{31}} ds$$

On side 1-2, $\Psi_1(s)$ is equal to the 1D Lagrange polynomial at node 1. On side 3-1, $\Psi_1(s)$ is equal to the 1D Lagrange polynomial at node 2. As a result:

$$Q_1 = \int_0^{h_{12}} \frac{h_{12} - s}{h_{12}} q_0 ds + \int_0^{h_{31}} \frac{s}{h_{12}} q_0 \frac{s}{h_{31}} ds$$

After integrating, we get:

$$Q_1 = \frac{q_0}{3} h_{31} + \frac{q_0}{2} h_{12}$$

For the second node:

$$Q_2 = \oint \Psi_2(s)q_n(s)ds = \int_0^{h_{12}} \Psi_2(s)q_n(s)ds$$

because $\Psi_2(s) = 0$ on side 3-1 and because $q_n(s) = 0$ on side 2-3. Moreover, on side 1-2, $\Psi_2(s)$ is equal to the 1D Lagrange polynomial at node 2, so that:

$$Q_2 = \int_0^{h_{12}} \frac{s}{h_{12}} q_0 ds$$

After integrating:

$$Q_2 = \frac{q_0 h_{12}}{2}$$

For the third node:

$$Q_3 = \oint \Psi_3(s) q_n(s) ds = \int_0^{h_{31}} \Psi_3(s) q_n(s) ds$$

because $\Psi_3(s) = 0$ on side 1-2 and because $q_n(s) = 0$ on side 2-3. Moreover, on side 3-1, $\Psi_3(s)$ is equal to the 1D Lagrange polynomial at node 1, so that:

$$Q_3 = \int_0^{h_{31}} \frac{h_{31} - s}{h_{31}} q_0 \frac{s}{h_{31}} ds$$

After integrating:

$$Q_3 = \frac{q_0 h_{31}}{6}$$

We check that $Q_1 + Q_2 + Q_3 = q_0 h_{12} + q_0 h_{31}/2$, which corresponds to the total load applied to the element.

4.11 Consider a problem described by the Poisson's equation:

$$-\nabla^2 u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f_0 \quad \text{in} \quad \Omega$$

in the square region shown in Figure 4.8. The boundary conditions are:

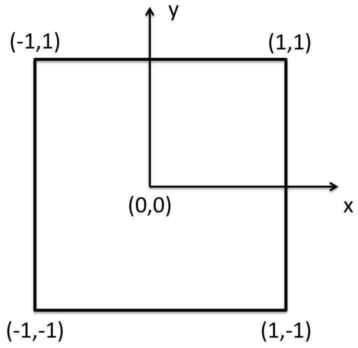


Figure 4.8 2D FEM to solve Poisson's equation

$$u(x, y) = 0 \quad \text{on} \quad \Gamma$$

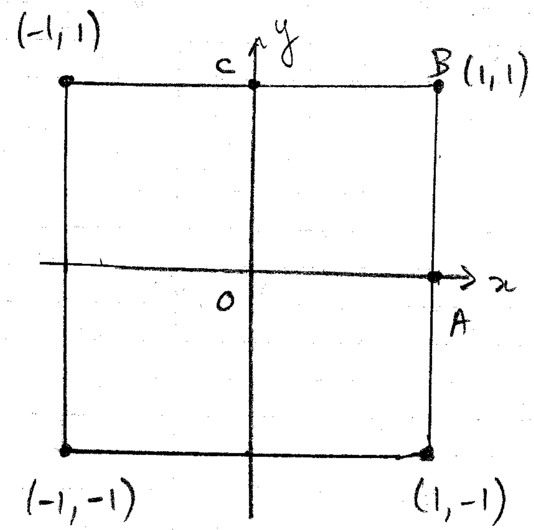
We wish to use the FEM to determine $u(x, y)$ on the domain Ω.

1. Show that it is sufficient to solve the problem on 1/8-th of the domain only to determine the solution everywhere in Ω.
2. Mesh this deduced domain with four linear triangular elements (justify).

3. Solve the FEM problem on the reduced domain (i.e. calculate the unknown nodal values of the primary variable).
4. Post-process the results of the FEM model (i.e. calculate the unknown boundary integrals of the secondary variable).

Solution: See the notes in the next 8 pages.

II. Commented Example (8.3.1 p 443)



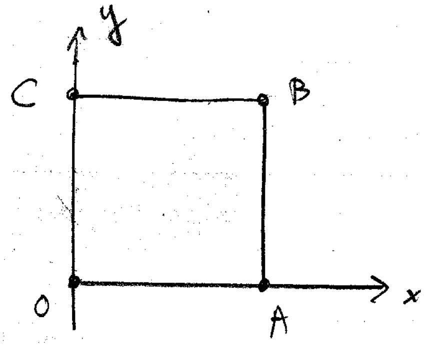
$$\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = f_0 & \text{in } \Omega \\ u(x,y) = 0 & \text{on } \Gamma \end{cases}$$

SV: $q_n = \partial u/\partial x \, n_x + \partial u/\partial y \, n_y$

STEP 1 : MESHING

⇒ symmetry about the x and y axes

⇒ possible to mesh OABC only and deduce the solution over Ω by symmetry

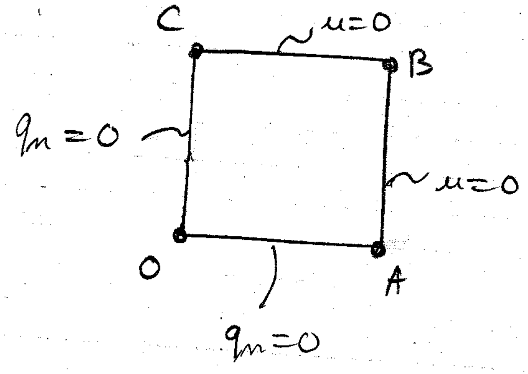


⇒ new boundary conditions

* AB, BC: external boundaries, no change: $u_{|AB} = u_{|BC} = 0$

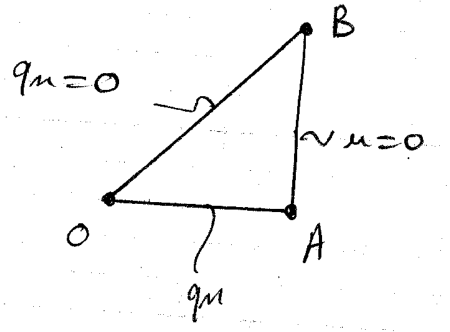
* OA: axis of symmetry ⊥ y
  $u(x,y)$ reaches an extremum:
  $$\left.\frac{\partial u}{\partial n}\right|_{OA} = -\left.\frac{\partial u}{\partial y}\right|_{OA} = 0 \quad (\vec{n} = -\vec{e_y} \text{ outward}),$$

* CO: axis of symmetry ⊥ x
  $u(x,y)$ reaches an extremum:
  $$\left.\frac{\partial u}{\partial n}\right|_{CO} = -\left.\frac{\partial u}{\partial x}\right|_{CO} = 0 \quad (\vec{n} = -\vec{e_x} \text{ outward}))$$



⇒ We have a new axis of symmetry: OB.

⇒ possible to mesh OAB only



New Boundary conditions:

* no change on OA and AB:

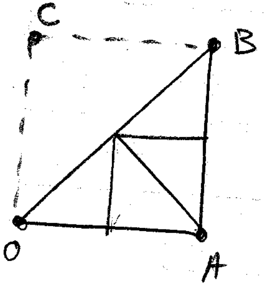
$$q_n|_{OA} = -\frac{\partial u}{\partial y}\Big|_{OA} = 0$$

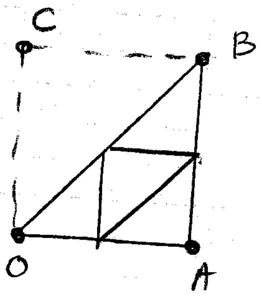
$$u|_{AB} = 0$$

* OB is an axis of symmetry, therefore u reaches an extremum on OB: $\frac{\partial u}{\partial n}\Big|_{OB} = q_n|_{OB} = 0$

Natural choice : triangular elements

①





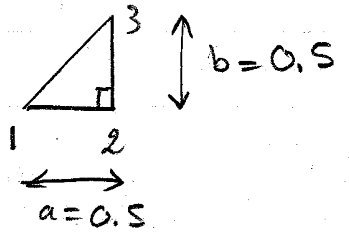
②

① will give a solution symmetric about OB AND about AC
② " " " " OB only

The physical problem is symmetric about OB only, so the mesh n° 2 is more desirable.

STEP 2 : FE MODEL

Using mesh ② with linear triangular elements, we can notice that all elements have the same geometry. If the same local numbering convention is used over all elements, the stiffness matrix will be the same for all elements:



$$[K^e] = \frac{1}{2ab}\begin{bmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{bmatrix}$$

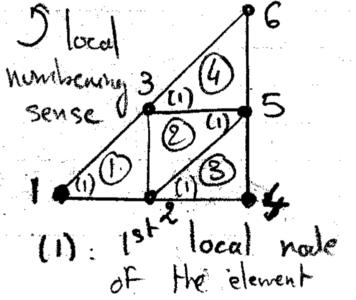
Because the distributed load is constant all over the domain and because all elements have the same geometry, we have:

$$\{F^e\} = \frac{f_0\, ab}{6}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$ For all elements.

Numerical values: $$[K^e] = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

$$\{F^e\} = \frac{f_0}{24}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

STEP 3 : ASSEMBLY



Connectivity Matrix:

e \ dof	1	2	3
1	1	2	3
2	5	3	2
3	2	4	5
4	3	5	6

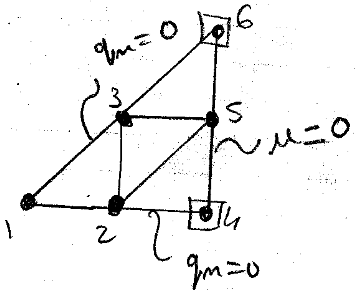
$$[K]\{U\} = \{F\} + \{Q\}$$

with :

$$[K] = \left[\begin{array}{cccccc} K^1_{11} & K^1_{12} & K^1_{13} & 0 & 0 & 0 \\ & (K^1_{22}+K^2_{33}+K^3_{11}) & (K^1_{23}+K^2_{32}) & K^3_{12} & K^3_{13} & 0 \\ & & (K^1_{33}+K^2_{22}+K^4_{11}) & 0 & K^4_{12} & K^4_{13} \\ & \text{sym.} & & K^3_{22} & K^3_{23} & 0 \\ & & & & (K^2_{11}+K^3_{33}+K^4_{22}) & K^4_{23} \\ & & & & & K^4_{33} \end{array}\right]$$

$$\{Q\} = \begin{Bmatrix} Q^1_1 \\ Q^1_2 + Q^2_3 + Q^3_1 \\ Q^1_3 + Q^2_2 + Q^4_1 \\ Q^3_2 \\ Q^2_1 + Q^3_3 + Q^4_2 \\ Q^4_3 \end{Bmatrix}$$

STEP 4 : BOUNDARY CONDITIONS

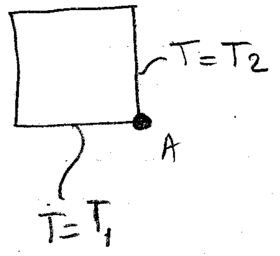


- essential bc : $U_4 = U_5 = U_6 = 0$
- natural bc : $Q_1 = Q_2 = Q_3 = 0$

Remark 1 : node 4, node 6 are on 2 boundaries. We could have chosen to impose $Q_4 = Q_6 = 0$ (according to the natural bc on sides 1-4 and 1-6), but applying essential b.c. first gives more rows to condense → Faster resolution

Remark 2: What if an edge belongs to 2 sides where different values of the dependent variable are applied?

ex:



with $T_1 \neq T_2$.

⇒ A is called a singular point

1) Choose an approximate b.c. at node A, ⚠ not the average. Generally, the largest of the 2 bc, to get conservative estimates

2) Refine the mesh at the vicinity of the singular point.

STEP 5: Resolution / Condensation

Same process as in 1D: delete the rows containing a known value of the PV:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ & K_{22} & K_{23} \\ \text{sym} & & K_{33} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1 + Q_1 \\ F_2 + Q_2 \\ F_3 + Q_3 \end{Bmatrix} - \underbrace{\overset{\text{known}}{\begin{Bmatrix} K_{14}\bar{U}_4 + K_{15}\bar{U}_5 + K_{16}\bar{U}_6 \\ K_{24}\bar{U}_4 + K_{25}\bar{U}_5 + K_{26}\bar{U}_6 \\ K_{34}\bar{U}_4 + K_{35}\bar{U}_5 + K_{36}\bar{U}_6 \end{Bmatrix}}}_{= 0 \text{ since in this problem } \bar{U}_4 = \bar{U}_5 = \bar{U}_6 = 0}$$

STEP 6: Post processing

$$\begin{Bmatrix} Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = - \begin{Bmatrix} F_4 \\ F_5 \\ F_6 \end{Bmatrix} + \begin{bmatrix} K_{41} & K_{42} & K_{43} \\ K_{51} & K_{52} & K_{53} \\ K_{61} & K_{62} & K_{63} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$

BUT if the coefficients of the stiffness matrix are not stored → post processing needed

Ex $Q_4 = Q_2^3 = \oint_{\Gamma_{e3}} q_n(s)\,\psi_2(s)\,ds$

$$Q_4 = \int_{1-2} q_n(x)\,\psi_2(x, y=0)\,dx + \int_{2-3} q_n(y)\,\psi_2(x=1, y)\,dy$$

$$+ \int_{3-1} q_n(s)\,\psi_2(x,y)\,ds$$

[$\psi_2(x,y)$: 2D interpolation fct for 2nd node in the elt]

with $q_n(s) = 0$ over 1-2 and $\psi_2(s) = 0$ over 3-1

$$\Rightarrow Q_4 = \int_0^{h_{2-3}} \left(+ \frac{\partial u}{\partial x} \underbrace{n_x}_{=1} \right) \underbrace{\left(1 - \frac{y}{h_{2-3}} \right)} dy$$

node 2 in triangle 3
is the 1st node of side 2-3
with the numbering sense ↺

$\frac{\partial u}{\partial x}$ is approximated by $\frac{\partial u_h^e}{\partial x} \simeq \sum_{j=1}^{3} u_j^e \frac{\partial \psi_j^e}{\partial x}$

Over element 3: $\frac{\partial u_h^e}{\partial x} \simeq \underset{\substack{\downarrow \\ U_2}}{u_1^3} \frac{\partial \psi_1}{\partial x} + \underset{\substack{\downarrow \\ U_4 \\ =0}}{u_2^3} \frac{\partial \psi_2}{\partial x} + \underset{\substack{\downarrow \\ U_5 \\ =0}}{u_3^3} \frac{\partial \psi_3}{\partial x}$

$$\Rightarrow Q_4 = \int_0^{h_{2-3}} \underbrace{\frac{\beta_1^{(3)}}{2A_3}}_{\frac{\partial \psi_1}{\partial x}} \times U_2 \left(1 - \frac{y}{h_{2-3}} \right) dy \quad , \beta_1^{(3)} = -0.5$$

$$Q_4 = \frac{-0.5}{2A_3} U_2 \underbrace{\int_0^{h_{2-3}} \left(1 - \frac{y}{h_{2-3}} \right) dy}_{\left[\left(1 - \frac{y}{h_{2-3}}\right)^2 \times \left(-\frac{h_{23}}{2} \right) \right]_0^{h_{23}} = \frac{h_{23}}{2}}$$

$$Q_4 = -0.5 \frac{h_{23}}{4A_3} U_2 = -\frac{1}{2} \times \underset{\substack{\downarrow \\ h_{23}}}{\frac{1}{2}} \times \frac{1}{4} \times \underset{\substack{\downarrow \\ 1/A_3}}{8} U_2 = \boxed{-0.5 U_2 = Q_4}$$

See notes 2-19

05/2019

Chapter 4 – Section I – Comprehensive Problem: Poisson's equation on a square domain

$$Q_6 = Q_3^{④} = \oint_{\Gamma ④} \psi_3^{④}(x,y)\, q_n(s)\, ds = \underbrace{\int_1^2 \psi_3^{④} q_n\, ds}_{\substack{=0 \\ \psi_3^{④}\big|_{1-2} = 0}} + \underbrace{\int_2^3 \psi_3^{④} q_n\, ds}_{\substack{s=y \\ \vec{n} = \vec{e_x}}} + \underbrace{\int_3^1 \psi_3^{④} q_n\, ds}_{\substack{=0 \\ q_n\big|_{3-1} = 0}}$$

$$Q_6 = \int_0^{n_{23}} \overset{2-3}{\psi_3}(y)\, \frac{\partial u}{\partial x}\, dy \quad \text{with } \frac{\partial u}{\partial x} \simeq \frac{\partial}{\partial x}\left[\sum_k u_k \frac{\partial \psi_k}{\partial x}\right] = u_3 \frac{\partial \psi_1^{④}}{\partial x} = u_3\, \beta_1$$

$$Q_6 = \int_0^{1/2} 2y \times u_3 \beta_1\, dy = 2 u_3 \beta_1 \times \frac{1}{2} \times \frac{1}{4}$$

$$\boxed{Q_6 = \frac{\beta_1}{4} u_3}$$

⊛ $Q_5 = Q_1^{②} + Q_2^{④} + Q_3^{③}$

$$Q_1^{②} = \oint_{\Gamma ②} \psi_1(x,y)\, q_n(s)\, ds = \underbrace{\int_1^2 \psi_1(x,y)\, q_n(s)\, ds}_{\substack{s=-x \\ \vec{n} = -\vec{e_y}}} + \underbrace{\int_3^1 \psi_1(x,y)\, q_n(s)\, ds}_{\vec{n} = \frac{\vec{e_x}}{\sqrt{2}} - \frac{\vec{e_y}}{\sqrt{2}}}$$

$$Q_1^{②} = -\int_1^2 \overset{1-2}{\hat{\psi}_1}(-x) \left[-\frac{\partial u}{\partial y}\right] dx + \int_3^1 \hat{\psi}_1^{3-1}(s) \left[\frac{1}{\sqrt{2}}\frac{\partial u}{\partial x} - \frac{1}{\sqrt{2}}\frac{\partial u}{\partial y}\right] ds$$

$$Q_2^{①} = \int_0^{h_{12}} \left(1+\frac{x}{h_{12}}\right)\left[u_2 \underbrace{\frac{\partial \varphi_3}{\partial x}}_{\beta_3} + u_3 \underbrace{\frac{\partial \varphi_2}{\partial x}}_{\beta_2}\right] dx + \int_3^{h_{31}=1/\sqrt{2}} \frac{s}{h_{31}} \times \frac{1}{\sqrt{2}}\left[\beta_3 u_2 + \beta_2 u_3 - \gamma_3 u_2 - \gamma_2 u_3\right] ds$$

(*) $$Q_2^{①} = \left(\beta_3 u_2 + \beta_2 u_3\right) \times \frac{3}{4} - \frac{1}{4}\left(\beta_3 u_2 + \beta_2 u_3 + \gamma_3 u_2 + \gamma_2 u_3\right)$$

$$Q_2^{④} = \int_{\Gamma ④} \varphi_2(x,y)\, q_n(s)\, ds = \underbrace{\int_1^2 \varphi_2(x,y)\, q_n\, ds}_{s=x,\ \vec{n}=-\vec{e_y}} + \underbrace{\int_2^3 \varphi_2(x,y)\, ds}_{s=y,\ \vec{n}=\vec{e_x}}$$

$$Q_2^{④} = -\int_0^{1/2} 2x \frac{\partial u}{\partial y} dx + \int_0^{1/2} (1-2y) \frac{\partial u}{\partial x} dy$$

$$Q_2^{④} = 2\gamma_1 u_3 \times \left(-\frac{1}{8}\right) + \beta_1 u_3 \times \frac{1}{4}$$

(*) $$Q_2^{④} = \left(\frac{\beta_1}{4} - \frac{\gamma_1}{4}\right) u_3$$

$$Q_3^{③} = \int_{\Gamma ③} \overset{③}{\varphi_3}(x,y)\, q_n(s)\, ds = \underbrace{\int_2^3 \varphi_3\, q_n(s)\, ds}_{s=y,\ \vec{n}=\vec{e_x}} + \underbrace{\int_3^1 \varphi_3\, q_n\, ds}_{\vec{n}=-\frac{\vec{e_x}}{\sqrt{2}}+\frac{\vec{e_y}}{\sqrt{2}}}$$

$$Q_3^{③} = \int_0^{1/2} 2y \frac{\partial u}{\partial x} dy + \int_1^{1/\sqrt{2}} \left(1-\frac{s}{h_{31}}\right) \times \frac{1}{\sqrt{2}}\left(-\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) ds$$

$$Q_3^{③} = 2\beta_1 u_2 \times \frac{1}{2} + \frac{1}{\sqrt{2}}\left(-\beta_1 u_2 + \gamma_1 u_2\right) \underbrace{\left(\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{2} \times \frac{1}{2}\right)}_{\frac{1}{2\sqrt{2}}}$$

$$Q_3^{③} = \beta_1 u_2 + \frac{1}{4}\left(-\beta_1 u_2 + \gamma_1 u_2\right)$$

(*) $$Q_3^{③} = \frac{3\beta_1 u_2}{4} + \frac{\gamma_1 u_2}{4}$$

4.12 Determine the Lagrange interpolation polynomials for:

- a triangular element that has two nodes per side ($k = 2$)

- a triangular element that has three nodes per side ($k = 3$)

- a triangular element that has four nodes per side ($k = 4$)

Solution: Here, we express the interpolation functions of the triangular elements as functions of the area coordinates, by using the interpolation property.

- Triangular element with two nodes per side:

$$\Psi_1(x, y) = L_1(s), \quad \Psi_2(x, y) = L_2(s), \quad \Psi_3(x, y) = L_3(s)$$

- Triangular element with three nodes per side (see Figure 4.9 for the node numbering):

$$\begin{aligned}
\Psi_1(x, y) &= 2L_1(s)\left(L_1(s) - \frac{1}{2}\right) \\
\Psi_2(x, y) &= 2L_2(s)\left(L_2(s) - \frac{1}{2}\right) \\
\Psi_3(x, y) &= 2L_3(s)\left(L_3(s) - \frac{1}{2}\right) \\
\Psi_4(x, y) &= 4L_1(s)L_2(s) \\
\Psi_5(x, y) &= 4L_2(s)L_3(s) \\
\Psi_6(x, y) &= 4L_1(s)L_3(s)
\end{aligned}$$

- Triangular element with four nodes per side (see Figure 4.9 for the node numbering):

$$
\begin{aligned}
\Psi_1(x,y) &= \frac{9}{2}L_1(s)\left(L_1(s)-\frac{1}{3}\right)\left(L_1(s)-\frac{2}{3}\right)\\
\Psi_2(x,y) &= \frac{9}{2}L_2(s)\left(L_2(s)-\frac{1}{3}\right)\left(L_2(s)-\frac{2}{3}\right)\\
\Psi_3(x,y) &= \frac{9}{2}L_3(s)\left(L_3(s)-\frac{1}{3}\right)\left(L_3(s)-\frac{2}{3}\right)\\
\Psi_4(x,y) &= \frac{27}{2}L_1(s)L_2(s)\left(L_1(s)-\frac{1}{3}\right)\\
\Psi_5(x,y) &= \frac{27}{2}L_1(s)L_2(s)\left(L_2(s)-\frac{1}{3}\right)\\
\Psi_6(x,y) &= \frac{27}{2}L_2(s)L_3(s)\left(L_2(s)-\frac{1}{3}\right)\\
\Psi_7(x,y) &= \frac{27}{2}L_2(s)L_3(s)\left(L_3(s)-\frac{1}{3}\right)\\
\Psi_8(x,y) &= \frac{27}{2}L_1(s)L_3(s)\left(L_3(s)-\frac{1}{3}\right)\\
\Psi_9(x,y) &= \frac{27}{2}L_1(s)L_3(s)\left(L_1(s)-\frac{1}{3}\right)\\
\Psi_{10}(x,y) &= 27L_1(s)L_2(s)L_3(s)
\end{aligned}
$$

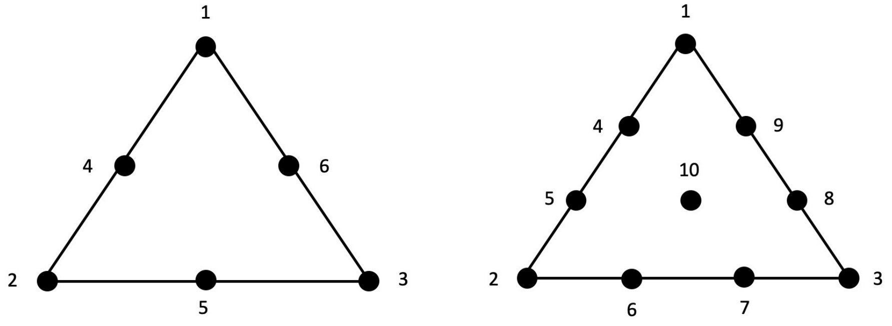


Figure 4.9 Node numbering adopted in Problem 4.12.

4.13 Determine the interpolation function Ψ_{14} for the triangular element shown in Figure 4.10. *Assume that nodes on the sides of the element are equally spaced.*

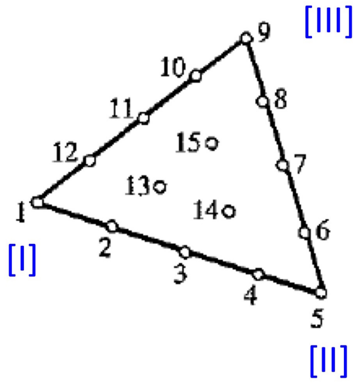


Figure 4.10 A higher-order triangular element. Image taken from [**?**]

Solution: We express the interpolation function at node 14 as a function of the area coordinates, by using the interpolation property:

$$\Psi_{14}(x, y) = 128 L_I(s) L_{II}(s) L_{III}(s) \left(L_{II}(s) - \frac{1}{4} \right)$$

in which we note the edges of the element as nodes I, II and III, as shown in the figure.

4.14 Calculate the Jacobian of each of the three elements of the mesh shown in Figure 4.11. Explain whether the geometry and numbering conventions are acceptable or not.

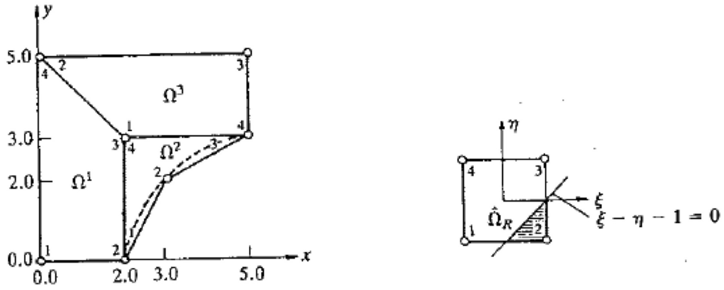


Figure 4.11 Meshing problem requiring calculating the Jacobian. Image taken from [**?**]

Solution: To calculate the Jacobian on each element, we first need to explain how the geometry is interpolated on each element, according to the following equations:

$$x(\xi,\eta)=\sum_{j=1}^{4}x_j\Psi_j(\xi,\eta),\quad y(\xi,\eta)=\sum_{j=1}^{4}y_j\Psi_j(\xi,\eta)$$

in which the interpolation functions $\Psi_j(\xi,\eta)$ are those of the linear quadrilateral master element:

$$\begin{aligned}\Psi_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta)\\ \Psi_2(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta)\\ \Psi_3(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta)\\ \Psi_4(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta)\end{aligned}$$

Then, the Jacobian is calculated for each element, as follows

$$J=\frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\eta}-\frac{\partial x}{\partial\eta}\frac{\partial y}{\partial\xi}$$

For element 1, $x_1=0$, $x_2=2$, $x_3=2$, $x_4=0$, $y_1=0$, $y_2=0$, $y_3=3$ and $y_4=5$. We find: $J_1=(4-\xi)/16$, which is always positive, since $\xi\in[-1,\,1]$. Element 1 is thus properly meshed. For element 2, $x_1=2$, $x_2=3$, $x_3=5$, $x_4=2$, $y_1=0$, $y_2=2$, $y_3=3$ and $y_4=3$. We find $J_2=2(1-\xi+\eta)/4$. So $J_2>0$ only if $\eta>\xi-1$, i.e. only when the point considered is in the convex hull of the element. Hence, element 2 is not properly meshed, because it is not convex. For element 3, $x_1=2$, $x_2=0$, $x_3=5$, $x_4=5$, $y_1=3$, $y_2=5$, $y_3=5$ and $y_4=3$. We find $J_3=-(4+\xi)/2$, which is always negative, since $\xi\in[-1,\,1]$. Hence, element 3 is not properly meshed. Here, this is because the node numbering convention chosen in the element goes clockwise, while the node numbering convention in the master element goes counter-clockwise (the same node numbering sense has to be chosen for both).

4.15 Determine the conditions on the location of node 3 of the quadrilateral element shown in Figure 4.12.

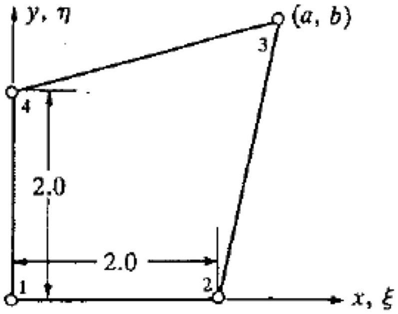


Figure 4.12 Finding an acceptable element shape. Image taken from (Reddy, 2004)

Solution: To calculate the Jacobian of the element, we first need to explain how the geometry is interpolated, according to the following equations:

$$x(\xi,\eta)=\sum_{j=1}^{4}x_j\Psi_j(\xi,\eta),\quad y(\xi,\eta)=\sum_{j=1}^{4}y_j\Psi_j(\xi,\eta)$$

Using the expressions of the interpolation functions (linear quadrilateral master element):

$$\begin{aligned}\Psi_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta)\\ \Psi_2(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta)\\ \Psi_3(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta)\\ \Psi_4(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta)\end{aligned}$$

and using the coordinates of the nodes: $x_1=0$, $x_2=2$, $x_3=a$, $x_4=0$, $y_1=0$, $y_2=0$, $y_3=b$ and $y_4=2$, we have:

$$\begin{aligned}x(\xi,\eta) &= \frac{1}{4}(1+\xi)\left[2(1-\eta)+a(1+\eta)\right]\\ y(\xi,\eta) &= \frac{1}{4}(1+\eta)\left[b(1+\xi)+2(1-\xi)\right]\end{aligned}$$

We then calculate the Jacobian:

$$J=\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta}-\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

$$J = a + b + (b-2)\xi + (a-2)\eta$$

At node 1 ($\xi = -1$, $\eta = -1$): $J = 4$, positive. At node 2 ($\xi = 1$, $\eta = -1$): $J = 2b$, which is positive, since according to the figure shown, $b > 0$. At node 3 ($\xi = 1$, $\eta = 1$): $J = 2(a + b - 2)$, which is positive only if $b > 2 - a$. At node 4 ($\xi = -1$, $\eta = 1$): $J = 2a$, which is positive, since according to the figure shown, $a > 0$. So to summarize, the element will be acceptable only if $b > 2 - a$, i.e., only if point 3 is above the line that links nodes 2 and 4. In other words, the element is only acceptable if it is convex.

4.16 Consider the isoparametric quadrilateral element shown iFigure 4.13. Use the Gauss-Legendre numerical integration scheme of the lowest order possible to calculate the following integrals:

$$\begin{aligned} S_{ij}^{00} &= \int_\Omega \Psi_i(x,y)\Psi_j(x,y)dxdy \\ S_{ij}^{12} &= \int_\Omega \frac{\partial \Psi_i(x,y)}{\partial x}\frac{\partial \Psi_i(x,y)}{\partial y}dxdy \end{aligned}$$

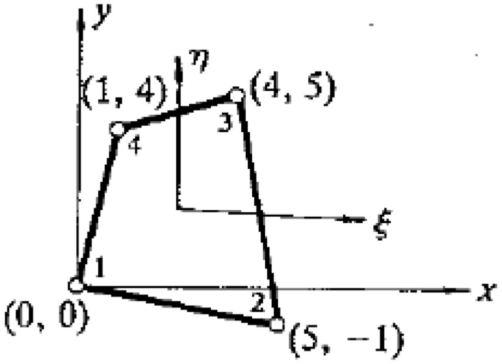


Figure 4.13 Evaluating integrals defined on an element of irregular shape. Image taken from (Reddy, 2004)

Solution: For the first integral:

$$S_{ij}^{00} = \int_{-1}^{1}\int_{-1}^{1} \Phi_i(\xi,\eta)\Phi_j(\xi,\eta)\, J d\xi d\eta$$

in which the expressions of the interpolation functions for the linear quadrilateral master element are:

$$\begin{aligned}\Phi_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta)\\ \Phi_2(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta)\\ \Phi_3(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta)\\ \Phi_4(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta)\end{aligned}$$

To evaluate S_{ij}^{00}, we thus need to calculate the Jacobian J on the element:

$$J = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

with:

$$x(\xi,\eta) = \sum_{j=1}^{4} x_j \Phi_j(\xi,\eta), \quad y(\xi,\eta) = \sum_{j=1}^{4} y_j \Phi_j(\xi,\eta)$$

Noting that $x_1 = 0$, $x_2 = 5$, $x_3 = 4$, $x_4 = 1$, $y_1 = 0$, $y_2 = -1$, $y_3 = 5$ and $y_4 = 4$, we get:

$$x(\xi,\eta) = \frac{1}{4}\left(10 + 8\xi - 2\xi\eta\right), \quad y(\xi,\eta) = \frac{1}{4}\left(8 + 10\eta + 2\xi\eta\right)$$

and we find:

$$J = \frac{1}{4}(20 + 4\xi - \xi\eta)$$

The integrand in S_{ij}^{00} is thus cubic in ξ and cubic in η. Therefore, we need to perform a Gauss Legendre numerical integration of order $r \geq (p+1)/2 = 2$. As an example, let us calculate S_{11}^{00}. We have:

$$S_{11}^{00} = \int_{-1}^{1}\int_{-1}^{1} \frac{1}{64}(1-\xi)^2(1-\eta)^2(20 + 4\xi - \xi\eta)d\xi d\eta$$

We note $P(\xi,\eta) = \frac{1}{64}(1-\xi)^2(1-\eta)^2(20+4\xi-\xi\eta)$. The Gauss-Legendre quadrature of order 2 approximates the integral as follows:

$$S_{11}^{00} \simeq P(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) + P(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) + P(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) + P(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$

The integral will be calculated exactly since P is a polynomial, with an order $p = 2r - 1$. Then, $S_{12}^{00} = S_{21}^{00}$ and S_{22}^{00} will be calcualted in the same way.
For the second integral:

$$S_{ij}^{12} = \int_{-1}^{1}\int_{-1}^{1}\left(J_{11}^{*}\frac{\partial \Phi_i(\xi,\eta)}{\partial \xi} + J_{12}^{*}\frac{\partial \Phi_i(\xi,\eta)}{\partial \eta}\right)\left(J_{21}^{*}\frac{\partial \Phi_i(\xi,\eta)}{\partial \xi} + J_{22}^{*}\frac{\partial \Phi_i(\xi,\eta)}{\partial \eta}\right) J d\xi d\eta$$

in which $[J^*]$ is the inverse of the Jacobian matrix. The Jacobian matrix is calculated as:

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8-2\eta & 2\eta \\ -2\xi & 10+2\xi \end{bmatrix}$$

The inverse of the Jacobian matrix is calculated as:

$$[J^*] = [J]^{-1} = \frac{1}{4J}\begin{bmatrix} 10+2\xi & -2\eta \\ 2\xi & 8-2\eta \end{bmatrix}$$

Each term in the brackets of the integrand of S_{ij}^{12} is the ratio of a plynomial that is linear in ξ and in η, by J. So the integrand is the product of J by a polynomial that is quadratic in ξ and in η. Overall, the integrand is thus cubic in ξ and in η, so a Gauss-Legendre quadrature of order 2 should allow calculating the integrals S_{ij}^{12} exactly.

4.17 Prove the equations given for the weak formulation of a plane elasticity problem.

Solution: The balance of momentum in directions x and y is expressed as:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = \rho\frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = \rho\frac{\partial^2 u_y}{\partial t^2}$$

The general form of the constitutive law is:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{Bmatrix}$$

Combining the balance and constitutive equations, one gets:

$$\frac{\partial}{\partial x}\left(c_{11}\epsilon_{xx} + c_{12}\epsilon_{yy}\right) + \frac{\partial}{\partial y}\left(2c_{66}\epsilon_{xy}\right) + f_x = \rho\frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left(2c_{66}\epsilon_{xy}\right) + \frac{\partial}{\partial y}\left(c_{21}\epsilon_{xx} + c_{22}\epsilon_{yy}\right) + f_y = \rho\frac{\partial^2 u_y}{\partial t^2}$$

Then, we use the definition of the small deformation tensor:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and we obtain:

$$\frac{\partial}{\partial x}\left(c_{11}\frac{\partial u_x}{\partial x} + c_{12}\frac{\partial u_y}{\partial y}\right) + \frac{\partial}{\partial y}\left(c_{66}\frac{\partial u_x}{\partial y} + c_{66}\frac{\partial u_y}{\partial x}\right) + f_x = \rho\frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left(c_{66}\frac{\partial u_x}{\partial y} + c_{66}\frac{\partial u_y}{\partial x}\right) + \frac{\partial}{\partial y}\left(c_{21}\frac{\partial u_x}{\partial x} + c_{22}\frac{\partial u_y}{\partial y}\right) + f_y = \rho\frac{\partial^2 u_y}{\partial t^2}$$

From there, we can write the weak formulation. From the balance equation in the x-direction: $\forall w_1(x,y) \sim \delta u_x(x,y)$:

$$\int_{\Omega_e} \left[h_e \left[\frac{\partial w_1}{\partial x} \left(c_{11} \frac{\partial u_x}{\partial x} + c_{12} \frac{\partial u_y}{\partial y} \right) + c_{66} \frac{\partial w_1}{\partial y} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \right] + \rho w_1 \ddot{u}_x \right] dxdy$$

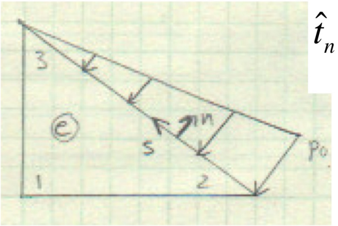
$$= \int_{\Omega_e} h_e w_1 f_x dxdy + \oint_{\Gamma_e} h_e w_1 \widehat{t}_x ds$$

From the balance equation in the y-direction: $\forall w_2(x,y) \sim \delta u_y(x,y)$:

$$\int_{\Omega_e} \left[h_e \left[\frac{\partial w_2}{\partial y} \left(c_{12} \frac{\partial u_x}{\partial x} + c_{22} \frac{\partial u_y}{\partial y} \right) + c_{66} \frac{\partial w_2}{\partial x} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \right] + \rho w_2 \ddot{u}_y \right] dxdy$$

$$= \int_{\Omega_e} h_e w_2 f_y dxdy + \oint_{\Gamma_e} h_e w_2 \widehat{t}_y ds$$

4.18 For the Finite Element in plane elasticity shown in Figure 4.14, determine the surface load vector $\{Q_e\}$.



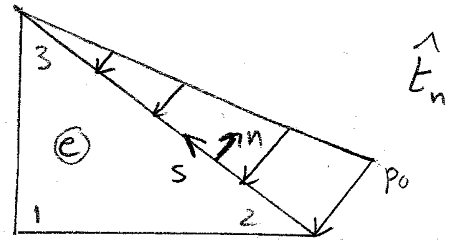
$$\hat{t}_n(s) = -\left(p_0 - p_0 \frac{s}{L_{23}}\right) = -p_0\left(1 - \frac{s}{L_{23}}\right)$$

Figure 4.14 Calculation of boundary integrals in plane elasticity.

Solution: See the solution in the notes provided in the next 3 pages.

V – Surface Loads ($\{Q\}$)

Example 11.5.1 p 619

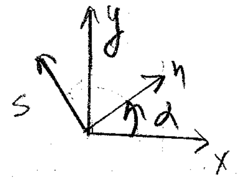


$$\hat{t}_n(s) = -\left(p_0 - p_0\frac{s}{L_{23}}\right) = -p_0\left(1-\frac{s}{L_{23}}\right)$$

$$\{Q^e\} = \oint_{\Gamma_e} [\Psi]^T \begin{Bmatrix}\hat{t}_x\\ \hat{t}_y\end{Bmatrix} dS \quad , \quad dS = dz\,ds$$

$$\{Q^e\} = h_e \int_{1-2} [\Psi]^T \begin{Bmatrix}\hat{t}_x\\ \hat{t}_y\end{Bmatrix} ds + h_e \int_{2-3} [\Psi]^T \begin{Bmatrix}\hat{t}_x\\ \hat{t}_y\end{Bmatrix} ds + h_e \int_{3-1} [\Psi]^T \begin{Bmatrix}\hat{t}_x\\ \hat{t}_y\end{Bmatrix} ds$$

Thickness (in z-direction)



$$\begin{Bmatrix}\hat{t}_x\\ \hat{t}_y\end{Bmatrix} = \underbrace{\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}}_{[R]^T} \begin{Bmatrix}\hat{t}_n\\ \hat{t}_s\end{Bmatrix}$$

On sides 1-2, 3-1 : $\hat{t}_n = \hat{t}_s = 0$

$$\{Q^e\} = h_e \int_{2-3} [\Psi]^T [R]^T \begin{Bmatrix}\hat{t}_n\\ \hat{t}_s\end{Bmatrix} ds$$

(Note: if the element were connected to other elements on sides 1-2 and 3-1, $\hat{t}_n$ and $\hat{t}_s$ would be compensated by the tractions at the boundaries of these neighboring elements — equilibrium of the SV)

with $\hat{t}_s = 0$, $\hat{t}_n = p_0\left(1-\frac{s}{L_{23}}\right)$ on side 2-3,

and $\Psi_1 = 0$ on side 2-3.

$$\{Q^e\} = h_e \int_{2-3} \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_1 \\ \psi_2 & 0 \\ 0 & \psi_2 \\ \psi_3 & 0 \\ 0 & \psi_3 \end{bmatrix} \begin{Bmatrix} -p_0\left(1-\frac{s}{L_{23}}\right)\cos\alpha \\ -p_0\left(1-\frac{s}{L_{23}}\right)\sin\alpha \end{Bmatrix} ds$$

$$\{Q^e\} = h_e \int_{2-3} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \psi_1^{1D}(s) & 0 \\ 0 & \psi_1^{1D}(s) \\ \psi_2^{1D}(s) & 0 \\ 0 & \psi_2^{1D}(s) \end{bmatrix} \begin{Bmatrix} -p_0\left(1-\frac{s}{L_{23}}\right)\cos\alpha \\ -p_0\left(1-\frac{s}{L_{23}}\right)\sin\alpha \end{Bmatrix} ds$$

$$\{Q^e\} = h_e \int_{2-3} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \left(1-\frac{s}{L_{23}}\right) & 0 \\ 0 & \left(1-\frac{s}{L_{23}}\right) \\ s/L_{23} & 0 \\ 0 & s/L_{23} \end{bmatrix} \begin{Bmatrix} -p_0\left(1-\frac{s}{L_{23}}\right)\cos\alpha \\ -p_0\left(1-\frac{s}{L_{23}}\right)\sin\alpha \end{Bmatrix} ds$$

$$\{Q^e\} = h_e \int_{s=0}^{s=L_{23}} \begin{Bmatrix} 0 \\ 0 \\ -p_0\left(1-\frac{s}{L_{23}}\right)^2\cos\alpha \\ -p_0\left(1-\frac{s}{L_{23}}\right)^2\sin\alpha \\ -p_0\frac{s}{L_{23}}\left(1-\frac{s}{L_{23}}\right)\cos\alpha \\ -p_0\frac{s}{L_{23}}\left(1-\frac{s}{L_{23}}\right)\sin\alpha \end{Bmatrix} ds$$

$$\{Q^e\} = -h_e p_0 \left\{ \begin{array}{c} 0 \\ 0 \\ \left[-\frac{L_{23}}{3}\left(1-\frac{s}{L_{23}}\right)^3 \cos\alpha \right]_0^{L_{23}} \\ \left[-\frac{L_{23}}{3}\left(1-\frac{s}{L_{23}}\right)^3 \sin\alpha \right]_0^{L_{23}} \\ -\left[\left(\frac{L_{23}}{2}\left(\frac{s}{L_{23}}\right)^2 - \left(\frac{s}{L_{23}}\right)^3 \times \frac{L_{23}}{3} \right) \cos\alpha \right]_0^{L_{23}} \\ -\left[\left(\frac{L_{23}}{2}\left(\frac{s}{L_{23}}\right)^2 - \frac{L_{23}}{3}\left(\frac{s}{L_{23}}\right)^3 \right) \sin\alpha \right]_0^{L_{23}} \end{array} \right\}$$

$$\{Q^c\} = -h_e p_0 \left\{ \begin{array}{c} 0 \\ 0 \\ \frac{L_{23}}{3}\cos\alpha \\ \frac{L_{23}}{3}\sin\alpha \\ -L_{23}\frac{\cos\alpha}{6} \\ -L_{23}\frac{\sin\alpha}{6} \end{array} \right\} = -\frac{h_e p_0 L_{23}}{6} \left\{ \begin{array}{c} 0 \\ 0 \\ 2\cos\alpha \\ 2\sin\alpha \\ -\cos\alpha \\ -\sin\alpha \end{array} \right\}$$

4.19 In the plane stress problem shown in Figure 4.15, determine the horizontal component of the load vector at node 16 (Q_{16x}) and the vertical component of the load vector at node 11 (Q_{11y}).

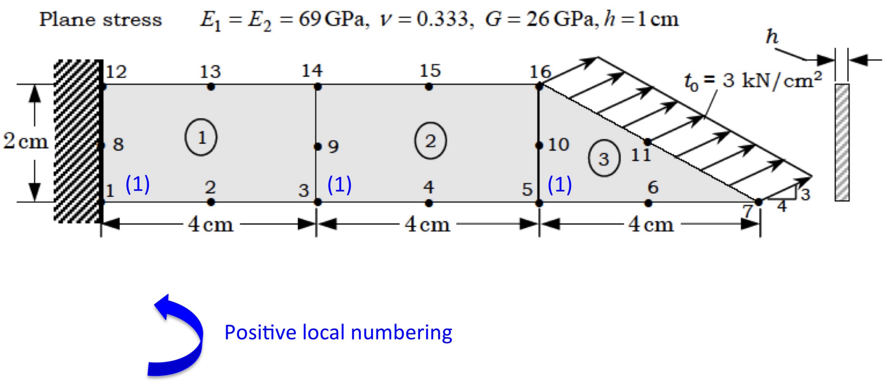
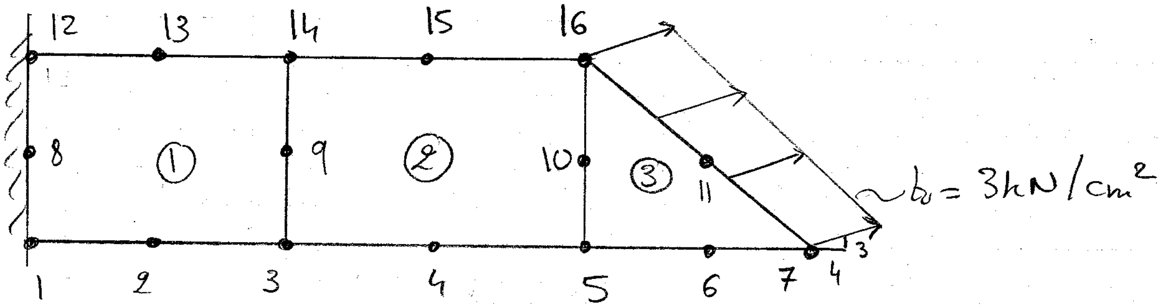


Figure 4.15 Calculation of boundary integrals in plane elasticity.

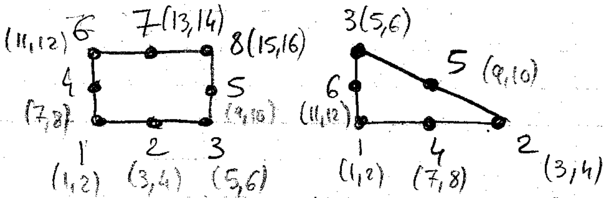
Solution: See the solution in the next 4 pages of notes.

Problem 11.3 p 630

Structure of thickness h.



Local numbering:



Objective: determine the horizontal component of the loads vector at node 16, and the vertical component of the loads vector at node 11: Q_{16x} , Q_{11y}.

$$\left.\begin{aligned} Q_{16x} &= Q_{31} \\ Q_{11y} &= Q_{22} \end{aligned}\right\} \text{(with the global numbering of the d.o.f.}$$

① Assembly:

$$\left\{\begin{aligned} Q_{16x} &= Q^2_{15} + Q^3_5 \\ Q_{11y} &= Q^3_{10} \end{aligned}\right.$$

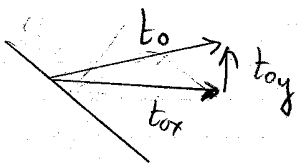
② Elementary loads vectors

$$\{Q^2\} = h \oint_{\Gamma_2} [\Psi]^T \begin{Bmatrix} \hat{t}_x \\ \hat{t}_y \end{Bmatrix} ds$$

$$= \underbrace{h\int_{1-2} [\Psi]^T \begin{Bmatrix} \hat{t}_x \\ \hat{t}_y \end{Bmatrix} ds}_{=0 \text{ (Free surface)}} + \underbrace{h\int_{2-3} [\Psi]^T \begin{Bmatrix} \hat{t}_x \\ \hat{t}_y \end{Bmatrix} ds}_{\substack{\text{traction exists but} \\ \text{vanishes at} \\ \text{assembly:} \\ \text{equilibrium of SV}}} + \underbrace{\int_{3-4} h[\Psi]^T \begin{Bmatrix} \hat{t}_x \\ \hat{t}_y \end{Bmatrix} ds}_{=0 \text{ (Free surface)}} + \underbrace{h\int_{4-1} [\Psi]^T \begin{Bmatrix} \hat{t}_x \\ \hat{t}_y \end{Bmatrix} ds}_{\substack{\text{balanced} \\ \text{SV at} \\ \text{inter-element} \\ \text{boundaries}}}$$

$$\{Q^3\} = \oint_{\Gamma_3} h[\psi]^T \begin{Bmatrix}\bar{t}_x\\ \bar{t}_y\end{Bmatrix} ds$$

$$= h\underbrace{\int_{1-2} [\psi]^T \begin{Bmatrix}\bar{t}_x\\ \bar{t}_y\end{Bmatrix} ds}_{=0 \text{ (free surface)}} + \int_{2-3} h[\psi]^T \begin{Bmatrix}\bar{t}_x\\ \bar{t}_y\end{Bmatrix} ds + \underbrace{\int_{3-1} h[\psi]^T \begin{Bmatrix}\bar{t}_x\\ \bar{t}_y\end{Bmatrix} ds}_{\text{balanced: equilibrium of SV at inter-element boundaries}}$$



$$\frac{t_{oy}}{t_{ox}} = \frac{3}{4} \quad , \quad t_o^2 = t_{ox}^2 + t_{oy}^2$$

$$t_o^2 = \left(1 + \frac{9}{16}\right) t_{ox}^2 = \frac{25}{16} t_{ox}^2 \Rightarrow t_{ox} = \frac{4}{5} t_o$$

$$t_{oy} = \frac{3}{4} \times \frac{4}{5} t_o \Rightarrow t_{oy} = \frac{3}{5} t_o$$

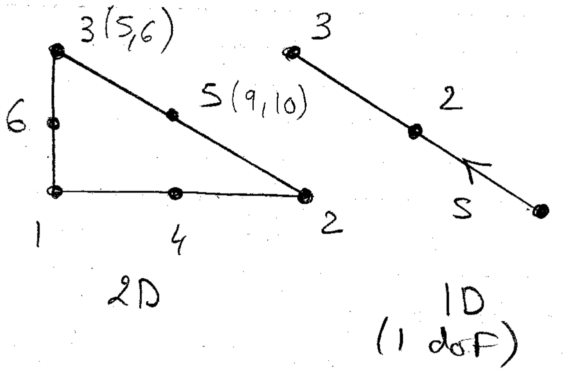
$$\{Q^3\} = \frac{t_o}{5} \int_{2-3} h\, [\psi]^T \begin{Bmatrix}4\\ 3\end{Bmatrix} ds$$

$$\Rightarrow \left| \begin{array}{l} Q_{16x} = Q^3_{\underset{\text{dof}}{\textcircled{5}}} = \left(h \frac{t_o}{5} \int_{2-3} \begin{bmatrix} \psi_3^{(3)}(x,y) & 0 \\ 0 & \psi^{(3)}_{\underset{\text{node}}{\textcircled{3}}}(x,y) \end{bmatrix} \begin{Bmatrix}4\\ 3\end{Bmatrix} ds \right) \cdot \begin{Bmatrix} \overset{\text{x component}}{\textcircled{1}} \\ 0 \end{Bmatrix} \\ Q_{11y} = Q^3_{\underset{\text{dof}}{\textcircled{10}}} = \left(h \frac{t_o}{5} \int_{2-3} \begin{bmatrix} \psi_5^{(3)}(x,y) & 0 \\ 0 & \psi^{(3)}_{\underset{\text{node}}{\textcircled{5}}}(x,y) \end{bmatrix} \begin{Bmatrix}4\\ 3\end{Bmatrix} ds \right) \cdot \begin{Bmatrix} 0 \\ \underset{\text{y component}}{\textcircled{1}} \end{Bmatrix} \end{array} \right.$$

dof 5 = x - component for node 3

dof 10 = y - component for node 5

③ Transformation of 2D interpolation functions into 1D interpolation functions on the sides



$$\varphi_3^{(3)}(x,y)\Big|_{2-3} \equiv \varphi_3^{1D}(s)$$

$$\varphi_3^{1D}(s) = \frac{s}{h_{23}}\left(\frac{2s}{h_{23}}-1\right)$$

$$\varphi_5^{(3)}(x,y)\Big|_{2-3} \equiv \varphi_2^{1D}(s)$$

$$\varphi_2^{1D}(s) = 4\,\frac{s}{h_{23}}\left(1-\frac{s}{h_{23}}\right)$$

④ Conclusion

$$\begin{cases} Q_{16_x} = \dfrac{h\,t_0}{5}\displaystyle\int_0^{h_{2-3}^{(3)}} \begin{Bmatrix} \dfrac{4s}{h_{23}}\left(\dfrac{2s}{h_{23}}-1\right) \\ \dfrac{3s}{h_{23}}\left(\dfrac{2s}{h_{23}}-1\right) \end{Bmatrix} \cdot \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} ds \\ \\ Q_{11_y} = \dfrac{h\,t_0}{5}\displaystyle\int_0^{h_{2-3}^{(3)}} \begin{Bmatrix} \dfrac{16s}{h_{23}}\left(1-\dfrac{s}{h_{23}}\right) \\ \dfrac{12s}{h_{23}}\left(1-\dfrac{s}{h_{23}}\right) \end{Bmatrix} \cdot \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} ds \end{cases}$$

$$\left\{ \begin{aligned} Q_{16x} &= h\frac{t_0}{5} \times \frac{4}{h_{23}^{(3)}} \int_0^{h_{23}^{(3)}} s\left(\frac{2s}{h_{23}} - 1\right) ds \\ Q_{11y} &= h\frac{t_0}{5} \times \frac{12}{h_{23}^{(3)}} \int_0^{h_{23}^{(3)}} s\left(1 - \frac{s}{h_{23}}\right) ds \end{aligned} \right.$$

$$\left\{ \begin{aligned} Q_{16x} &= \frac{4 t_0 h}{5 h_{23}^{(3)}} \left[\frac{2}{3h_{23}} s^3 - \frac{s^2}{2} \right]_0^{h_{23}^{(3)}} \\ Q_{11y} &= \frac{12 t_0 h}{5 h_{23}^{(3)}} \left[\frac{s^2}{2} - \frac{s^3}{3h_{23}} \right]_0^{h_{23}^{(3)}} \end{aligned} \right.$$

$$\left\{ \begin{aligned} Q_{16x} &= \frac{4 t_0 h}{5} \times \frac{1}{h_{23}^{(3)}} \times \left(h_{23}^{(3)}\right)^2 \left(\frac{2}{3} - \frac{1}{2}\right) \Rightarrow \\ Q_{11y} &= \frac{12 t_0 h}{5} \times \frac{1}{h_{23}^{(3)}} \left(h_{23}^{(3)}\right)^2 \left(\frac{1}{2} - \frac{1}{3}\right) \Rightarrow \end{aligned} \right. \boxed{\begin{aligned} Q_{16x} &= \frac{2 t_0 h\, h_{23}^{(3)}}{15} \\ Q_{11y} &= \frac{2 t_0 h\, h_{23}^{(3)}}{5} \end{aligned}}$$

4.20 Homework 3 - Problem 1

The Finite Element Method based on Ritz Method is used to study an insulating wall constituted of three homogeneous layers, each of which being characterized by a thermal conductivity k_i (Fig. 4.16). There is no energy production (no source) in the wall. The outer surfaces are exposed to heat exchanges with the atmosphere. These convective transfers are modeled as boundary conditions on heat flow: $Q = -\beta(T - T_\infty)$. The film coefficient is β_L on the left side, and β_R on the right side (Fig. 4.16). The objective of this problem is to determine the temperature distribution inside the wall by using three one-dimensional linear Finite Elements.

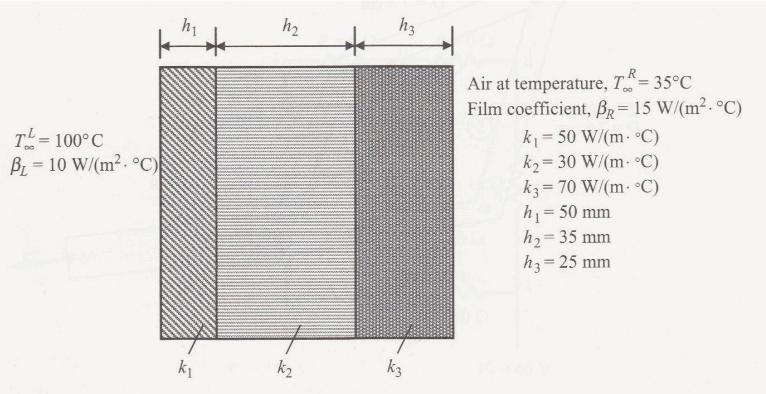


Figure 4.16 Heat Transfer Problem.

1. What is the equation of 1D heat transfer in the most general case? Explain each term of this equation. Explain why in this particular problem, the elementary governing equation reduces to:

$$-\frac{d}{dx}\left(k_i A \frac{dT}{dx}\right) = 0, \qquad i = 1, 2, 3 \tag{4.1}$$

2. Consider an element defined on $]x_a, \, x_b[$. Write the weak formulation for this element, up to the integration by parts.

3. Give the "nodal" conditions (boundary conditions and concentrated loads). Indicate whether the boundary conditions are essential or natural.

4. Provide the elementary stiffness matrices and elementary force vectors. *Provide the general form of the stiffness matrix and force coefficients (using Ψ_j to denote interpolation functions). Then provide the numerical values of the coefficients of the stiffness matrix.*

5. Draw the connectivity table and assemble the elementary equations obtained in question 4.

6. Write the system of condensed equations. Rearrange the system of equations in order to have all unknown variables on the left hand-side of the matrix equation. *Do not solve.*

Solution:

1. In the most general case, the governing equation for 1D heat transfer writes:

$$-\frac{d}{dx}\left(kA\frac{dT}{dx}\right) + \beta P\left(T(x) - T_\infty\right) = Ag(x) \tag{4.2}$$

The first term of equation 4.2 represents conductive flow, the second term represents convective flow (heat exchanges with the atmosphere), and the right-hand side represents the influence of heat sources. There is no heating source to consider in this problem, thus: $g(x) = 0$. There is no convection along the perimeter of the out-of-plane cross-section of the wall. Convection is included in the boundary conditions because heat exchanges with the atmosphere are only possible at the left and right faces of the wall. Therefore, the governing equation reduces to a heat conduction equation:

$$-\frac{d}{dx}\left(k_i A\frac{dT}{dx}\right) = 0, \qquad i = 1, 2, 3 \tag{4.3}$$

2. Weighted Integral Statement:

$$\forall w \simeq \delta T, \qquad -\int_{x_a}^{x_b} w(x)\frac{d}{dx}\left(k_i A\frac{dT}{dx}\right)dx = 0, \qquad i = 1, 2, 3 \tag{4.4}$$

Integration by Parts:

$$\forall w \simeq \delta T, \qquad \int_{x_a}^{x_b} \frac{dw}{dx}k_i A\frac{dT}{dx}dx = \left[w(x)k_i A\frac{dT}{dx}\right]_{x_a}^{x_b}, \qquad i = 1, 2, 3 \tag{4.5}$$

3. The primary variable is temperature. The secondary variable is heat flow, defined as: $Q^i = k_i A\frac{dT}{dx}$ for element i. At internal nodes, no concentrated load is applied: $Q_2 = Q_3 = 0$. Two natural boundary conditions are applied, on the left and right sides of the wall. Two convective flows are imposed: $Q(0) = Q_L = -\beta_L(T(0) - T_\infty^L)$ and $Q(h_1 + h_2 + h_3) = Q_R = -\beta_R(T(h_1 + h_2 + h_3) - T_\infty^R)$. In terms of nodal conditions: $Q_1 = -\beta_L(T_1 - T_\infty)$ and $Q_4 = -\beta_R(T_4 - T_\infty)$.
Note: Actually, the boundary conditions mix imposed values for heat flux ($Q(0)$, $Q(h_1 + h_2 + h_3)$) and imposed values of temperature ($T(0)$, $T(h_1 + h_2 + h_3)$). Such boundary conditions are called "mixed boundary conditions".

4. Each layer is modeled by a one-dimensional linear Finite Element. In a local coordinate system, the interpolation functions are expressed as:

$$\begin{cases} \Psi_1^e(\overline{x}) = \left(1 - \frac{\overline{x}}{h_e}\right) \\ \\ \Psi_2^e(\overline{x}) = \frac{\overline{x}}{h_e} \end{cases} \tag{4.6}$$

in which h_e is the length of the element. Here, the elements have different lengths: $h_1 \neq h_2$, $h_2 \neq h_3$ and $h_3 \neq h_1$. The weak formulation 4.5 provides the typical elementary equation which has to be used in this Finite Element model:

$$\forall k = 1, 2, \quad \left[\int_0^{h_i} k_i A \frac{d\Psi_k}{d\overline{x}} \frac{d\Psi_j}{d\overline{x}} d\overline{x} \right] T_j = Q_k^i, \qquad i = 1, 2, 3 \tag{4.7}$$

with $Q_k^i = +/- \left[\Psi_k(x) k_i A \frac{dT}{dx} \right]$ (the sign depends on the node considered after the integration by parts). There is no contribution of a volumetric force: $\{F^i\} = \{0\}$ for i=1,2,3. With the interpolation functions used in this problem (equation 4.6), and with A and k_i parameters being constant, the typical elementary stiffness matrix turns to be:

$$[K^i] = \frac{k_i\, A}{h_i} \begin{bmatrix} 1 & -1 \\ & \\ -1 & 1 \end{bmatrix}, \qquad i = 1, 2, 3 \tag{4.8}$$

5. The connectivity table for this Finite Element model is provided in Tab. 4.1. Assembling the elementary equations provides the global matrix equation of this problem:

Table 4.1 Connectivity Table for Problem 1.

Element	T_1	T_2
1	1	2
2	2	3
3	3	4

$$A \begin{bmatrix} k_1/h_1 & -k_1/h_1 & 0 & 0 \\ -k_1/h_1 & k_1/h_1 + k_2/h_2 & -k_2/h_2 & 0 \\ 0 & -k_2/h_2 & k_2/h_2 + k_3/h_3 & -k_3/h_3 \\ 0 & 0 & -k_3/h_3 & k_3/h_3 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 \end{Bmatrix} \tag{4.9}$$

6. The "nodal" conditions are the following (see question 3): $Q_1 = -\beta_L(T_1 - T_\infty)$, $Q_2 = Q_3 = 0$, $Q_4 = -\beta_R(T_4 - T_\infty)$. There is no boundary condition imposed on the primary variable. Therefore, the system of equations cannot be condensed. After

introducing the boundary conditions, the FE equation 4.9 writes:

$$A\begin{bmatrix} k_1/h_1+\beta_L/A & -k_1/h_1 & 0 & 0 \\ -k_1/h_1 & k_1/h_1+k_2/h_2 & -k_2/h_2 & 0 \\ 0 & -k_2/h_2 & k_2/h_2+k_3/h_3 & -k_3/h_3 \\ 0 & 0 & -k_3/h_3 & k_3/h_3+\beta_R/A \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}=\begin{Bmatrix} -\beta_L T_\infty \\ 0 \\ 0 \\ -\beta_R T_\infty \end{Bmatrix} \tag{4.10}$$

4.21 Homework 3 - Problem 2

A field problem is governed by the following differential equation:

$$-\nabla^2 u(x,y) = f_0$$

The problem is solved with a linear triangular element. The nodal values of the dependent variable are the following:

$$u_1 = 389.79, \qquad u_2 = 337.19, \qquad u_3 = 395.08$$

The interpolation functions of the element are given by:

$$\begin{aligned} \Psi_1(x,y) &= \frac{1}{8.25}(12.25 - 2.5x - 1.5y) \\ \Psi_2(x,y) &= \frac{1}{8.25}(-1.5 + 3x - 1.5y) \\ \Psi_3(x,y) &= \frac{1}{8.25}(-2.5 - 0.5x + 3y) \end{aligned}$$

The flux vector is approximated as:

$$\mathbf{q}(x,y) = \nabla\left(u_{FEM}(x,y)\right) = \frac{\partial u_{FEM}}{\partial x}\mathbf{i} + \frac{\partial u_{FEM}}{\partial y}\mathbf{j}$$

1. Find the component of the flux in the direction of the vector $4\mathbf{e_x} + 3\mathbf{e_y}$ at ($x = 3$, $y = 2$).
2. A point source of magnitude Q_0 is located at point (x_0, y_0) = (3,2) inside the triangular element. Determine the contribution of the point source to the element source vector. Express your answer in terms of Q_0.

Solution:

1. The flux vector is approximated as:

$$\mathbf{q}(x,y) = \nabla\left(u_{FEM}(x,y)\right) = \frac{\partial u_{FEM}}{\partial x}\mathbf{i} + \frac{\partial u_{FEM}}{\partial y}\mathbf{j}$$

which can be expressed as:

$$\mathbf{q}(x,y) \;=\; \sum_{k=1}^{3}\left[u_k\frac{\partial\Psi_k}{\partial x}\mathbf{i} + u_k\frac{\partial\Psi_k}{\partial y}\mathbf{j}\right]$$

Using the expressions of the interpolation functions and of the nodal values provided in the problem:

$$\mathbf{q}(x,y) \;=\; 389.79\left(\frac{-2.5}{8.25}\mathbf{i} - \frac{1.5}{8.25}\mathbf{j}\right) + 337.19\left(\frac{3}{8.25}\mathbf{i} - \frac{1.5}{8.25}\mathbf{j}\right) + 395.08\left(\frac{-0.5}{8.25}\mathbf{i} + \frac{3}{8.25}\mathbf{j}\right)$$

We find:

$$\mathbf{q}(x,y) \;=\; -19.45\mathbf{i} + 11.49\mathbf{j}$$

We notice that the expression of the flux vector does not depend on the position (x,y) in the element. The component of the flux vector on $4\mathbf{i} + 3\mathbf{j}$ is obtained by projection. We find: $\mathbf{q}\cdot(4\mathbf{i}+3\mathbf{j}) = -43.33$.

2. A point source Q_0 at (x_0, y_0) can be expressed as a distributed flux $Q_0\delta(x - x_0, y - y_0)$, in which $\delta(x', y')$ represents here the Dirac function (equals one when $x' = y' = 0$, and zero otherwise). According to the weak formulation, the force terms are obtained by integration, as follows:

$$F_i = \int_{\Omega_e}\Psi_i(x,y)Q_0\delta(x-x_0, y-y_0)dxdy = \Psi_i(x_0,y_0)Q_0$$

With $x_0 = 3$ and $y_0 = 2$, from the expressions of we find: $F_1 = 1.75\,Q_0/8.25$, $F_2 = 4.5\,Q_0/8.25$, $F_3 = 2\,Q_0/8.25$.

CHAPTER 5

ELEMENTS OF PORO-ELASTICITY

PROBLEMS

5.1 [FEM for 1D steady fluid flow] Consider the steady laminar flow of a viscous incompressible fluid with constant density in a long annular region between two coaxial cylinders of radii R_i and R_0 (see Figure 5.1). The differential equation for this case is given by:

$$-\frac{1}{r}\frac{d}{dr}\left(r\,\mu\,\frac{dw}{dr}\right) = f_0, \quad f_0 = \frac{P_1 - P_2}{L}$$

where w is the velocity along the cylinders (i.e., the z component of velocity), μ is the viscosity, L is the length of the region along the cylinders in which the flow is fully developed, and P_1 and P_2 are the pressures at $z = 0$ and $z = L$, respectively (P_1 and P_2 represent the combined effect of static pressure and gravitational force). The boundary conditions are:

$$w(r = R_0) = w(r = R_i) = 0$$

1. Write the weak formulation of the problem.
2. Consider two linear elements over the segment $[R_0,\ R_i]$. Write the two elementary equations that govern the problem, using Ritz method.
3. Assemble and condense the system of equations.
4. Solve the system of finite element equations for the primary variable and write the expression of approximate solution over the segment $[R_0,\ R_i]$.

5. Post-process the finite element results to calculate the unknown nodal secondary variables.
6. Repeat questions 2 and 3 with one quadratic element. Do not solve.

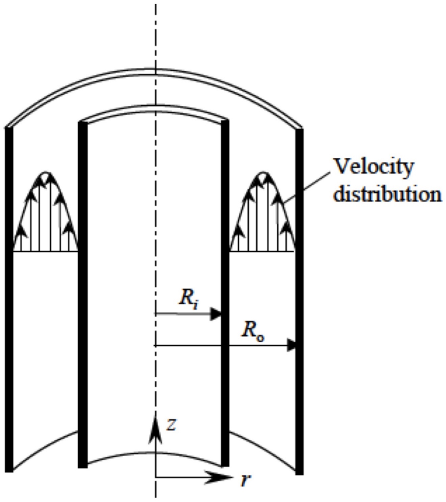


Figure 5.1 Viscous incompressible fluid flow in an annular area.

Solution:

1. **Weak formulation:**

$$\forall u \sim \delta w, \quad -2\pi \int_{r_a}^{r_b} u(r) \frac{1}{r} \frac{d}{dr} \left(r\, \mu \frac{dw}{dr} \right) r\, dr \;=\; 2\pi \int_{r_a}^{r_b} u(r) f_0\, r\, dr$$

$$\forall u \sim \delta w, \quad \int_{r_a}^{r_b} \frac{du}{dr} \left(r\, \mu \frac{dw}{dr} \right) dr \;=\; \int_{r_a}^{r_b} u(r) f_0\, r\, dr \;+ \left[u(r) \left(r\, \mu \frac{dw}{dr} \right) \right]_{r_a}^{r_b}$$

Secondary variable:

$$q(r) \;=\; r\, \mu \frac{dw}{dr}$$

2. **Elementary equations:**
For each element defined on the segment $[r_a, r_b]$:

$$\psi_1(r) = \frac{r_b - r}{r_b - r_a}, \quad \psi_2(r) = \frac{r - r_a}{r_b - r_a}$$

$$\forall i = 1, 2, \quad \sum_{j=1}^{2} w_j \int_{r_a}^{r_b} \frac{d\psi_i}{dr} \left(r\, \mu \frac{d\psi_j}{dr} \right) dr \;=\; \int_{r_a}^{r_b} \psi_i(r) f_0\, r\, dr \;+ [\psi_i(r) q(r)]_{r_a}^{r_b}$$

$$[K^e] \;=\; \frac{\mu\,(r_a + r_b)}{2\,(r_b - r_a)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F^e\} = \frac{f_0}{6}\left\{\begin{array}{l} -2(r_a)^2 + (r_b)^2 + r_a r_b \\ -(r_a)^2 + 2(r_b)^2 - r_a r_b \end{array}\right\}$$

$$\{Q^e\} = \left\{\begin{array}{l} -q(r_a) \\ +q(r_b) \end{array}\right\}$$

For the two elements in this problem:

$$[K^{(1)}] = \frac{\mu\,(3R_i + R_0)}{2\,(R_0 - R_i)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K^{(2)}] = \frac{\mu\,(R_i + 3R_0)}{2\,(R_0 - R_i)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

3. **Assembly:**

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} \end{bmatrix}\left\{\begin{array}{l} w_1 \\ w_2 \\ w_3 \end{array}\right\} = \left\{\begin{array}{l} Q_1^{(1)} \\ Q_2^{(1)} + Q_1^{(2)} \\ Q_2^{(2)} \end{array}\right\} + \left\{\begin{array}{l} F_1^{(1)} \\ F_2^{(1)} + F_1^{(2)} \\ F_2^{(2)} \end{array}\right\}$$

After some computation...

$$\tfrac{\mu}{2(R_0-R_i)}\begin{bmatrix} (3R_i + R_0) & -(3R_i + R_0) & 0 \\ -(3R_i + R_0) & 4(R_i + R_0) & -(R_i + 3R_0) \\ 0 & -(R_i + 3R_0) & (R_i + 3R_0) \end{bmatrix}\left\{\begin{array}{l} w_1 \\ w_2 \\ w_3 \end{array}\right\}$$

$$= \left\{\begin{array}{l} Q_1 \\ Q_2 \\ Q_3 \end{array}\right\} + \tfrac{f_0 \times (R_0 - R_i)}{24}\left\{\begin{array}{l} 5R_i + R_0 \\ 6(R_i + R_0) \\ R_i + 5R_0 \end{array}\right\}$$

The nodal conditions are $w_1 = w_3 = 0$, $Q_2 = 0$. The condensation process yields one equation for one variable (w_2):

$$K_{22}\,w_2 = F_2$$

$$\frac{2\mu(R_i + R_0)}{R_0 - R_i}\,w_2 = \frac{f_0 \times (R_0 - R_i)(R_i + R_0)}{4}$$

4. **Resolution and approximation:**
We solve for w_2:

$$w_2 = \frac{f_0 \times (R_0 - R_i)^2}{8\mu}$$

Approximation for each element:

$$w^e(r) \simeq w_1^e \psi_1^e(r) + w_2^e \psi_2^e(r)$$

Specifically:

$$\forall r \in [R_i,\,(R_0 + R_i)/2], \quad w(r) \simeq w_2 \frac{R_0 + R_i - 2r}{R_0 - R_i}$$

$$\forall r \in [(R_i + R_0)/2,\ R_0], \quad w(r) \simeq w_2 \frac{2r - (R_0 + R_i)}{R_0 - R_i}$$

5. **Post-processing:**
   We approximate Q_1 and Q_3:

$$Q_1 = Q_1^{(1)} = \left(r\mu \frac{dw}{dr} \right)_{r=r_a^{(1)}}$$

$$Q_1 \simeq -\frac{2\mu w_2 R_i}{R_0 - R_i} = -\frac{f_0 R_i (R_0 - R_i)}{4}$$

$$Q_3 = Q_2^{(2)} = \left(r\mu \frac{dw}{dr} \right)_{r=r_b^{(2)}}$$

$$Q_2 \simeq \frac{2\mu w_2 R_i}{R_0 - R_i} = \frac{f_0 R_i (R_0 - R_i)}{4}$$

6. **One quadratic element:**

$$\frac{\pi\mu}{3L} \begin{bmatrix} 3R_0 + 11R_i & -4R_0 - 12R_i & R_0 + R_i \\ -4R_0 - 12R_i & 16(R_0 + R_i) & -12R_0 - 4R_i \\ R_0 + R_i & -12R_0 - 4R_i & 11R_0 + 3R_i \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$

$$= \frac{\pi f_0 L}{3} \begin{Bmatrix} R_i \\ 2R_i + 2R_0 \\ R_0 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \end{Bmatrix}$$

5.2 [FEM for 1D steady fluid flow] Consider the irrotational flow of an ideal fluid about a circular cylinder with its axis perpendicular to the plane of flow, which takes place between two long horizontal walls (see figure 5.2). The governing equation is:

$$-\nabla^2 u = 0 \tag{5.1}$$

Is it more computationally efficient to model the flow problem with the stream function ($u = \Psi$) or the flow potential ($u = \Phi$)?

Solution: See the next four pages of notes.

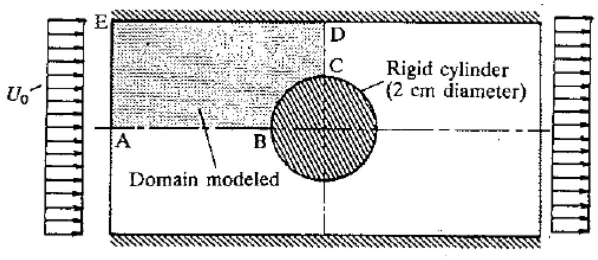


Figure 5.2 Flow around a non-penetrable obstacle in 2D. Image taken from (Reddy, 2004).

$$F_8 = \int_{\Omega_5} \varphi_1^{(5)}(x,y)\, Q_2\, \delta(x-x_{P_2}, y-y_{P_2})\, dxdy$$

$$+ \int_{\Omega_{10}} \varphi_2^{(10)}(x,y)\, Q_1\, \delta(x-x_{P_1}, y-y_{P_1})\, dxdy$$

Step 5: Resolution / Condensation

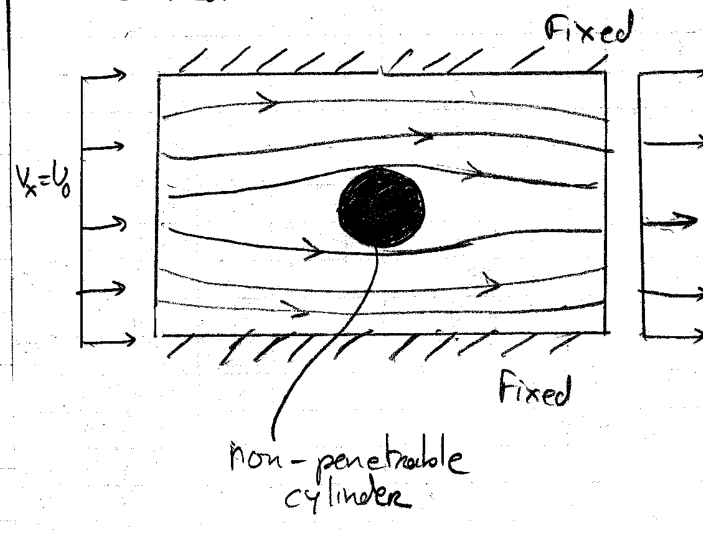
6 essential b.c.

total number of d.o.F. : (14 nodes) × (1 dof/node) = 14

⇒ after condensation, we should obtain a system of 8 equations, to solve for $\emptyset_2, \emptyset_3, \emptyset_4, \emptyset_7, \emptyset_8, \emptyset_9, \emptyset_{12}, \emptyset_{13}$.

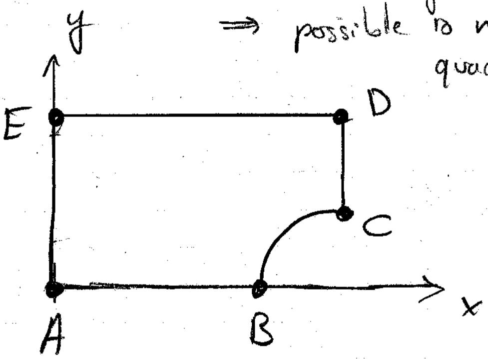
Ex. 8.5.5 p 479

Confined Flow Around a Circular Cylinder:



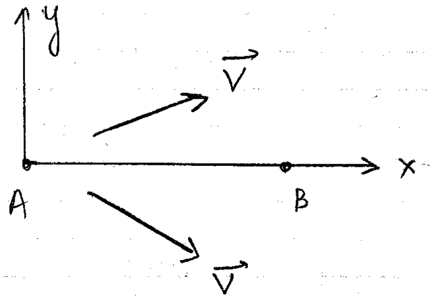
Symmetries: about x & y axes
- symmetric in y-direction
- anti-symmetric in x-direction

⇒ possible to model one quadrant of the domain only



Boundary Conditions:

- AB:



$v_x(x,y) = v_x(x,-y)$ symmetric

$v_y(x,y) = -v_y(x,-y)$ anti-symmetric

At $y=0$ (on the horizontal axis):

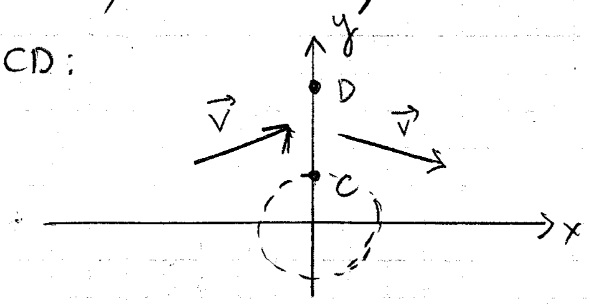
$$v_y(x, y=0) = 0 \Longleftarrow \begin{cases} v_y(x, y=0^-) = -v_y(x, y=0^+) \\ \text{(anti-symmetric)} \\ v_y(x, y=0^-) = v_y(x, y=0^+) \\ \text{continuity of the velocity field} \end{cases}$$

$$\Downarrow$$

$$\boxed{v_y = 0 \text{ on } AB}$$

- BC: the cylinder cannot be penetrated by the fluid, i.e. $\boxed{v_n = 0 \text{ on } BC}$ (no fluid velocity normal to the boundary BC)

- CD:



$v_x(x,y') = v_x(-x,y')$ symmetric

$v_y(x,y') = -v_y(-x,y')$ anti-symmetric

At $x=0$ (on the vertical axis)

$$\begin{cases} v_y(0^-, y') = -v_y(0^+, y') \text{ - antisymmetric} \\ v_y(0^-, y') = v_y(0^+, y') \text{ - continuous} \end{cases}$$

$\Rightarrow v_y(x=0, y') = 0$

$\Rightarrow \boxed{v_y = 0 \text{ on } CD}$

- DE: Fixed wall: fluid flow is impossible beyond the wall $\Rightarrow$ non-penetration condition $\Rightarrow v_n = 0$ (no fluid velocity normal to the wall) with $n = y$ $\Rightarrow \boxed{v_y = 0 \text{ on } DE}$

- EA: Fluid velocity is specified: $\boxed{v_x = U_0,\ v_y = 0 \text{ on } EA}$

Summary

	v_x, v_y	$\Psi(x,y)$	$\phi(x,y)$
AB	$v_y = 0$	$\partial\Psi/\partial x = 0$, $n = -y$	$\partial\phi/\partial y = 0$, $n = -y$
BC	$v_n = 0$	$\partial\Psi/\partial s = 0$	$\partial\phi/\partial n = 0$
CD	$v_y = 0$	$\partial\Psi/\partial x = 0$, $n = x$	$\partial\phi/\partial y = 0$, $n = x$
DE	$v_y = 0$	$\partial\Psi/\partial x = 0$, $n = y$	$\partial\phi/\partial y = 0$, $n = y$
EA	$v_x = U_0$, $v_y = 0$	$\partial\Psi/\partial x = 0$, $\partial\Psi/\partial y = U_0$, $n = -x$	$\partial\phi/\partial x = -U_0$, $\partial\phi/\partial y = 0$, $n = -x$

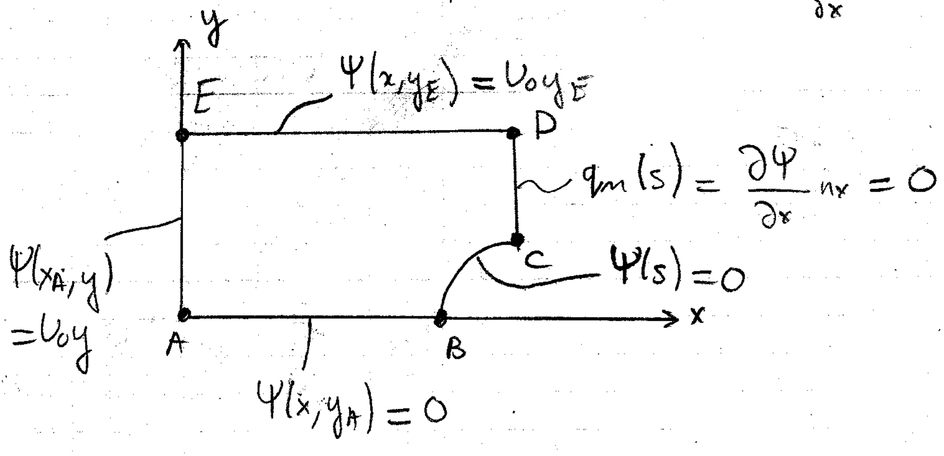
Stream Function Formulation

- on EA: $\Psi(x,y) = U_0 y + \Psi(0,0)$

  solution defined ± a constant

  $\Rightarrow$ by convention, one can take $\Psi(0,0) = 0$

  $\Rightarrow \Psi(x,y) = U_0 y$ on EA, and $\Psi(x_A, y_A) = 0$, $\Psi(x_E, y_E) = U_0 y_E$.
- on DE: $\partial\Psi/\partial x = 0 \Rightarrow \Psi(x,y) = \Psi(x, y_E) = \Psi(x_E, y_E) = U_0 y_E$
- on AB: $\partial\Psi/\partial x = 0 \Rightarrow \Psi(x,y) = \Psi(x, y_A) = \Psi(x_A, y_A) = 0$
- on BC: $\partial\Psi/\partial s = 0 \Rightarrow d\Psi/ds = 0 \Rightarrow \Psi(s) = \Psi(s_B) = \Psi(x_B, y_B)$
  $= \Psi(x_A, y_A) = 0$
- on CD: $\partial\Psi/\partial x = 0$, with $n = x \Rightarrow \frac{\partial\Psi}{\partial x} n_x = q_n = 0$



- DE, EA, AB, BC: essential bc
- CD: natural b.c.

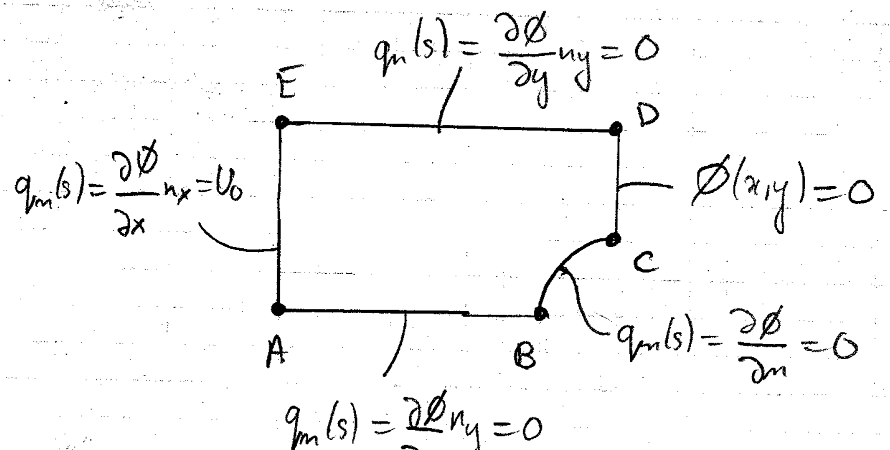
Velocity Potential Formulation

- on CD: $\phi(x,y) = \phi(x=x_c, y) = \text{constant} = \phi(x_c, y_c)$
  Solution defined ± a constant ⇒ by convention, one can take
  $\phi(x_c,y_c) = 0 \Rightarrow \phi(x,y) = 0$ on CD
- on DE: $\partial\phi/\partial y = 0$, with $n=y \Rightarrow \frac{\partial\phi}{\partial y} n_y = q_n(s) = 0$
- on BC: $\partial\phi/\partial n = 0 \rightarrow q_n(s) = 0$
- on AB: $\partial\phi/\partial y = 0$ with $n=y \Rightarrow \frac{\partial\phi}{\partial y} n_y = q_n(s) = 0$
- on EA: $\partial\phi/\partial y = 0 \Rightarrow \phi(x,y) = \phi(x_A,y) = \text{constant} = \underbrace{\phi(x_A,y_A)}_{\text{unknown}}$

  cannot be set to zero since we already set the reference at $\phi(x_c,y_c)=0$.

  On the other hand: $\partial\phi/\partial x = -U_0$ with $n=-x$

  $\Rightarrow q_n(s) = \frac{\partial\phi}{\partial x} n_x = U_0$



- DE, EA, AB, BC: natural bc
- CD: essential bc

} ⇒ more advantageous to resort to the stream function, to have more essential bc, and thus, a more condensed system of equations

5.3 [FEM for 1D steady fluid flow] Consider the groundwater flow problem governed by the following equation:

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial \Phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_{22}\frac{\partial \Phi}{\partial y}\right) = f(x, y) \tag{5.2}$$

Two pumps are used to extract the water brought by a river, modeled as a lineic fluid source. The boundary and loading conditions are shown in the Figure 5.3. Propose a FEM model made of linear triangular elements to approximate the distribution of water fluxes.

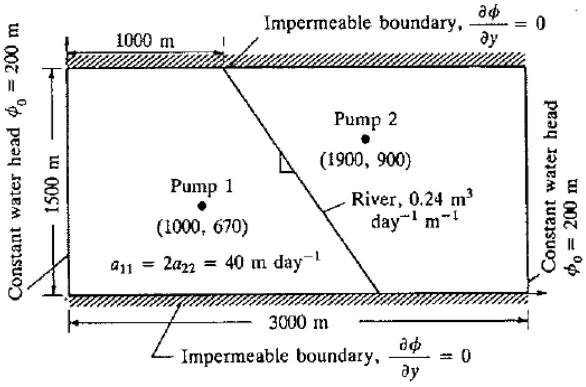
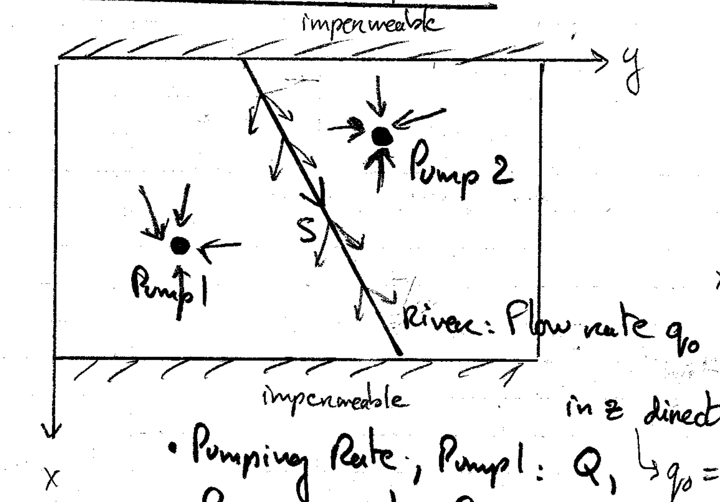


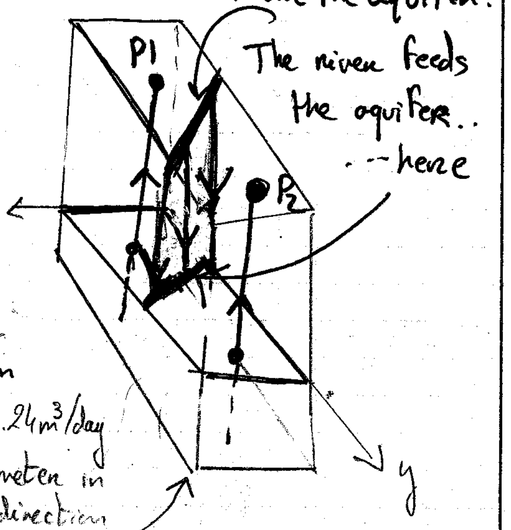
Figure 5.3 Seepage problem in 2D. Image taken from (Reddy, 2004).

Solution: See the next six pages of notes.

Ex. 8.5.4 p 474

Top View of an aquifer





The pumps "remove" some fluid from the aquifer.

The river feeds the aquifer.. ...here

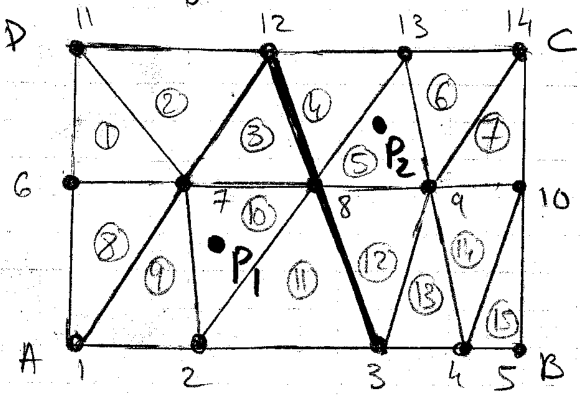
in z direction ↳ $q_0 = 0.24 m^3/day$ per meter in z direction

- Pumping Rate, Pump 1: Q_1
- Pumping Rate, Pump 2: Q_2

Step 1: Meshing:

"Follow" the river to apply q_0 at inter-element boundaries

⟹ triangular elements seem a good choice.



- Ex of rough mesh, with linear triangular elements
- P_1 and P_2 do not have to be nodes of the mesh

Step 2: FE Model

Governing Equation: $-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_{22}\frac{\partial \phi}{\partial y}\right) = f(x,y) \text{ in } \Omega$

Boundary Conditions: • AB, CD: impermeable → velocity ⊥ AB, CD is zero

ie $V_x = 0$

$\rightarrow \frac{\partial \phi}{\partial x}\underbrace{n_x}_{=1} = 0$

(natural bc)

Note: a_{11} can be ≠ from a_{22} if $k_x \neq k_z$ (permeability in x & z directions)

- BC, DA: assumption: constant water head

$\Rightarrow \vec{v}$ // boundary $\doteq 0$

$\Rightarrow v_x = 0$

$\Rightarrow \frac{\partial \phi}{\partial x} = 0$, ie for y fixed (at BC or DA):

$\phi = \phi_0 =$ constant (essential bc)

$$[K^e]\{\phi^e\} = \{F^e\} + \{Q^e\}$$

$$K_{ij}^e = \int_{\Omega^e} \left(a_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + a_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy$$

$$F_i^e = \int_{\Omega^e} \psi_i(x,y)\, f(x,y)\, dxdy$$

$$Q_i^e = \oint_{\Gamma_e} \psi_i(s)\, q_n(s)\, ds \quad \text{with } q_n(s) = \frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y$$

Step 3: Assembling (same process as usual; 3dof/elt)

Step 4: Boundary Conditions and Applied Loads

- consider essential bc first, to have more rows of the assembled system of equations to condense:

→ for the nodes on BC and DA, $\phi_i = \phi_0$:

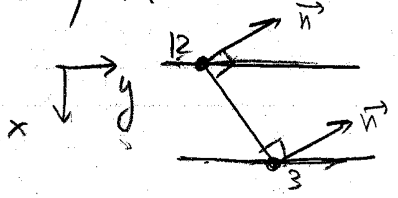
$$\phi_1 = \phi_5 = \phi_6 = \phi_{10} = \phi_{11} = \phi_{14} = \phi_0$$

- natural boundary conditions:

over AB and CD, $Q_i = 0$.

$$Q_2 = Q_4 = Q_{13} = 0$$

⚠ at nodes 3 and 12, $Q \neq 0$ because of the river flow



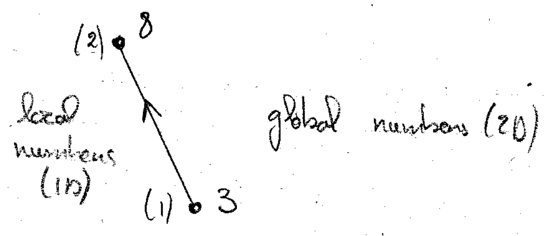
At 3 and 12,
Flow ⊥ x = 0
but Flow // $\vec{n}$ is $\neq 0$

- equilibrium of the S.V. (internal nodes)

$Q_7 = Q_9 = 0$ (no flow applied)

Q_3; Q_8; Q_{12} : influence of the river infiltration rate

$$Q_3^{\text{internal}} = Q_3^{\text{external}} = \int_0^{h_{3-8}} \varphi_1^{1D}(s)\, q_0\, ds$$



$$Q_3 = \int_0^{h_{3-8}} \left(1 - \frac{s}{h_{3-8}}\right) q_0\, ds$$

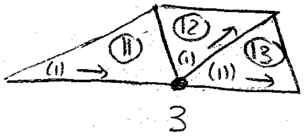
$$Q_3 = \left[-\frac{h_{3-8}}{2} \left(1 - \frac{s}{h_{3-8}}\right)^2 q_0 \right]_0^{h_{3-8}} = \frac{q_0\, h_{3-8}}{2}$$

In the same way:

$$\begin{cases} Q_8 = \dfrac{q_0\, h_{3-8}}{2} + \dfrac{q_0\, h_{8-12}}{2} \\ Q_{12} = q_0 \dfrac{h_{8-12}}{2} \end{cases}$$

$$\Rightarrow Q_3 + Q_8 + Q_{12} = q_0 \left(h_{3-8} + h_{8-12}\right) = Q^{ext}_{total}$$

$\boxed{\text{NOTE}}$ $Q_3 = Q_2^{11} + Q_1^{12} + Q_1^{13}$



$$Q_3 = \oint_{\Gamma_{11}} \varphi_2^{2D}(x(s), y(s))\, q_n(s)\, ds + \oint_{\Gamma_{12}} \varphi_1^{2D}(x(s), y(s))\, q_n(s)\, ds + \oint_{\Gamma_{13}} \varphi_1^{2D}(x(s), y(s))\, q_n(s)\, ds$$

$$Q_3 = \underbrace{\int_0^{h^{11}_{2-3}} \varphi_1^{1D}(s)\, q_n(s)\, ds}_{\text{elt } 11} + \underbrace{\int_0^{h^{12}_{23}} \varphi_2^{1D}(s)\, q_n(s)\, ds}_{\text{elt } 12} + \underbrace{0}_{\text{elt 13 (no load)}}$$

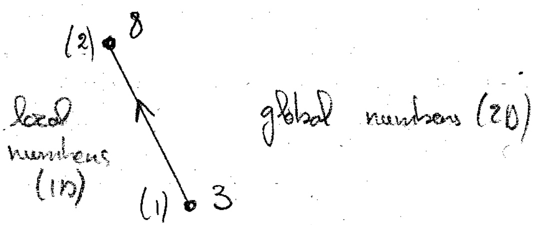
with $h^{11}_{23} = h^{12}_{23} = h_{3-8}$

$$Q_3 = \int_0^{h_{3-8}} \varphi_1^{1D}(s)\, q_n(s)\, ds + \int_0^{h_{38}} \varphi_2^{1D}(s)\, q_n(s)\, ds$$

$$Q_3 = \int_0^{h_{3-8}} q_n(s) \left(\varphi_1^{1D} + \varphi_2^{1D}\right) ds$$

Q_3; Q_8; Q_{12} : influence of ~~the~~ the river infiltration rate

$$Q_3^{\text{internal}} = Q_3^{\text{external}} = \int_0^{h_{3-8}} \varphi_1^{1D}(s)\, q_0\, ds$$



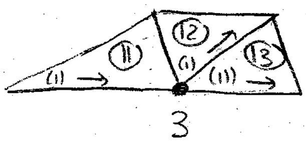
$$Q_3 = \int_0^{h_{3-8}} \left(1 - \frac{s}{h_{3-8}}\right) q_0\, ds$$

$$Q_3 = \left[\frac{-h_{3-8}}{2} \left(1 - \frac{s}{h_{3-8}}\right)^2 q_0 \right]_0^{h_{3-8}} = \frac{q_0\, h_{3-8}}{2}$$

In the same way: $\left\{ \begin{array}{l} Q_8 = \dfrac{q_0\, h_{3-8}}{2} + \dfrac{q_0 h_{8-12}}{2} \\ Q_{12} = q_0\, \dfrac{h_{8-12}}{2} \end{array} \right.$

$$\Rightarrow Q_3 + Q_8 + Q_{12} = q_0 \left(h_{3-8} + h_{8-12}\right) = Q_{\text{total}}^{\text{ext}}$$

$\boxed{\text{NOTE}}$ $Q_3 = Q_2^{11} + Q_1^{12} + Q_1^{13}$



$$Q_3 = \oint_{\Gamma_{11}} \varphi_2^{2D}(x(s), y(s))\, q_n(s)\, ds + \oint_{\Gamma_{12}} \varphi_1^{2D}(x(s), y(s))\, q_n(s)\, ds + \oint_{\Gamma_{13}} \varphi_1^{2D}(x(s), y(s))\, q_n(s)\, ds$$

$$Q_3 = \underbrace{\int_0^{h_{2-3}^{11}} \varphi_1^{1D}(s)\, q_n(s)\, ds}_{\text{elt } 11} + \underbrace{\int_0^{h_{23}^{12}} \varphi_2^{1D}(s)\, q_n(s)\, ds}_{\text{elt } 12} + \underbrace{0}_{\text{elt } 13 \text{ (no load)}}$$

with $h_{23}^{11} = h_{23}^{12} = h_{3-8}$

$$Q_3 = \int_0^{h_{3-8}} \varphi_1^{1D}(s)\, q_n(s)\, ds + \int_0^{h_{38}} \varphi_2^{1D}(s)\, q_n(s)\, ds$$

$$Q_3 = \int_0^{h_{3-8}} q_n(s) \underbrace{\left(\varphi_1^{1D} + \varphi_2^{1D}\right)}_{=1 \text{ (partition of unity)}} ds$$

$$Q_3 = \int_0^{h_{3-8}} q_n(s)\, ds = \frac{h_{38}}{2} q_0 \Rightarrow \boxed{q_n(s) = \frac{q_0}{2}}$$

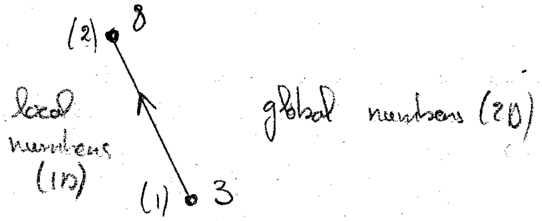
($\frac{1}{2}$ load taken by each elt)

- equilibrium of the S.V. (internal nodes)

$Q_7 = Q_9 = 0$ (no flow applied)

Q_3; Q_8; Q_{12} : influence of the river infiltration rate

$$Q_3^{internal} = Q_3^{external} = \int_0^{h_{3-8}} \varphi_1^{10}(s)\, q_0\, ds$$



$$Q_3 = \int_0^{h_{3-8}} \left(1 - \frac{s}{h_{3-8}}\right) q_0\, ds$$

- "Flow loading" due to the pumps

"concentrated loads" located in elements 5 (P_2) and 10 (P_1)

Element 5 → nodes 8, 9, 13

Element 10 → nodes 2, 8, 7

$$F_2 = \int_{\Omega_{10}} \varphi_1^{(10)}(x,y)\, Q_1\, \delta(x - x_{P_1}, y - y_{P_1})\, dx dy$$

same approach for F_7, F_9, F_{13}

$$F_8 = \int_{\Omega_5} \varphi_1^{(5)}(x,y)\, Q_2\, \delta(x-x_{P_2}, y-y_{P_2})\, dxdy$$

$$+ \int_{\Omega_{10}} \varphi_2^{(10)}(x,y)\, Q_1\, \delta(x-x_{P_1}, y-y_{P_1})\, dxdy$$

Step 5: Resolution / Condensation

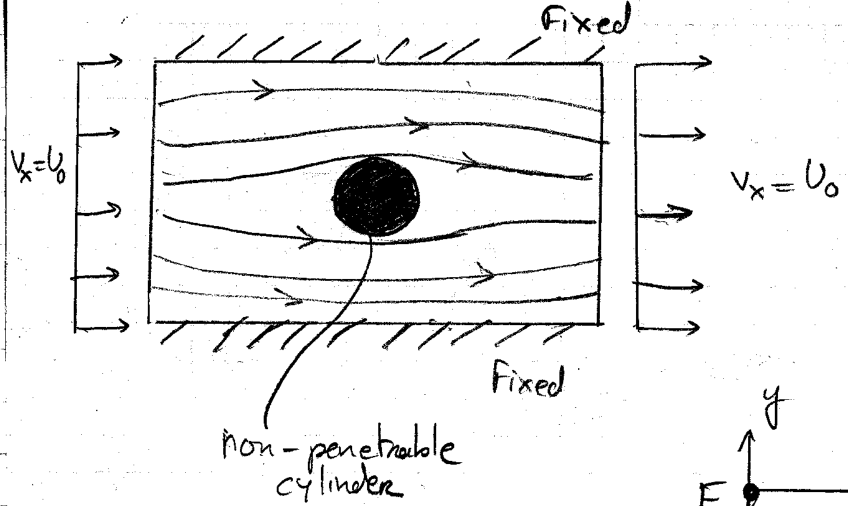
6 essential b.c.

total number of d.o.F. : (14 nodes) × (1 dof/node) = 14

⇒ after condensation, we should obtain a system of 8 equations, to solve for $\phi_2, \phi_3, \phi_4, \phi_7, \phi_8, \phi_9, \phi_{12}, \phi_{13}$.

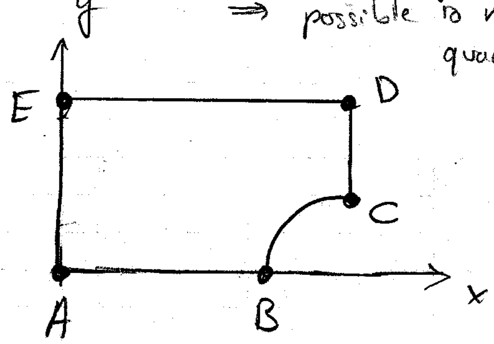
Ex. 8.5.5 p 479

Confined Flow Around a Circular Cylinder:



Symmetries: about x & y axes
- symmetric in y-direction
- anti-symmetric in x-direction

⇒ possible to model one quadrant of the domain only



5.4 To account for a percolation porosity, the following modified Kozeny-Carman relation was established:

$$K = B\frac{(\phi - \phi_c)^3}{(1+\phi_c-\phi)^2}d^2$$

1. Calculate the permeability of a sandstone sample which has porosity 0.32 and an average grain size of 100 μm. Assume $B = 15$ and $\phi_c = 0.035$.
2. Compare the permeabilities κ_1 and κ_2 of two sandstones that have the same porosity and pore microstructure, but different average grain sizes, $d_1 = 80\mu m$ and $d_2 = 240\mu m$.

Solution:

1. The units of B in this equation are such that expressing d in microns gives permeability in milliDarcy. Then:

$$K = B\frac{(\phi - \phi_c)^3}{(1+\phi_c-\phi)^2}d^2 = 15\frac{(0.32-0.035)^3}{(1+0.035-0.32)^2} \times 100^2 = 6.79\,\text{Darcy}$$

2. Assuming that B and ϕ_c are the same for both sandstones since they have the same pore microstructure, we can express the ratio of their permeabilities as:

$$\frac{\kappa_1}{\kappa_2} = \frac{d_1^2}{d_2^2} = \frac{80^2}{240^2} = \frac{1}{9}$$

   The sandstone with larger average grain size has a higher permeability (by a factor of 9), even though both have the same total porosity.

5.5 We recall that the 1D laminar flow in a pipe of circular cross-section of radius r is:

$$q = -\frac{\pi\, r^4}{8\,\eta}\frac{\Delta p}{L}$$

In a tubular pore of circular cross-section, with radius r, Darcy's law is expressed as:

$$q = -K\frac{A}{\eta}\frac{\Delta p}{L} = K\frac{\pi\, r^2}{\eta}\frac{\Delta p}{L}$$

Consider a unit rock volume that contains N tubular pores of circular cross-section, following an isotropic distribution, with a radius size distribution $p(r)$. Show that the intrinsic permeability K has the following expression:

$$\kappa = \frac{\Phi}{8}\frac{1}{\int_0^\infty f(r)dr}\int_0^\infty r^2 f(r)dr$$

In which Φ is the porosity of the rock sample, and in which the radius volume frequency $f(r)$ is defined as $f(r) = N\,L\,\pi\,r^2\,p(r)$.

Solution: For a porous medium that contains N pores of different radii, with a radius size distribution $p(r)$, the equation of laminar flow is:

$$q = -\frac{N\pi \int_0^\infty r^4 p(r)dr}{8\eta}\frac{\Delta p}{L} \tag{5.3}$$

And Darcy law is expressed as:

$$q = -\kappa \frac{1}{\eta}\frac{N\pi \int_0^\infty r^2 p(r)dr}{\Phi}\frac{\Delta p}{L} \tag{5.4}$$

Multiplying Equations 5.4 and 5.3 by L, and then combining them, provides:

$$\frac{\Phi}{8}\int_0^\infty r^2 f(r)dr = \kappa \int_0^\infty f(r)dr$$

which gives the required expression of permeability:

$$\kappa = \frac{\Phi}{8}\frac{1}{\int_0^\infty f(r)dr}\int_0^\infty r^2 f(r)dr$$

5.6 Show that for the pipe model illustrated in Figure 5.4, the intrinsic permeability above the percolation threshold has the following expression:

$$K = \frac{\pi}{32}\frac{\overline{\lambda}\overline{r}^4}{\overline{l}^3}$$

in which $\overline{\lambda}$ is the average capillary length and $\overline{r}$ is the average capillary radius. *Assume that the distribution of capillaries centers along the z-axis is homogeneous and isotropic, and that the probability density functions of r, λ, θ and ϕ are isotropic and statistically independent.*

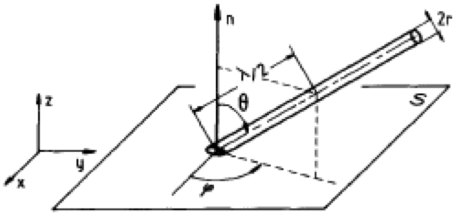


Figure 5.4 Pipe model: Unit section S intercepts pipes of various orientations (θ, ϕ), radius r and length λ. $\overline{l}$ is the average spacing between two pipes. Image taken from (Gueguen and Dienes, 1989)

Solution: See the paper by Gueguen and Dienes (1989) provided in the following pages.

Mathematical Geology, Vol. 21, No. 1, 1989

Transport Properties of Rocks from Statistics and Percolation[1]

Y. Gueguen[2] and J. Dienes[3]

Two simplified microstructural models that account for permeability and conductivity of low-porosity rocks are compared. Both models result from statistics and percolation theory. The first model assumes that transport results from the connection of 1D objects or "pipes"; the second model assumes that transport results from the connection of 2D objects or "cracks." In both cases, statistical methods permit calculation of permeability k and conductivity σ, which are dependent on three independent microvariables: average pipe (crack) length, average pipe radius (crack aperture), and average pipe (crack) spacing. The degree of connection is one aspect of percolation theory. Results show that use of the mathematical concept of percolation and use of the rock physics concept of tortuosity are equivalent. Percolation is used to discuss k and σ near the threshold where these parameters vanish. Relations between bulk parameters (permeability, conductivity, porosity) are calculated and discussed in terms of microvariables.

KEY WORDS: permeability, electrical conductivity, porosity, microstructure.

INTRODUCTION

Although "it is obvious that no simple correlation between porosity and permeability can exist" (Scheidegger, 1974), the search for such correlations is pervasive in rock physics. The reason is probably that tentative correlations are attractive to develop, and they are frequently successful. Success is guaranteed by the fact that undetermined constants are always introduced and adjusted from experimental data. As suggested by Scheidegger, if a correlation should exist, it is between structure and permeability. Structure is a term that needs to be defined accurately: it means here "the microstructure of the porosity." Structure cannot be defined quantitatively by a single parameter. It has to be described by a set of statistical distributions of microstructural parameters, each of them being specific of pore geometry.

---

[1] Manuscript received 27 January 1987; accepted 28 October 1987.

[2] Laboratoire de Physique des Matériaux, Ecole et Observatoire de Physique du Globe, 67084 Strasbourg Cedex, France.

[3] Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545.

Transport properties, i.e., permeability and conductivity, are calculated using statistical methods and percolation theory (Broadbent and Hammersley, 1957). The results are given for a rock in which the microstructure of the porosity is described by a random distribution of pipes. A similar calculation was reported previously for cracks (Dienes, 1982) and has proved to be useful in some situations (Gueguen et al., 1986). The results show that concepts of tortuosity and percolation are equivalent. Both the pipe and the crack models are used to compute relationships between bulk parameters (permeability, porosity, conductivity) in terms of microvariables.

PERMEABILITY FROM STATISTICS AND PERCOLATION

Many theoretical models are available to describe fluid flow in rocks (Scheidegger, 1974; Brace, 1977; Dullien, 1979; Seeburger and Nur, 1984; Koplik et al., 1984; Wilke et al., 1985). Here, permeability k for two models, a pipe and a crack model, is calculated.

Statistics

The pipe model presented here is the following. A set of pipes of variable radii r and lengths λ is isotropically distributed (Fig. 1). The model is similar to that of Haring and Greenkorn (1970), but it could be modified to take into account anisotropy. The statistical calculation of permeability (Appendix A) is

$$k = \frac{\pi}{32} n_0 \overline{\lambda}\, \overline{r^4}$$

where $\overline{r^4}$ is the fourth-order moment of the radius distribution and n_0 the number of pipes per unit volume.

A simplified version of the pipe model is sufficient for our purpose. An approximate expression for $(\overline{r^4})$ is used. The radius distribution is assumed to be narrow so that $(\overline{r^4})$ is close to $(\overline{r})^4$. The average spacing between pipes $\overline{\ell}$ is

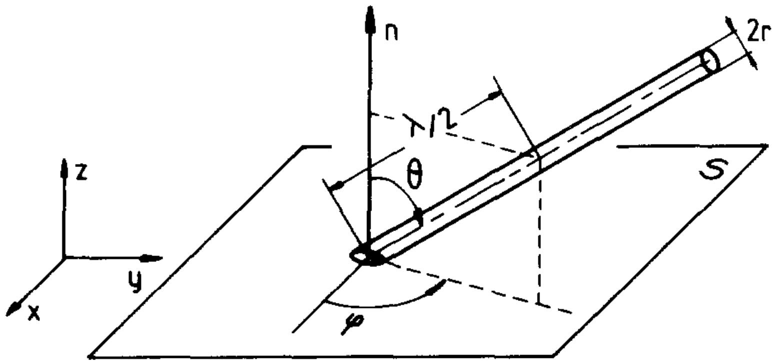


Fig. 1. Pipe model. Unit section S intercepts pipes of various orientation $\vec{l}(\theta, \phi)$, radius r, and length λ; $\vec{n}$ is normal to S and $\bar{\ell}$ is average spacing between pipes.

introduced so that $n_0 \simeq 1/\bar{\ell}^3$. With this approximation,

$$k = \frac{\pi}{32}\frac{\overline{\lambda r^4}}{\bar{\ell}^3}$$

Only those pipes that are connected have to be retained in the statistical calculation of volume flow. Therefore,

$$k = \frac{\pi}{32} f \frac{\overline{\lambda r^4}}{\bar{\ell}^3} \qquad (1)$$

where f is the fraction of connected pipes: $0 \leq f \leq 1$. The meaning of "connected pipes" is that of percolation theory: it refers to pipes that are part of an "infinite path." Calculation of f is possible with a percolation model.

A second microstructural model corresponds to a distribution of 2D objects (cracks). Although this situation is certainly relevant to crystalline rocks, crack models are uncommon. Dienes (1982) has shown that an isotropic distribution of cracks with radius c, number density n_0, and aspect ratio $A = w/c$ results in a permeability,

$$k = \frac{4\pi}{15} A^3 \, n_0 \, \overline{c^5} \, \theta f$$

where $n_0 \, \overline{c^5}$ is the fifth moment of the crack number density. Factor θ accounts for the hydrodynamics of flow through a system of cracks with varying thickness, but seems to differ little from unity in the cases examined. The factor f accounts for the fraction of cracks that belong to an infinite network and is to be determined from percolation theory. Three independent microstructural parameters are introduced: $\bar{c}$ (average crack radius), $\bar{w}$ (average half-crack aperture), and $\bar{\ell}$ (average crack spacing) (Fig. 2). By restricting to the isotropic case and using approximations similar to the previous ones as far as statistical dis-

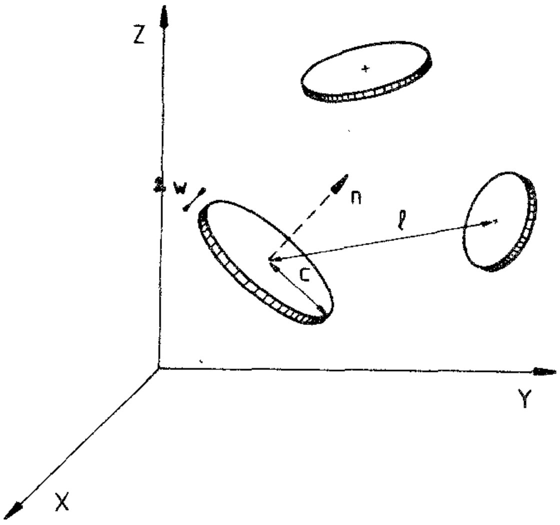


Fig. 2. Crack model: isotropic distribution of cracks. Each crack is characterized by its radius c and aperture $2w$: $\vec{n}$ is normal to a crack and $\bar{\ell}$ is average crack spacing.

tributions are concerned

$$k = \frac{4\pi}{15} f \frac{\overline{w}^3\, \overline{c}^2}{\overline{\ell}^3} \tag{2}$$

Percolation

Pipe or crack centers are assumed to be distributed on a random network. Let p be the probability that two pipes or two cracks intersect. From percolation theory (Stauffer, 1985), f is known to be a function of p. Below a threshold probability p_c, $f = 0$. Above p_c, f increases rapidly until $f = 1$. In the transition region where $p \# p_c$, $f \propto (p - p_c)^n$, where $n \simeq 2$ (De Gennes and Guyon, 1978; Englman et al. 1983). Unfortunately, f can be calculated exactly from p only if simple lattices are assumed. A classical approximation is that of the Bethe lattice where each pipe or crack has z neighbors. Then, $p_c = 1/(z - 1)$ (Stauffer, 1985). Assuming $z = 4$ (as in Dienes, 1982), $p_c = 1/3$. Near this threshold

$$f \simeq 54\, (p - p_c)^2 \tag{3}$$

This result is obtained from Eq. (27) (Dienes, 1982) where p is close to $1/3$ (Fig. 3).

The above result has yet to be completed, however, in so far as p is a function of three microvariables $\overline{\lambda}$, $\overline{r}$, $\overline{\ell}$ or $\overline{w}$, $\overline{c}$, $\overline{\ell}$. Several ways are known to calculate p.

One method (David, 1985) relies on calculation of the number of intersections of a given population of objects (pipes or cracks) with a random line. [This calculation can be found in Dienes (1978)]. Another way is to use the concept of excluded volume (De Gennes, 1976). The excluded volume V_e is the average volume around one object (pipe or crack) within which a second object must have its center in order for the two objects to intersect.

In the case of two cylindrical pipes, $V_e \simeq 2\, \overline{\lambda}^2\, \overline{r}$. Given that $1/\overline{\ell}^3$ is the pipe number density, the average number of intersections per pipe is $V_e/\overline{\ell}^3$.

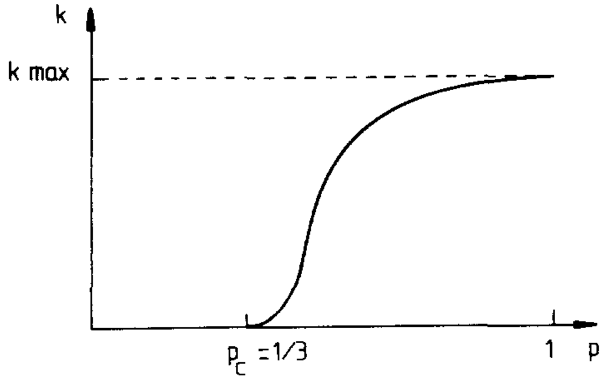


Fig. 3. Variation of the permeability k, as a function of the probability of intersection, p. Percolation threshold is noted p_c; $k_{\max}$ is obtained when $p = f = 1$.

Assuming as above $z = 4$,

$$p \simeq \frac{\overline{\lambda}^2 \overline{r}}{2\overline{\ell}^3} \tag{4}$$

Together with Eqs. (1) and (3), Eq. (4) gives the solution for k in terms of microvariables r, λ, ℓ. Relation (4) gives moreover the additional condition for having a nonzero permeability,

$$\frac{\overline{\lambda}^2 \, \overline{r}}{2 \, \overline{\ell}^3} \geq \frac{1}{3}$$

As could be expected, the probability that any pipe intersects another pipe is small if the pipes are short (small $\overline{\lambda}$) or if they are spaced far apart (large $\overline{\ell}$).

Probability p for two cracks to intersect has been determined previously from different methods (David, 1985; Charlaix et al., 1984) (Appendix B). These last authors used the calculation of De Gennes (1976) that gives the excluded volume for two discs: $V_e = \pi^2 \, \overline{c}^3$. Given that $1/\overline{\ell}^3$ is the crack density, the probability p is,

$$p \simeq \frac{\pi^2}{4} \frac{\overline{c}^3}{\overline{\ell}^3} \tag{5}$$

assuming $z = 4$.

As previously, Eqs. (2), (3), and (5) allow k to be expressed in terms of three microvariables w, c, ℓ.

The value of $k_{\max}$ is given by Eq. (2) with $f = 1$. Nonzero permeability is observed for $p > \frac{1}{3}$, i.e., $\overline{c}/\overline{\ell} > 0.5$ (Gueguen et al., 1986).

Tortuosity

Porosity is derived easily for each model as

$$\phi = \pi \frac{\overline{r}^2 \overline{\lambda}}{\overline{\ell}^3} \qquad \text{(pipe model)} \ (6)$$

and

$$\phi = 2\pi \frac{\overline{c}^2 \overline{w}}{\overline{\ell}^3} \qquad \text{(crack model)} \ (7)$$

From Eq. (1) and (6), a possible relation between k and ϕ is

$$k = \frac{f}{32} \overline{r}^2 \, \phi \qquad \text{for the pipe model}$$

This equation is similar to Eq. (11) of Walsh and Brace (1984), also derived for a pipe model, which is

$$k = \frac{1}{b}\frac{1}{\tau^2} m^2 \phi$$

where τ is "tortuosity," m is hydraulic radius, and b is a constant.

Comparison of both equations leads to

$$\frac{1}{32} f = \frac{1}{b}\frac{1}{\tau^2}$$

This means that the tortuosity concept that is used frequently in rock physics and the percolation concept are related. This result is not really surprising and agrees with those from many other authors concerned with percolation and random media (e.g., Redner, 1983). If $f \to 0$, $\tau \to \infty$; thus, connections no longer exist throughout the medium and $k \to 0$. A large tortuosity corresponds to a situation where pore connection is poor. In terms of percolation theory, "infinite paths" through the medium exist, but only a few. The opposite situation, a small tortuosity, corresponds to the existence of many "infinite paths." The value of τ in the case of ideal connection and isotropy is $\tau = 2$. In terms of percolation, ideal connection means $f = 1$; thus, $b = 8$.

RELATIONS BETWEEN BULK PARAMETERS IN TERMS OF MICROVARIABLES

The above equations concerned with permeability k and porosity ϕ may be completed with similar results on electrical conductivity σ. The bulk parameters (k, σ, ϕ) can be expressed in terms of microvariables $(\bar{r}, \bar{\lambda}, \bar{\ell})$ or $(\bar{c}, \bar{w}, \bar{\ell})$ previously used.

Bulk parameters

Formation factor F is used frequently in rock physics. By definition, $F = \sigma_f / \sigma$ where σ_f is the fluid conductivity and σ is the rock conductivity. Calculation of σ goes along the same lines as calculation of k gives

$$F = 4 f^{-1} \phi^{-1} \tag{8}$$

Results for the three bulk parameters k, F, ϕ (Table 1) are expressed as functions of three microvariables so that theoretically the relations (Table 1) may be inverted to get $\bar{r}$, $\bar{\lambda}$, $\bar{\ell}$ or $\bar{w}$, $\bar{c}$, $\bar{\ell}$. Table 1 should be completed by Eqs. (3), (4), and (5) when the rock is close to the percolation threshold p_c. Below p_c, $f = 0$; well above p_c, $f = 1$. Depending on how the equations are combined, several permeability–porosity relations can be derived also.

One of them has been discussed above for the pipe model ($k = f/32\ \bar{r}^2\ \phi$). However, this is not the only possible $k - \phi$ relation because three variables

Table 1. Permeability, Conductivity, and Porosity for Two Models

Model	k	$F = \sigma_f/\sigma$	ϕ
Pipe model	$\frac{\pi}{32} f \frac{\bar{\lambda}\,\bar{r}^4}{\bar{\ell}^3}$	$4f^{-1}\,\phi^{-1}$	$\pi \frac{\bar{r}^2\,\bar{\lambda}}{\bar{\ell}^3}$
Crack model	$\frac{4\pi}{15} f \frac{\bar{w}^3\,\bar{c}^2}{\bar{\ell}^3}$	$4f^{-1}\,\phi^{-1}$	$2\pi \frac{\bar{c}^2\,\bar{w}}{\bar{\ell}^3}$

are involved in both k and ϕ. Using this relation together with Eq. (8),

$$kF = \frac{\bar{r}^2}{8} \qquad \text{or} \qquad \bar{r} = 2\,(2\,kF)^{1/2}$$

Therefore, $\bar{r}$ is obtained simply from kF. Other microvariables can be obtained in a similar way by appropriate combinations of k, F, and ϕ. Because F is proportional to ϕ, however, only $\bar{r}$ and $\bar{\lambda}/\bar{\ell}^3$ can be obtained separately for the pipe model and expressed as functions of k and F when $f = 1$. This is not the case close to the percolation threshold where Eqs. (3) and (4) can be used and

$$\bar{\ell} = (8\pi\, kF\bar{\lambda}/\phi)^{1/3}$$

$$\text{with} \quad \bar{\lambda} = 2\pi\,\phi^{-1}\,(2\,kF)^{1/2}\left\{1/3 + \left[2/(27\,F\,\phi)^{1/2}\right]\right\}$$

so that the three microvariables are obtained in that situation.

In the case of the crack model, a possible relation between k and ϕ is, from Eqs. (2) and (8),

$$k = \frac{2}{15} f\,\bar{w}^2\,\phi$$

thus

$$kF = \frac{8}{15}\,\bar{w}^2 \qquad \text{or} \qquad \bar{w} = \frac{1}{4}\,(30\,kF)^{1/2}$$

Then, $\bar{w}$ and $\bar{c}^2/\bar{\ell}^3$ may be obtained separately as functions of k and F when $f = 1$. Again, close to the percolation threshold, Eqs. (3) and (5) can be used and $\bar{c}$ and $\bar{\ell}$ can be derived:

$$\bar{c} = 2(\pi\phi)^{-1}\,(30\,kF)^{1/2}\left\{1/3 + \left[2/(27\,F\phi)^{1/2}\right]\right\}$$

$$\bar{\ell} = \left(\frac{\pi^2}{4}\right)^{1/3}\,\bar{c}\left\{1/3 + \left[2/(27\,F\phi)^{1/2}\right]\right\}^{-1/3}$$

Testing of the Models

The above models should be tested in both general and particular situations. The average microvariables ($\bar{\lambda}$ or $\bar{c}$, $\bar{r}$ or $\bar{w}$, $\bar{\ell}$) can be measured from photomicrographs or stained thin sections. Bulk properties F, k, and ϕ can be measured also. Therefore, to what extent the models are faithfully reproducing observations should be possible to determine. An important distinction should be made between situations where the rock is close to the percolation threshold and situations where it is well above it. Only in the first case does Eq. (3) apply. In the second case, a complete inversion is not possible unless additional assumptions are used. Preliminary testing has been done in a few cases, either on little-porosity Fontainebleau sandstones (Gueguen et al., 1986) or in large-porosity (North Sea) chalks (Jakubowski, 1986). In both cases, the crack model predicted permeabilities are in good agreement with those observed.

Particular Situations

Given that the three bulk parameters are calculated from three independent microvariables, complete inversion to get the microvariables from the three bulk parameters is possible as long as these variables enter k, ϕ, σ in different combinations. This is not the case, as seen, if $f = 1$. If additional conditions on the microvariables exist, however, complete inversion may be possible even for the situation $f = 1$. In that case, particular specific relations also can be derived between k, F, and ϕ.

An example of such a particular situation is that corresponding to a variable pipe or crack density ($\bar{\ell}$ variable), the two other microvariables being held constant. Examining how F varies in such a case near the percolation threshold by using Eqs. (3)–(8) and choosing as the unique variable ϕ instead of $\bar{\ell}$,

$$F \propto (\phi - \phi_c)^{-2} \phi^{-1}$$

where

$$\phi_c = \frac{2\pi}{3} \frac{\bar{r}}{\lambda} \qquad \text{(pipe model)}$$

or

$$\phi_c = \frac{4}{3\pi} \frac{\bar{w}}{\bar{c}} \qquad \text{(crack model)}$$

This result is close to Archie's law $F = \phi^{-m}$.

Another example of a particular situation can be found if the pressure variation of k and σ is considered. Assume that pipe radius and crack aperture are the two variables that are the most sensitive to pressure. Therefore, if only r (pipe model) or w (crack model) vary, with f constant ($f = 1$),

$$k \propto r^4, \quad \phi \propto r^2, \quad F \propto \phi^{-1}, \quad k \propto F^{-2} \quad \text{(pipe model)}$$

$$k \propto w^3, \quad \phi \propto w, \quad F \propto \phi^{-1}, \quad k \propto F^{-3} \quad \text{(crack model)}$$

Walsh and Brace (1984) have argued previously that, in such a case, $k \propto F^{-n}$ and that n must fall in the range $1 \leq n \leq 3$. Their data are presented here ($n = -\log k/\log F$):

	n
Westerly granite	1.97
Westerly granite (300°)	2.0
Westerly granite (500°)	2.2
Chelmsford Granite	2.13
Henderson augen gneiss	2.15
Marysville granite	2.7
Raft River siltstone	2.4
Frederic diabase (700°)	2.1
Pottsville Sandstone	2.6
Pigean Cove Granite	2.1

When several measurements are available for a rock, an average n is calculated so that this list is slightly different from Table 1 in Walsh and Brace's paper. Experimental data show that $2 \leq n \leq 3$, which is in agreement with the two limits set by the pipe model ($n = 2$) and the crack model ($n = 3$). The possibility $n = 1$ is excluded by our models, but not by Walsh and Brace. They noted, however, that, if $n \simeq 1$, this would imply a good sensitivity in tortuosity-pore aperture (w or r) dependence. This means, in terms of percolation theory, that the rock is close to the percolation threshold. In this case, k changes rapidly near the percolation threshold because f varies ($f \neq 1$).

CONCLUSIONS

The main conclusions can be summarized as:

(1) Simple models, using 1D objects (pipes) or 2D objects (cracks), can be useful to discuss correlations (pore microstructure)-(transport properties). Pore microstructure is described by statistical distributions of three microvariables that have been assumed to be isotropic. The anisotropic case could be investigated in a similar way.

(2) Use of percolation theory and tortuosity are, in a certain sense, equivalent as found previously by others. In both cases, unfortunately, parameters (f or τ) that cannot easily be measured from photomicrographs must be used. Potentially, an advantage lies in using percolation theory, because a large body of information exists describing the statistics of isolated objects as discussed

(for example, by Stauffer, 1985). Though percolation oversimplifies the physical situation assuming either free flow or no flow in a narrow passage, it is far from being completely understood and is, in fact, the subject of active research. The subject of anisotropic percolation, for example, has received virtually no attention.

(3) Pipe and crack models have been used to derive permeability-conductivity-porosity relationships. Depending on the particular situations that are considered, various possible relations are predicted between bulk parameters.

(4) Models should be tested by measuring microvariables from micrographs and measuring bulk properties in the laboratory. Preliminary testing has been done in a few cases successfully.

Note Added in Proof

Dr. Stauffer has drawn our attention to an important recent publication that was not known to us (*Fragmentation, Form and Flow in Fractured Media*, 1986). This book contains several papers that are concerned with flow in fractured media. In particular, the paper by Charlaix et al. (1986) contains some results that are presented and discussed also in the present paper (critical behavior of k, probability p in terms of disc radius). An important difference, however, is that Charlaix et al. did not use a statistical approach. They examined a ``crack model'' corresponding to a broad distribution of crack apertures. Our approach is more appropriate for a distribution that is not broad. In fact, Charlaix et al. (1986) have applied the Ambegoakar et al. (1971) percolation model to the calculation of ``crack model'' permeability. A similar approach has been published simultaneously and independently in the case of a ``pipe model'' (Katz and Thompson, 1987). Katz and Thompson have considered a population of pipes with a broad distribution of pipe radii. They also applied the Ambegaokar percolation model to the calculation of ``pipe model'' permeability. The main differences between those models and ours are linked with the types of statistical distribution that have to be considered. Here statistics and percolation are combined whereas the quoted authors used only percolation and do not express the bulk parameters in terms of microvariables.

APPENDIX A: STATISTICAL CALCULATION OF *k* FOR PIPE MODEL

The volume flow through a capillaric pipe of radius r is given by the Poiseuille law:

$$\vec{Q} = \frac{-\pi\, r^4}{8\eta} (\nabla p \cdot \vec{t})\ \vec{t}$$

where $\vec{t}$ is a unit vector along the pipe axis ($\nabla p \cdot \vec{t}$ is the pressure gradient along the pipes), and η is fluid viscosity. The flow through a section (S) of porous medium defined by a plane z = constant (Fig. 1) is considered. The area of the section is unity and this unit section intercepts a number of pipes that are indexed by 1, 2, . . . α, . . . N. The normal to the section is $\vec{n}$. Total volume flow through it is $\Sigma_{\alpha=1}^{N} \vec{Q}^{\alpha} \cdot \vec{n}$, where $\vec{Q}^{\alpha}$ is the flow through pipe α. By comparing this result to Darcy's law,

$$\sum_{\alpha=1}^{N} \frac{-\pi(r^{\alpha})^{4}}{8\eta} [p_{,i}\, t_i^{\alpha}]\, [t_j^{\alpha}\, n_j] = \frac{-k_{ij}}{\eta} p_{,i}\, n_j$$

Thus, the permeability is

$$k_{ij} = \left(\frac{\pi}{8}\right) \sum_{\alpha=1}^{N} (r^{\alpha})^{4}\, t_i^{\alpha}\, t_j^{\alpha}$$

An isotropic distribution of pipes is assumed and substitutes an integral expression for the discrete summation.

Let $n(r, \lambda, \theta, \varphi, z)$ be the number of pipes per unit volume having a radius r, a length λ, an orientation defined by θ and φ, and a center at z (Fig. 1). Then, the number of pipes N having a radius r and an orientation (θ, φ) that intercepts the unit area is

$$N(r, \theta, \varphi) = 2 \int_0^{\infty} d\lambda \int_0^{\lambda/2 \cos\theta} n(r, \lambda, \theta, \varphi, z)\, dz$$

because a pipe with fixed values of θ and λ intercepts (S) only if its center is at $z \leq \lambda/2 \cos\theta$ (Fig. 1). Assuming isotropy and statistical independence,

$$N(r, \theta, \varphi) = 2 \int_0^{\infty} d\lambda \int_0^{\lambda/2 \cos\theta} f(r) g(\lambda) \frac{n_0}{2\pi}\, dz = \bar{\lambda} f(r) \cos\theta \frac{n_0}{2\pi}$$

The distribution of centers of pipes along z is assumed to be homogeneous and isotropic:

$$\bar{\lambda} = \int_0^{\infty} \lambda g(\lambda)\, d\lambda$$

and

$$n(r, \lambda, \theta, \varphi, z) = f(r)\, g(\lambda)\, n_0 \frac{1}{2\pi}$$

where n_0 is the number of pipes per unit volume, $f(r)$ is the fraction of them having radius r, and $g(\lambda)$ is the fraction having length λ. The factor $1/2\pi$ normalizes the angular distribution.

With these distributions, the expression for k becomes

$$k = \frac{\pi}{8} \frac{n_0}{2\pi} \bar{\lambda} \int_0^{\infty} r^4 f(r)\, dr \int_0^{\pi/2} (\cos^3 \theta) 2\pi \sin \theta \, d\theta$$

$$= \frac{\pi}{32} n_0 \bar{\lambda} \overline{r^4}$$

where $(\overline{r^4})$ is the fourth-order moment of the radius distribution.

APPENDIX B

Dienes (1978) has shown that the mean free path for a random line in a field of thin, circular cracks is $\lambda = 2/\pi \bar{c}^2 \bar{\ell}^3$. Assuming that intersections are described by a Poisson process, the probability that a crack is isolated is $q^4 = e^{-\nu}$ where $\nu = L/\lambda$ and L is the circumference of the average crack, i.e., $L = 2\pi \bar{c}$, and $p = (1 - q)$.

Then, $(1 - p)^4 = e^{-\nu}$ where $\nu = \pi^2 \bar{c}^3/\bar{\ell}^3$. This frequency ν is exactly identical to that obtained from the excluded volume argument. The difference between the above calculated value of p and Eq. (5) stems from the fact that an approximation has been used to derive p from ν in Eqs. (4) and (5). Using the assumption of the Bethe lattice with 4 neighbors leads to $q^4 = e^{-\nu}$, i.e., to the above equation. When ν is small, however, $e^{-\nu/4} \simeq 1 - \nu/4$ and $p \simeq \nu/4$.

ACKNOWLEDGMENTS

This work has been supported by the NATO Scientific Affairs Division. The authors thank I. Lerche for his constructive criticisms, which have helped to improve the original manuscript.

REFERENCES

Ambegaokar, V.; Halperin, B. I.; and Langer, J. S., 1971, Hopping Conductivity in Disordered Systems: Phys. Rev., v. B4, p. 2612–2620.

Brace, W. F., 1977, Permeability from Resistivity and Pore Shape: J. Geophys. Res., v. 82, p. 23, 3343–3349.

Broadbent, S. E. and Hammersley, J. M., 1957, Percolation Processes I, Crystals and Mazes: Proc. Cambridge Philos. Soc., v. 63, p. 629–641.

Charlaix, E.; Guyon, E.; and Rivier, N., 1984, A Criterion for Percolation Threshold in a Random Array of Plates: Solid State Commun., v. 50, p. 999–1002.

Charlaix, E.; Guyon, E.; and Roux, S., 1986, Critical Effects in the Permeability of Heterogeneous Media: p. 316–324 *in* Fragmentation, Form and Flow in Fractured Media: Annals of Israël Physical Society, v. 8.

David, C., 1985, Modèle d'Evolution de la Perméabilité d'une Roche: Rapport de D.E.A., Institut de Phys. du Globe, Strasbourg, 78 p.

De Gennes, P. G., 1976, The Physics of Liquid Crystals: Oxford University Press, Oxford, p. 38.

De Gennes, P. G. and Guyon, E., 1978, Lois Générales pour l'Injection d'un Fluide dans un Milieu Poreux Aléatoire: J. Mécanique, v. 77, p. 403–432.

Dienes, J. K., 1978, A Statistical Theory of Fragmentation: p. 51–55 *in* Y. S. Kim (Ed.), Proceedings of the 19th U. S. Symposium on Rock Mechanics.

Dienes, J. K., 1982, Permeability, Percolation and Statistical Crack Mechanics: p. 86–94 *in* R. E. Goodman and F. E. Heuze (Eds.), Issues in Rock Mechanics: American Institute of Mining, Metallurgical and Petroleum Engineers, New York.

Dullien, F. A. L., 1979, Porous Media: Fluid Transport and Pore Structure: Academic Press, New York, 396 p.

Englman, R.; Gur, Y.; and Jaeger, Z., 1983, Fluid Flow Through a Crack Network in Rocks: J. Appl. Mech., v. 50, p. 707–711.

Gueguen, Y.; David, C.; and Darot, M., 1986, Models and Time Constants for Permeability Evolution in Cracked Rocks: Geophys. Res. Lett., v. 13, p. 460–463.

Haring, R. E. and Greenkorn, R. A., 1970, A Statistical Model of a Porous Medium with Non Uniform Pores: A. I. Chem. Eng., v. 16, p. 477–483.

Jakubowski, M., 1986, Contribution à l'Etude Pétrographique et Pétrophysique de la Craie des Réservoirs Pétroliers due Complexe d'Ekofisk: Thèse d'Etat Univ. de Toulouse, 189 p.

Katz, A. J. and Thompson, A. H., 1987, Prediction of Rock Relectrical Conductivity from Mercury Injection Measurements: J. Geophys. Res., v. 92, p. 599–607.

Koplik, J.; Lin, C.; and Vermette, M., 1984, Conductivity and Permeability from Microgeometry: J. Appl. Phys., v. 56, p. 3127–3131.

Redner, S., 1983, Percolation and Conduction in Random Resistor–Diode Networks: p. 447–475 *in* G. Deutscher, R. Zallen, and J. Adler (Eds.), Percolation, Structure and Processes: Annals of the Israel Physical Society, v. 5.

Scheidegger, A. E., 1974, The Physics of Flow Through Porous Media: University of Toronto Press, Toronto, 353 p.

Seeburger, D. A. and Nur, A., 1984, Pore Space Model for Rock Permeability and Bulk Modulus: J. Geophys. Res., v. 89, p. 527–536.

Stauffer, D., 1985, Introduction to Percolation Theory: Taylor and Francis, London, 124 p.

Walsh, J. B. and Brace, W. F., 1984, The Effect of Pressure on Porosity and the Transport Properties of Rock: J. Geophys. Res., v. 89, p. 9425–9431.

Wilke, S.; Guyon, E.; and de Marsily, G., 1985, Water Penetration Through Fractured Rocks: Test of a Tridimensional Percolation Description: Math. Geology, v. 17, p. 17–27.

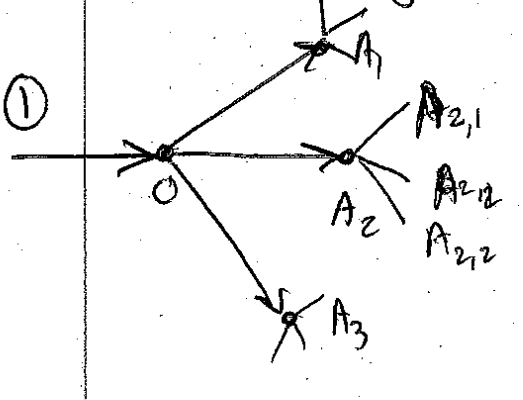
5.7 Consider a Bethe Lattice with $z = 4$.

1. What is the percolation threshold p_c for this network?
2. What is the average number of sites T to which the origin is connected through a single branch A ? *(T is the average size of the cluster for each branch).*
3. Determine the form of the relationship that links S to p and p_c.
4. Determine P, the fraction of sites that are part of the infinite cluster. *Introduce Q, the probability that a path starting from the origin is interrupted somewhere.*

Solution: See the following five pages of notes.

Percolation theory – Bethe lattice

①



No returning flow possible.

- prob. to flow from O to A_1 : p
- ———————— A_2 : p
- ———————— A_3 : p

Σ: proba that the flow does not stop @ A_1 ?

More generally: $(z-1)p$.

At the next site (A_1 or A_2 or A_3), again, the proba that the flow does not stop is $(z-1)p$.

$\longrightarrow$ tot. proba. that flow does not stop: $\left[(z-1)p\right]^2$

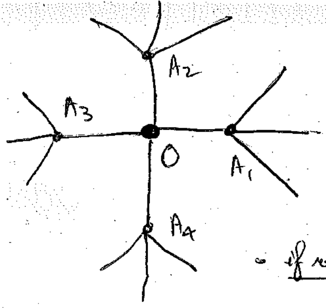
For n branching levels, the proba that the flow does not stop is:

$$P_n \equiv \left[(z-1)p\right]^{n+1}$$

$$P_n \xrightarrow[n\to+\infty]{} \neq 0 \iff (z-1)p>1 \implies \boxed{p_c = \frac{1}{z-1}}$$

For $z=4$: $p > \frac{1}{3}$.

②



O is surrounded by 4 sites (A_1, A_2, A_3, A_4) that may be occupied or not occupied.

- *if not occupied*: In average, +0 site connected to O with probability $(1-p)$
- *if occupied*: In average, +1 site connected to O with probability p.

Each occupied site is itself surrounded by 4 sites:

- 1 is O, which cannot contribute to the flow-out of the current site (no returning flow)
- 3 are in average connected to T sites (by def. of T), with probability p of being occupied.

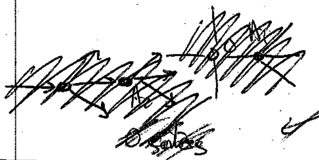
⇒ Each occupied site connected to O, is in average connected to $3Tp$ sites other than O.

⇒ Each occupied site brings $(1+3T)_p$ connections to O.

$$\Rightarrow T = 0\times(1-p) + (1+3T)p = (1+3T)p.$$

$$\Rightarrow \boxed{T = \frac{p}{1-3p}}$$

③ Cluster starting @ O: this time, we consider the entire network, regardless of the direction of the flow



~~• probability 1−p that one of the 4 neighbouring sites is unoccupied.~~

$p(1+3T)$ sites ← { • probability p occupied.

~~• each occupied site~~

See Fig. below

T_p sites { • probability p that O is occupied and connected to upstream neighbours.

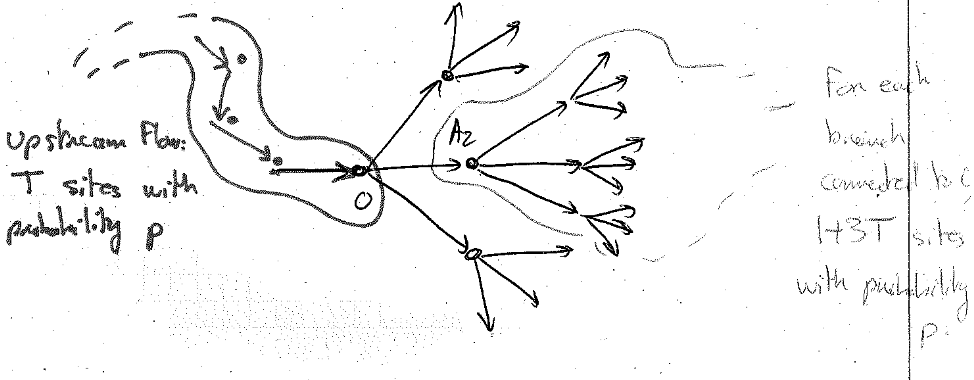
$$\boxed{S = (1+4T)_p}$$

Recalling that $T = \frac{p}{1-3_p}$: $S = \left(1 + \frac{4p}{1-3_p}\right)p$.

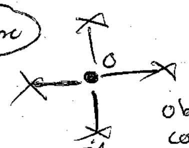
$$\boxed{S = \left(\frac{1+p}{1-3_p}\right)p}$$

For $z=4$, $p_c = \frac{1}{3}$, therefore: $S = \frac{(1+p)p}{3(p_c - p)}$

$$\boxed{\text{When } p \to p_c, \quad S \equiv (p_c - p)^{-1}}$$



④ (p>pc)

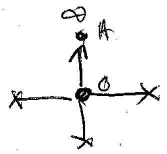


0 branch connected to O

1 possibility: Q^4

↓

flow interrupted (O isolated)



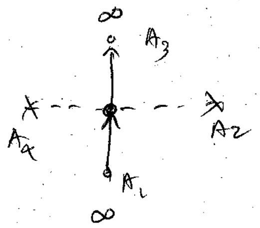
only 1 branch connected to O

4 possibilities

$4(1-Q)Q^3$

↓

flow interrupted at O (not at A).



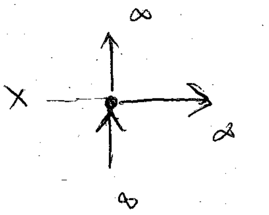
2 branches connected out of the 4.

3+2+1 = 6 possibilities

$6Q^2(1-Q)^2$

↙ P (interrupted) ↘ p(not interrupted).

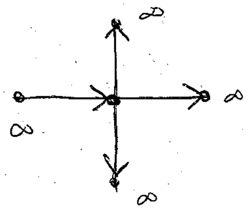
3 branches connected out of the 4,



4 possibilities

$4(1-Q)^3Q$

4 branches connected to O:



1 possibility: $(1-Q)^4$

⇒ Total number of sites connected to an ∞ cluster, on average:

$$\boxed{P = 6Q^2(1-Q)^2 + 4Q(1-Q)^3 + (1-Q)^4}$$

Proba that the ~~[illegible]~~ path through O is stopped somewhere

= proba that O – A is interrupted : $(1-p)$ (proba A not occupied)

+ proba that $(A - A_i)_{i=1,2,3}$ is interrupted : $p \times Q^3$

proba A occupied × probas A_i had not continuous –

$$\Longrightarrow \boxed{Q = 1-p + pQ^3}$$

$$\underset{1}{Q} = 1 \quad \text{or} \quad \underset{2}{Q} = -\frac{1}{2} + \sqrt{\frac{1}{p} - \frac{3}{4}} \quad \text{or } Q_3 \text{ not physical}$$

$\Rightarrow P = 0 \quad (p < p_c)$

$\downarrow$

$p > p_c$

$\downarrow$

$$\begin{cases} P = 6Q^2(1-Q)^2 + 4Q(1-Q)^3 + (1-Q)^4 \\ Q = -\frac{1}{2} + \sqrt{\frac{1}{p} - \frac{3}{4}} \end{cases}$$

$$\Longrightarrow \boxed{P \simeq (p-p_c)^2}$$

5.8 Consider a geomaterial in which the pores are penny-shaped, with the geometric parameters explained in Figure 5.5.

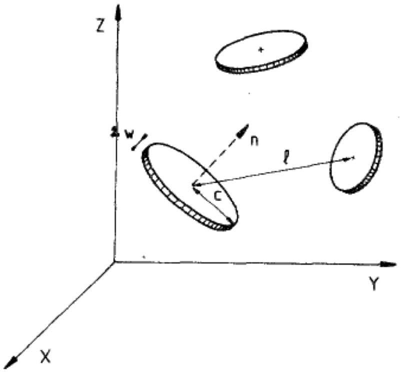


Figure 5.5 Flow in a network of penny-shaped pores. Image taken from (Gueguen and Dienes, 1989).

1. Calculate the value of l (average distance between fractures) at the percolation threshold. Assume that the fractures are disk shaped and of radius $c = 200\mu m$. *Assume that each fracture is a site that is part of a Bethe lattice.*
2. Calculate the maximum value of permeability for a population of identical fractures with radius $c = 200\mu m$ and aperture $w = 1\mu m$. Given: The permeability of a network of penny-shaped fractures is:

$$k = \frac{4\pi}{15} f \frac{\overline{w}^3 \overline{c}^2}{\overline{l}^3}$$

   In which f is the portion of fractures that form an infinite cluster, and $\overline{x}^n$ is the n-th moment of probability of the variable x.

Solution: See the solution of Problem 5.6 (paper by Gueguen and Dienes, 1989).

5.9 Prove the axis-symmetric consolidation equation shown in the course.

Solution: In the demonstration of the 1D consolidation equation, we show that:

$$\frac{\partial p_w}{\partial t} = c_v \boldsymbol{\nabla}^2 p_w \tag{5.5}$$

with:

$$\boldsymbol{\nabla}^2 p_w = (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla}(p_w)) : \boldsymbol{\delta}$$

in which $\boldsymbol{\delta}$ is the second-order identitiy tensor. In axis-symmetric conditions, using cylindrical coordinates:

$$\boldsymbol{\nabla}(p_w) = \frac{\partial p_w}{\partial r}\boldsymbol{e_r}$$

$$\begin{aligned}
\boldsymbol{\nabla} \otimes \boldsymbol{\nabla}(p_w) &= \boldsymbol{\nabla}\left(\frac{\partial p_w}{\partial r}\boldsymbol{e_r}\right) \\
&= \boldsymbol{\nabla}\left(\frac{\partial p_w}{\partial r}\right) \otimes \boldsymbol{e_r} + \frac{\partial p_w}{\partial r}\boldsymbol{\nabla}\left(\boldsymbol{e_r}\right) \\
&= \frac{\partial^2 p_w}{\partial r^2}\boldsymbol{e_r} \otimes \boldsymbol{e_r} + \frac{1}{r}\frac{\partial p_w}{\partial r}\frac{\partial \boldsymbol{e_r}}{\partial \theta} \otimes \boldsymbol{e_\theta} \\
&= \frac{\partial^2 p_w}{\partial r^2}\boldsymbol{e_r} \otimes \boldsymbol{e_r} + \frac{1}{r}\frac{\partial p_w}{\partial r}\boldsymbol{e_\theta} \otimes \boldsymbol{e_\theta}
\end{aligned}$$

From there, we find that:

$$\boldsymbol{\nabla}^2 p_w = \frac{\partial^2 p_w}{\partial r^2} + \frac{1}{r}\frac{\partial p_w}{\partial r} \tag{5.6}$$

The combination of Equations 5.5 and 5.6 provides the result sought:

$$\frac{\partial p_w}{\partial t} = c_v\left(\frac{\partial^2 p_w}{\partial r^2} + \frac{1}{r}\frac{\partial p_w}{\partial r}\right)$$

5.10 Consider a rock specimen filled with oil (mass density ρ_n, pressure p_n, saturation degree S_n) and water (mass density ρ_w, pressure p_w, saturation degree S_w). The specimen is cylindrical, with impermeable lateral boundaries. The specimen is subjected to a constant water flow at the base. The initial saturation degree of oil is $S_n^0 = 0.8$. We assume that fluid flow is purely 1D and occurs only in the direction of the axis of the specimen. The capillary pressure is assumed to be negligible, i.e. $\forall z,\ p_n(z) = p_w(z) = p(z)$. The porosity of the specimen is assumed to remain constant.

1. Provide the governing equations of the water and oil phases.
2. Assume that the fluids are incompressible. Show that the sum of the fluid velocities $v(z) = v_n(z) + v_w(z)$ is uniform throughout the sample, i.e. that $v(z)$ does not depend on the position z in the sample.
3. Show that:
$$\frac{dp}{dz} = \frac{-v(z) + (K_w\rho_w/\mu_w + K_n\rho_n/\mu_n)\,g}{K_w/\mu_w + K_n/\mu_n}$$
Explain why dp/dz only depends on S_w.

Solution: See the following two pages of notes.

$$\vec{v} = -\frac{KK^r}{\mu}\left\{\vec{\text{grad}}\, p - \rho \vec{g}\right\} \rightarrow v = -\frac{KK^r}{\mu}\left[\frac{dp}{dz} + \rho g\right]$$

$$\text{div}\, \rho \vec{v} + \frac{\partial}{\partial t}\left(\rho n S\right) = 0.$$

$$S_w + S_n = 1$$

$$p_c = 0 \Rightarrow p_w = p_n = p.$$

---

2 —

$$\begin{cases} v_w = -\dfrac{\overset{K_w}{\boxed{K K_w^r}}}{\mu_w}\left[\dfrac{dp}{dz} + \rho_w g\right] & \text{①} \\[2ex] v_n = -\dfrac{\overset{K_n}{\boxed{KK_n^r}}}{\mu_n}\left[\dfrac{dp}{dz} + \rho_n g\right]. & \text{②} \end{cases}$$

$$\begin{cases} \dfrac{\partial v_w}{dz} + n\dfrac{\partial S_w}{\partial t} = 0. & \text{③} \\[2ex] \dfrac{\partial v_n}{dz} + n\dfrac{\partial S_n}{\partial t} = 0 & \text{④} \end{cases}$$

$$S_w + S_n = 1. \qquad \text{⑤}$$

$$\begin{cases} v = v_w + v_n \\ f_w = v_w / v \\ f_n = v_n / v \end{cases}$$

Inconnues v_w, v_n, p, S_w, S_n

$$③ + ④ \rightarrow \frac{\partial v_w}{\partial z} + \frac{\partial v_n}{\partial z} + n\left(\frac{\partial S_w}{\partial t} + \frac{\partial S_n}{\partial t}\right) = 0$$

$$\frac{\partial v}{dz} + n\frac{\partial}{\partial t}\left(S_w + S_n\right) = 0$$

$$⑤ \rightarrow \frac{\partial v}{dz} = 0. \Rightarrow \boxed{v = c^{te}.}$$

v ne dépend pas de z ⇒ v
identique dans toutes les sectio
l'échantillon, en particulier d
la section $z = 0$.
Dans cette section, la saturation po
instantanément à l'instant 0 de
valeur $S_w = 0.2$ à la valeur $S_w =$

$$①+② \rightarrow v = v_w + v_n = -\left(\frac{K_w}{\mu_w} + \frac{K_n}{\mu_n}\right)\frac{dp}{dz} - \left(\frac{K_w\rho_w}{\mu_w} + \frac{K_n\rho_n}{\mu_n}\right)g$$

$$\Rightarrow \boxed{\frac{dp}{dz} = -\frac{v + \left(\frac{K_w\rho_w}{\mu_w} + \frac{K_n\rho_n}{\mu_n}\right)g}{\frac{K_w}{\mu_w} + \frac{K_n}{\mu_n}}}$$

et ensuite S_w reste constant
La vitesse v_w est imposée p
le débit constant → $v_w = c$
la vitesse v_n est nulle pui
l'huile se trouve alors à la
saturation résiduelle ⇒ K_n
⇒ $v = v_w + v_n = c^{te}$ indép
du temps dans la face d'e
⇒ $\underline{v = c^{te}\text{ indep. du } z \text{ et}}$

v, ρ_w, μ_w, ρ_n, μ_n, g constants
K_w, K_n dep. de S_w seulement

→ $\frac{dp}{dz}$ ne dépend que de S_w.

→ v_w, v_n ne dépendent que de S_w

5.11 The capillary rise in a tube can be calculated by using Jurin's equation:

$$\gamma_w \, h = p_n - p_w$$

and Laplace's equation:

$$p_n - p_w = \frac{2 t_{n/w} \cos \theta_w}{r}$$

in which h is the height of the capillary rise, γ_w is the specific weight of the wetting fluid in the tube (e.g., water), p_w is the pressure of the wetting fluid (water), p_n is the pressure of the non-wetting fluid (e.g., air), r is the radius of the tube, $t_{n/w}$ is the surface tension in the meniscus (e.g., at the water/air interface) and θ_w is the wetting angle, i.e. the angle that exists between the normal to the tube wall and the tangent to the meniscus. For a water/air meniscus under ambient temperature, we have $t_{n/w} = 75$ mN/m. The porous network inside a soil specimen is often modeled as a distribution of capillary tubes that are parallel to each other and that are not connected to one another, so that the equations of Jurin and Laplace can be applied. With this assumption in mind, let us consider a cylindrical soil specimen, 5 cm in diameter and 100 cm in height, made of solid grains, liquid water and gaseous air. The soil porous network is modeled by the bundle of tubes described in Table 5.1. Furthermore, it is assumed that the tubes are perfectly wettable, i.e., $\theta_w = 0^o$.

1. Calculate the porosity of the specimen.
2. Express the capillary rise in a single tube of diameter D (in symbolic formula). Calculate the capillary rise h in each type of tube listed in Table 5.1.
3. Express the relationship between the capillary pressure and the degree of saturation in water of the specimen. Plot the Water Retention Curve of the specimen.

Table 5.1 Porous network modeled as a bundle of tubes

Tube diameter (μm)	Number of tubes
0.1	500
0.5	1,000
1	5,000
5	50,000
10	100,000
50	50,000
100	5,000
500	1,000
1000	500

Solution: See the following two pages of notes.

$\boxed{\text{exercice 2}}$ Remontée capillaire dans un sol

✓ 1) $n = \frac{V_V}{V_T}$

$V_T = \pi \times (2,5.10^{-2})^2 \times 1 = 1,96.10^{-3}\ m^3$

$V_V = \pi \times (0,1.10^{-6})^2 \times 1 \times 500$

$+ \pi \times (0,5.10^{-6})^2 \times 1 \times 1000$

$\vdots$

Ø (µm)	$V_{unit.}$ (m^3)	$V_{T/Ø}$ (m^3)
0,1	$7,85.10^{-15}$	$3,93.10^{-12}$
0,5	$1,96.10^{-13}$	$1,96.10^{-10}$
1	$7,85.10^{-13}$	$3,92.10^{-9}$
5	$1,96.10^{-11}$	$9,82.10^{-7}$
10	$7,85.10^{-11}$	$7,85.10^{-6}$
50	$1,96.10^{-9}$	$9,82.10^{-5}$
100	$7,85.10^{-9}$	$3,93.10^{-5}$
500	$1,96.10^{-7}$	$1,96.10^{-4}$
1000	$7,85.10^{-7}$	$3,93.10^{-4}$
		$7,05.10^{-4}$ 738 cm^2

$n = \frac{7,05.10^{-4}}{1,96.10^{-3}}$ $\boxed{n \simeq 0,36}$ $n = 0,38.$

✓ 2) $Sp = u_a - u_w = \frac{2A}{R} = \frac{2A\cos\theta}{\underset{\uparrow\ D/2}{r}} = \gamma_w h$

$\gamma_w h = \frac{4A\cos\theta}{D}$

θ : angle air/liquide (ménisques)

capillaires parfaitement mouillables → $\theta = 0$

→ $\boxed{h = \frac{4A}{D\gamma_w}}$

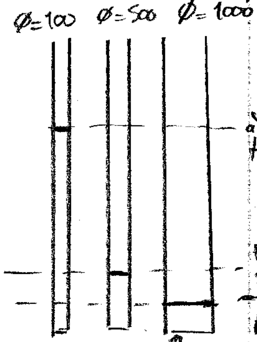
$h = \frac{4 \times 75.10^{-3}}{D \times 10^4} = \frac{3.10^{-5}}{D}.$

√3)

ϕ (μm)	h (m)	S_R (%)
0,1	300	~0
0,5	60	~0
1	30	~0
5	6	0,001
10	3	0,012
50	0,6	0,147
100	0,3	0,2
500	0,06	0,47
1000	0,03	1

↑ sens des calculs

4)
$$S_R = \frac{-\text{Stotale des types de tube de } \phi > \phi \text{ considéré} + \text{Stotale ech.}}{\text{Stotale ech.}}$$



le + gos tube est saturé à 3cm

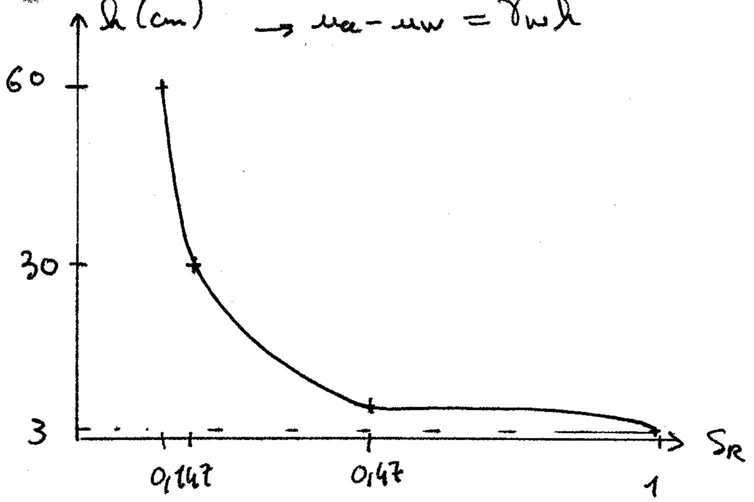
avec les volumes :

à 30cm 2 types de tubes ne st pas sat. $\frac{7,38-3,9}{7,38} = 0,47$ $\qquad 0,44 = \frac{7,05-3,93}{7,05}$

à 6 cm, ts les tubes sauf les + gros st saturés $\frac{7,38-3,9-2}{7,38} = 0,2.$ $\qquad 0,16 = \frac{7,05-3,93-1,96}{7,05}$

à 3cm, ts les tubes st sat

$\rightarrow u_a - u_w = \gamma_w h$



5.12 Let us consider two non-deformable cylinders of radius R and of length 1 m in the direction orthogonal to the sheet of paper (see Figure 5.6). At the contact between the two cylinders, a water meniscus exists. We assume that the material that makes the cylinders is perfectly wettable, i.e., $\theta_w = 0$. The meniscus section is an arc of circle of radius r. The width of the meniscus relative to the axis that links the two centers of the cylinders is noted l.

1. Calculate l as a function of r and R. Derive the relation between the water capillary pressure in the meniscus, R and $\alpha = R/r$. From there, calculate the attraction force f_u between the two cylinders because of the presence of the meniscus (provide the expression of f_u as a function of α and R).

2. Generalize the result found in question 1 above to a regular cubic packing of cylinders. Give the expression of the effective stress. What are the limitations of this model?

3. Calculate the porosity, void ratio and dry specific weight of the material (assuming a regular cubic packing). Assume $\rho_s = 27$ kN/m^3 (for the solid grains).

4. The capillary cohesion is defined as:

$$c_s = \sigma'_n \tan \Phi'$$

   in which Φ' is the friction angle and σ'_n is the normal component of the effective stress (where here, the normal direction is the direction of the segment that links the centers of both cylinders). Calculate the capillary cohesion of the medium as a function of the water capillary pressure, for a friction angle of 30^o. Plot the variations of the cohesion for $R = 1\mu m$.

5. Calculate the increase in material strength in the vertical direction as a function of the capillary pressure. Assume hte following:

   - The stress path is triaxial (σ_H=cst);
   - The external forces are given as: $f_{ext} = 2R \times \left(0.75\sigma_v^2 + 0.25\sigma_h^2\right)$;
   - The orientation angle of the external forces compared to the vertical is given as: $\delta_{ext} = 60 - arctan(0.577 \times (\sigma_H/\sigma_v))$ (in degrees);
   - The material strength of the dry medium is 100 kPa, which corresponds to an external force oriented by an angle of 30^o compared to the vertical (normal).

   To proceed with the calculation, assume a value of σ_v and calculate the capillary force such that σ_v reaches the material strength.

Solution: See the following two pages of notes.

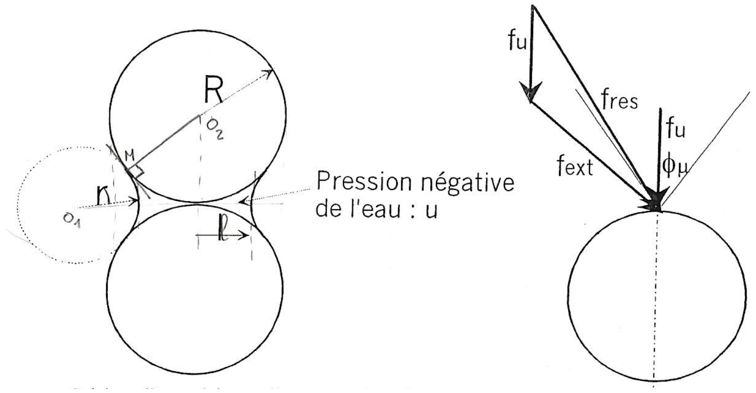


Figure 5.6 The menisci between two solid cylinders and the corresponding inter-granular forces. Image taken after Fleureau's course notes.

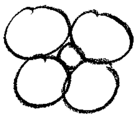
SNS.A4

Corrigé exo. Rouleaux

1) $\ell = -r\left[1-\sqrt{1+\frac{2R}{r}}\right] = -r\,b$ avec $b = 1-\sqrt{1+2\alpha}$ $\alpha = \frac{R}{r}$

$-u = \frac{A}{r} = \frac{A\alpha}{R}$ $f_u = -u \times 2\ell = \frac{A}{\cancel{r}} \times 2(-\cancel{r}\,b) = -2Ab$

2) $\sigma'_u = \frac{f_u}{2R} = -\frac{Ab}{R}$



limite lorsque tous les ménisques se touchent.

Limites du calcul : $R^2 = r(r+2R) \rightarrow r = 0.414\,R$

lim lorsque [illegible]
ie R+r ~R.

3) $n = 0.215$ $e = 0.27$ $\gamma_d = 21.2\ \mathrm{kN/m^3}$

alors $(R+r)^2 = 2R^2$

4) $\underset{\text{déf}}{C_{cap}} = \sigma'_u \;\mathrm{tg}\,\phi' = -\frac{Ab}{R}\,\mathrm{tg}\,30°$

$\sigma'_u = -\frac{Ab}{R} = -\frac{A}{R}\left[1-\sqrt{1+2\alpha}\right]$ avec $\alpha = -\frac{Ru}{A}$

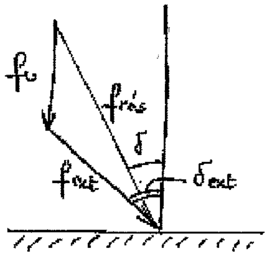
$R = 10^{-6}\,m$
$A = 75\times10^{-3}\ N/m$

5) données
$$\begin{cases} f_{ext} = 2R\left(0.75\,\sigma_V^2 + 0.25\,\sigma_H^2\right) \\ \delta_{ext} = 60 - \mathrm{Arctg}\left(0.577\,\sigma_H/\sigma_V\right) \end{cases}$$

- en l'absence d'eau $\delta_{ext} = 30° \rightarrow \sigma_H/\sigma_V = 1 \rightarrow \sigma_H = 100\,kPa$.

puisque la limite élastique est égale à σ_V

- en présence d'eau : $\mathrm{tg}\,\delta = \frac{f_{ext}\sin\delta_{ext}}{f_u + f_{ext}\cos\delta_{ext}} = \frac{f_{rés}\sin\delta}{f_u + f_{ext}\cos\delta_{ext}}$



$$\rightarrow f_u = f_{ext}\left[\frac{\sin\delta_{ext}}{\mathrm{tg}\,\delta} - \cos\delta_{ext}\right] \quad (1)$$

$\delta = 30°$ si $\sigma_V = \sigma_{vpc}$ (limite élastique : énoncé)

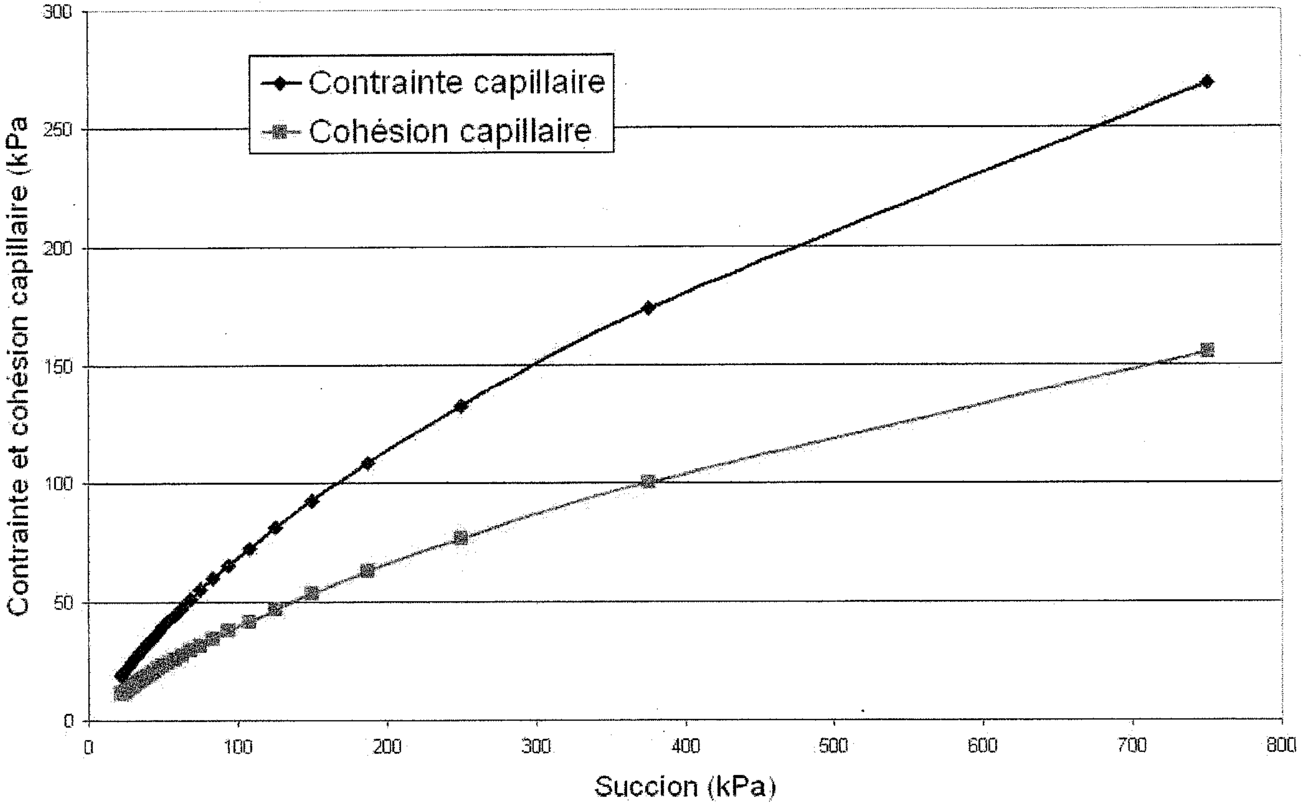
$$f_u \underset{\uparrow Q.1}{=} 2A\left[-1+\sqrt{\underset{\uparrow \text{dép de } u}{1+2\alpha}}\right] \rightarrow \sqrt{1+2\alpha} = \frac{f_u+2A}{2A} = 1+\frac{f_u}{2A}$$

$$\rightarrow 2\alpha = \left(1+\frac{f_u}{2A}\right)^2 - 1 \underset{\uparrow \text{déf de } \alpha}{=} -\frac{2Ru}{A} \Rightarrow -u = \frac{A}{2R}\left[\left(1+\frac{f_u}{2A}\right)^2 - 1\right] (2)$$

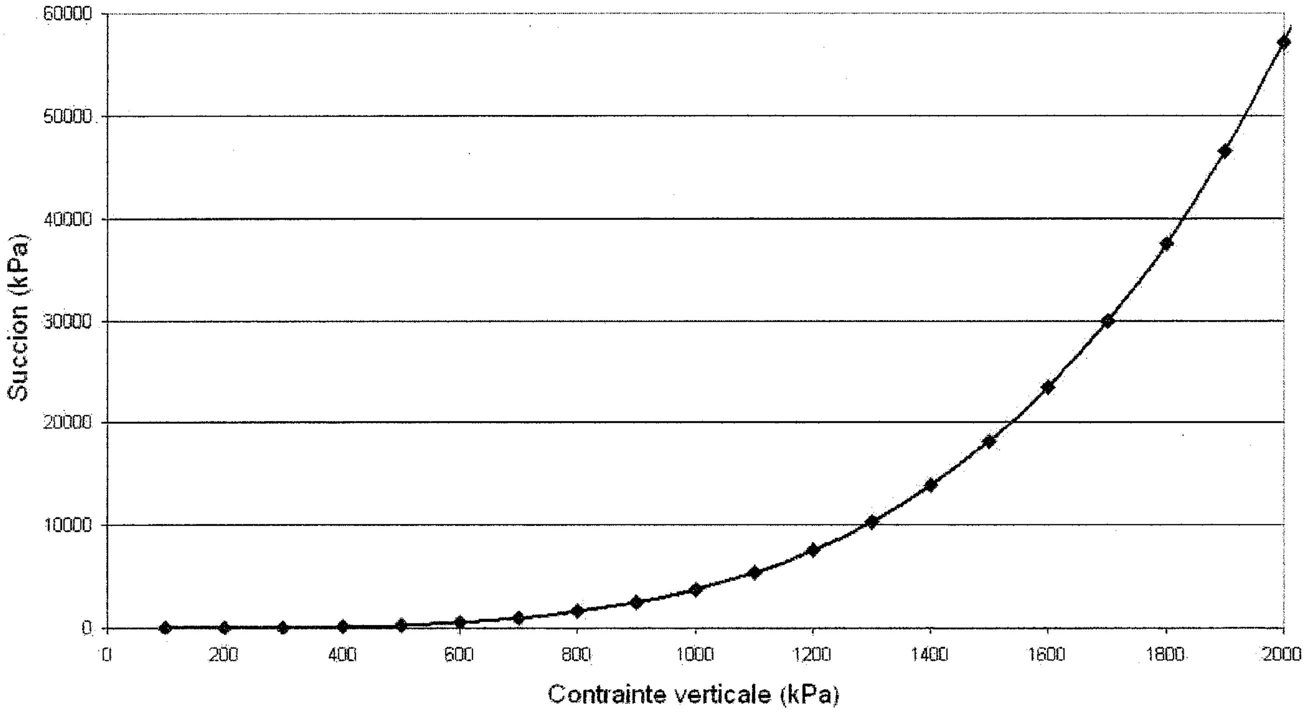
Principe du calcul : On se donne σ_V, on en déduit $\underbrace{\delta_{ext} \text{ et } f_{ext}}$
La formule (1) permet de calculer f_u, la formule (2) de calculer $-u$.
($\sigma_H = c^{te} = 100\,kPa$).

$\sigma_H = c^{te} = 100\,kPa$

ie on ↑ f_{ext}



Surface de charge en succion



5.13 Homework 3 - Problem 3

A varved clay consists of successive layers of silt and clay each of which has isotropic flow properties (see Figure 5.7).

1. Derive expressions for the average (gross) permeability of the varved clay system parallel (k_t) and perpendicular (k_n) to the direction of deposition.

2. Show that for any system of this type, $k_t \geq k_n \geq 0$.

3. Noting that the values of k_t and k_n are principal values of the permeability tensor, determine the components of the permeability tensor in a reference frame X' (see Figure 5.7).

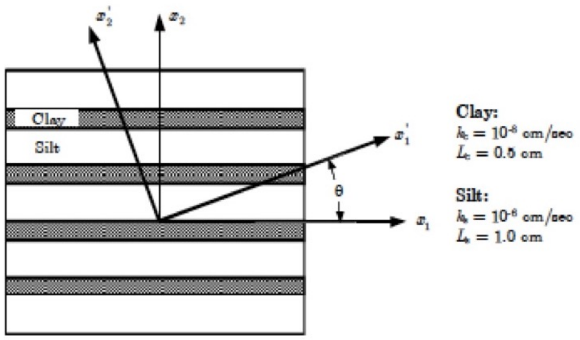


Figure 5.7 Flow through an element of varved clay.

Solution:

1. We note Δh_s (respectively Δh_c) the difference of pressure head from the top to the bottom of a silt layer (respectively, clay layer). The flow in the normal direction is the same across all layers, thus:

$$q_n = k_s \frac{\Delta h_s}{L_s} A = k_c \frac{\Delta h_c}{L_c} A$$

in which $A = B \times l$, where B is the width of the domain (in the plane of the sheet of paper) and l is the depth of the domain (in the direction orthogonal to the sheet of paper). Additioanally, the average normal permeability of the soil domain is defined as:

$$q_n = k_n \frac{\Delta \overline{h}}{L} A$$

in which $L = 5L_s + 4L_c$ (total height of the soil domain) and $\Delta \overline{h} = 5\Delta h_s + 4\Delta h_c$ (total difference of pressure head between the top and bottom of the soil domain). By combining the two above equations, we get:

$$k_n = \frac{(5L_s + 4L_c)}{5\frac{L_s}{k_s} + 4\frac{L_c}{k_c}}$$

The flow in the tangential direction is:

$$q_t = 5k_s \frac{\Delta h}{B} l \times L_s \; + 4k_c \frac{\Delta h}{B} l \times L_c$$

in which Δh is the difference of pressure head between the left and right hand sides of the soil domain. Additionally, the average tangential permeability of the soil domain is defined as:

$$q_t = k_t \frac{\Delta h}{B} l \times (5L_s + 4L_c)$$

By combining the two equations above, we get:

$$k_t = \frac{(5k_s L_s + 4k_c L_c)}{(5L_s + 4L_c)}$$

2. Since the given permeabilities k_s and k_c are positive, and the given lengths are positive, the two permeability components k_n and k_t are positive. Now, let us compare the two:

$$\frac{k_t}{k_n} = \frac{(5k_s L_s + 4k_c L_c)\left(5\frac{L_s}{k_s} + 4\frac{L_c}{k_c}\right)}{(5L_s + 4L_c)^2}$$

We have:

$$(5k_s L_s + 4k_c L_c)\left(5\frac{L_s}{k_s} + 4\frac{L_c}{k_c}\right) = 25L_s^2 + 16L_c^2 + 20L_s L_c\left(\frac{k_s}{k_c} + \frac{k_c}{k_s}\right)$$

Suppose $k_s = \alpha k_c$. Then:

$$(5k_s L_s + 4k_c L_c)\left(5\frac{L_s}{k_s} + 4\frac{L_c}{k_c}\right) = 25L_s^2 + 16L_c^2 + 20L_s L_c\left(\frac{\alpha^2 + 1}{\alpha}\right)$$

We have $\alpha^2 + 1 - 2\alpha = (\alpha - 1)^2 \geq 0$ so $\alpha^2 + 1 \geq 2\alpha$ and therefore, $\frac{\alpha^2+1}{\alpha} \geq 2$. As a result:

$$(5k_s L_s + 4k_c L_c)\left(5\frac{L_s}{k_s} + 4\frac{L_c}{k_c}\right) \geq 25L_s^2 + 16L_c^2 + 40L_s L_c = (5L_s + 4L_c)^2$$

We conclude that $\frac{k_t}{k_n} \geq 1$ and therefore:

$$k_t \geq k_n \geq 0$$

3. To calculate the components of the permeability tensor in the reference frame shown, we can apply the transformation equations that were established for stress transformation in 2D. Using Question 2 for the sign of $k_t - k_n$:

$$\begin{aligned}
k_{xx} &= \frac{k_n + k_t}{2} + \frac{k_t - k_n}{2}\cos(2\theta) \\
k_{yy} &= \frac{k_n + k_t}{2} - \frac{k_t - k_n}{2}\cos(2\theta) \\
k_{xy} &= \frac{k_t - k_n}{2}\sin(2\theta)
\end{aligned}$$

5.14 Homework 3 - Problem 4

The combination of the water momentum balance equation, the solid mass conservation equation and the water mass conservation equation provides:

$$\left(\frac{(\alpha-\phi)}{K_s}+\frac{\phi}{K_w^*}\right)\frac{D^s p_w}{Dt}+\alpha\, div\left(\frac{\partial \mathbf{u}}{\partial t}\right)=\frac{\mathbf{K_w}}{\mu_w}\nabla^2 p_w$$

... which can be solved for the fields of displacements $\mathbf{u}$ and water pore pressure p_w if combined with the momentum balance equation:

$$\mathbf{D_e} \vdots \nabla^2 \mathbf{u} - \alpha \nabla p_w = \mathbf{0}$$

Starting from these two governing equtions above, show that the 1D consolidation equation is expressed as:

$$\frac{\partial p_w}{\partial t}=\frac{k}{c_v}\frac{\partial^2 p_w}{\partial z^2}$$

with $\mathbf{K_w}=k\boldsymbol{\delta}$, $c_v=\mu_w(S+\alpha^2 m_v)$, $S=\left(\frac{(\alpha-\phi)}{K_s}+\frac{\phi}{K_w^*}\right)$ and $m_v=\frac{1}{K+\frac{4}{3}G}$. Assume that the solid skeleton has a linear isotropic elastic behavior.

Solution: We start with the governing equation:

$$\left(\frac{(\alpha-\phi)}{K_s}+\frac{\phi}{K_w^*}\right)\frac{D^s p_w}{Dt}+\alpha\, div\left(\frac{\partial \mathbf{u}}{\partial t}\right)=\frac{\mathbf{K_w}}{\mu_w}\nabla^2 p_w$$

Using the notations introduced in the problem:

$$S\frac{D^s p_w}{Dt}+\alpha\frac{\partial \epsilon_v}{\partial t}=\frac{k\boldsymbol{\delta}}{\mu_w}\nabla^2 p_w \tag{5.7}$$

The time derivative relative to the solid skeleton is:

$$\frac{D^s p_w}{Dt}=\frac{\partial p_w}{\partial t}+\nabla p_w\cdot \mathbf{v_s}$$

with, according to the constitutive laws adopted in this course:

$$\nabla p_w=\frac{K_w^*}{\rho_w}\nabla \rho_w$$

We assume that the permeability of the porous medium is high enough that we can neglect the gradient of water mass density. Therefore, in one dimension, equation 6.8 becomes:

$$S\frac{\partial p_w}{\partial t}+\alpha\frac{\partial \epsilon_{zz}}{\partial t}=\frac{k}{\mu_w}\frac{\partial^2 p_w}{\partial z^2} \tag{5.8}$$

For a linear isotropic elastic solid skeleton, we have:

$$\boldsymbol{\sigma}+\alpha p_w\boldsymbol{\delta}=\left(K-\frac{2}{3}G\right)\epsilon_v\boldsymbol{\delta}+2G\boldsymbol{\epsilon}$$

For a 1D consolidation test:

$$\sigma_{zz} + \alpha p_w = \left(K - \frac{2}{3}G\right)\epsilon_{zz} + 2G\boldsymbol{\epsilon}_{zz} = \left(K + \frac{4}{3}G\right)\epsilon_{zz} = \frac{1}{m_v}\epsilon_{zz} \tag{5.9}$$

Now combining equations 5.8 and 5.9:

$$\left(S + \alpha^2\, m_v\right)\frac{\partial p_w}{\partial t} + \alpha\, m_v \frac{\partial \sigma_{zz}}{\partial t} = \frac{k}{\mu_w}\frac{\partial^2 p_w}{\partial z^2}$$

In a consolidation test, the imposed stress does not vary over time, therefore:

$$\left(S + \alpha^2\, m_v\right)\frac{\partial p_w}{\partial t} = \frac{k}{\mu_w}\frac{\partial^2 p_w}{\partial z^2}$$

From which we deduce the required consolidation equation:

$$\frac{\partial p_w}{\partial t} = \frac{k}{\left(S + \alpha^2\, m_v\right)\mu_w}\frac{\partial^2 p_w}{\partial z^2} = \frac{k}{c_v}\frac{\partial^2 p_w}{\partial z^2}$$

5.15 Homework 3 - Problem 5

We subject a water-saturated solid specimen to a triaxial axisymmetric test. We assume that the state of stress in the specimen is uniform and that the loading platens at the top and bottom of the specimen are frictionless. Before applying any stress to the specimen, the pore pressure is uniform, and equal to p_0. The vertical stress applied on the top platen is noted σ_1, and the confining pressure applied on the lateral faces of the specimen is noted σ_3.

1. First consider that the test is undrained. Determine the stresses, pore pressures and strains in the specimen, in terms of σ_1, σ_3 and p_0.

2. Now consider that the test is drained. Repeat question 1, in the short-term and in the long-term.

Solution:

1. The total stress is given (boundary conditions), so we are looking for the pore pressure and the strains in the specimen (both are assumed to be uniform in the specimen). In undrained conditions, the specimen does not change in volume, since the solid grains and the water are both considered incompressible. There is no change of pore volume during the test. Therefore, the pore pressure of the specimen increases by the same amount as the mean stress applied at the boundary. Under the loading conditions, the pore pressure is therefore:

$$p_w = p_0 - \frac{1}{3}(\sigma_1 + 2\sigma_3)$$

in which compression counted negative. To find the strains, we can use Biot's constitutive relationships, which, in the absence of total volumetric deformation, boil down to:

$$\sigma_1 + \alpha p_w = 2G\epsilon_1$$
$$\sigma_3 + \alpha p_w = 2G\epsilon_3$$

in which G is the shear modulus of the specimen. The two equations above can be solved for ϵ_1 and ϵ_3.

2. Like for the undrained tests, the pore pressure in a drained test in the short term is:

$$p_w = p_0 - \frac{1}{3}(\sigma_1 + 2\sigma_3)$$

in which compression counted negative. The strains are obtained by solving the following system of equations, which comes form the constitutive relationships:

$$\sigma_1 + \alpha p_w = \left(K - \frac{2}{3}G\right)(\epsilon_1 + 2\epsilon_3) + 2G\epsilon_1$$
$$\sigma_3 + \alpha p_w = \left(K - \frac{2}{3}G\right)(\epsilon_1 + 2\epsilon_3) + 2G\epsilon_3$$

in which G is the shear modulus of the specimen and K is the drained bulk modulus of the specimen (solid grains only). In the long term, the excess pore pressure is dissipated and comes back to its initial value at equilibrium:

$$p_w = p_0$$

Then, the strain components can be obtained by solving the following system of equations, which comes from the constitutive relationships:

$$\sigma_1 + \alpha p_w = \left(K - \frac{2}{3}G\right)(\epsilon_1 + 2\epsilon_3) + 2G\epsilon_1$$
$$\sigma_3 + \alpha p_w = \left(K - \frac{2}{3}G\right)(\epsilon_1 + 2\epsilon_3) + 2G\epsilon_3$$

5.16 Homework 4 - Problem 1

1. Let us suppose that you are performing a triaxial compression test, in which $\sigma_{11} = \sigma_I$ and $\sigma_{22} = \sigma_{33} = \sigma_{II}$. Write the stress/strain relationship that governs the mechanical behavior of the solid skeleton during the test.

2. Let us suppose that you are performing a drained isotropic compression test, in which $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_I$, $p = 0$. We give the following constitutive relationship, which stems from thermodynamic principles:

$$\delta\Phi = \alpha Tr(\boldsymbol{\epsilon}) + \frac{K_s}{\alpha - \Phi_0} p_w$$

in which Φ is the current porosity, Φ_0 is the initial porosity, K_s is the bulk modulus of the solid phase, and α is the Biot coefficient. Write the stress/strain and porosity/pressure relationships for the drained isotropic compression test.

3. Now suppose that you are performing an undrained isotropic compression test, in which $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_I$, $\Delta v_f = 0$ (where v_f is the volume of the fluid in the specimen, assumed to be incompressible). Derive the stress/strain relationship and the stress/pore pressure relationship for the undrained isotropic compression test.

4. Discuss possible experimental plans to find the bulk and shear moduli of the solid skeleton, as well as the poroelasticity coefficients α and $N = K_s/(\alpha - \Phi_0)$.

Solution:

1. Linear isotropic behavior of the solid skeleton:

$$\boldsymbol{\sigma} + \alpha\, p_w \;=\; \left(K - \frac{2}{3}G\right) \epsilon_v \boldsymbol{\delta} \;+ 2G\boldsymbol{\epsilon}$$

The deviatoric part of the equation above is:

$$\boldsymbol{s} \;=\; 2G\boldsymbol{e} \tag{5.10}$$

where $\boldsymbol{s}$ is the deviatoric stress and $\boldsymbol{e}$ is the deviatoric strain. For the triaxial loading described:

$$\boldsymbol{s} \;=\; \sigma_I \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_{II} \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \sigma_{II} \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 - \frac{1}{3}(\sigma_I + 2\sigma_{II})(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3)$$

$$\boldsymbol{s} \;=\; \frac{2}{3}(\sigma_I - \sigma_{II}) \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \frac{1}{3}(\sigma_{II} - \sigma_I) \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \frac{1}{3}(\sigma_{II} - \sigma_I) \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

Similarly, for the strain:

$$\boldsymbol{e} \;=\; \frac{2}{3}(\epsilon_I - \epsilon_{II}) \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \frac{1}{3}(\epsilon_{II} - \epsilon_I) \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \frac{1}{3}(\epsilon_{II} - \epsilon_I) \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

Taking the component 11 of equation 5.10:

$$\sigma_I - \sigma_{II} = 2G(\epsilon_I - \epsilon_{II})$$

2. For the solid skeleton, with $p_w = 0$:

$$\boldsymbol{\sigma} = \left(K - \frac{2}{3}G\right) \epsilon_v \boldsymbol{\delta} \;+ 2G\boldsymbol{\epsilon}$$

Since the loading is isotropic:

$$\boldsymbol{\sigma} \;=\; \sigma_1 \boldsymbol{\delta} \;=\; \left(K - \frac{2}{3}G\right) (3\epsilon_I)\boldsymbol{\delta} \;+ 2G\epsilon_I \boldsymbol{\delta}$$

And finally:

$$\sigma_I \;=\; 3K\epsilon_I$$

For the fluid:

$$\delta\Phi = \alpha Tr(\boldsymbol{\epsilon}) + \frac{K_s}{\alpha - \Phi_0} p_w$$

For a drained isotropic test, $p_w = 0$ and:

$$\delta\Phi = 3\alpha\epsilon_I$$

3. For the solid skeleton, in direction 1:

$$\sigma_I + \alpha p_w \; = \; \left(K - \frac{2}{3}G\right) \times 3\epsilon_I + 2G \times \epsilon_I \; = \; 3K\epsilon_I \tag{5.11}$$

   For the fluid, in direction 1, we have $\delta\Phi = 0$ since $\Delta v_f = 0$ and since the fluid is incompressible. Therefore:

$$0 \; = 3\alpha\epsilon_I + \frac{K_s}{\alpha - \Phi_0} p_w$$

   from which we get:

$$p_w \; = \; -\frac{3\alpha(\alpha - \Phi_0)}{K_s}\epsilon_I \tag{5.12}$$

   Combining equations 5.11 and 5.12, we get:

$$\sigma_I \; = \; \left[K + \frac{(\alpha - \Phi_0)\alpha^2}{K_s}\right] \times 3\epsilon_I \tag{5.13}$$

   We note K_u the undrained bulk modulus:

$$K_u = K + \frac{(\alpha - \Phi_0)\alpha^2}{K_s}$$

   Now combining equations 5.12 and 5.13, we get the sought stress/pore pressure relationship:

$$p_w \; = \; -\alpha\frac{(\alpha - \Phi_0)}{K_s}\frac{1}{K_u}\sigma_I$$

4. In each of the three experiments above, we control stress and measure strain (or vice versa), and we control the volume of the fluid (or porosity, in the case of an incompressible fluid), and we measure the pore pressure (or vice versa). Hence, each of the five equations established in the previous questions provides a relationship from which poro-elasticity coefficients can be calculated. In particular, the triaxial compression test provides:

$$\sigma_I - \sigma_{II} = 2G(\epsilon_I - \epsilon_{II})$$

   which allows calculating the shear modulus of the solid skeleton. The drained isotropic compression test provides:

$$\sigma_I \; = \; 3K\epsilon_I$$
$$\delta\Phi = 3\alpha\epsilon_I$$

   which allows one to calculate the bulk modulus of the solid skeleton and the Biot's coefficient. Lastly, the undrained isotropic compression test provides:

$$p_w \; = \; -\alpha\frac{(\alpha - \Phi_0)}{K_s}\frac{1}{K_u}\sigma_I$$

with:

$$K_u = K + \frac{(\alpha - \Phi_0)\alpha^2}{K_s}$$

which, after some intermediate calculations, allows one to find the coefficient N.

5.17 Homework 4 - Problem 2

Polluted water is to be flushed from a confined aquifer of height L and hydraulic cnductivity k (see Figure 5.8). The method employed is to pump water out of well A. The water is then treated (off-site) and successively pumped back (at the same rate) into the aquifer well B, located at a distance $20d$ (where d is the diameter of the well bore). Assuming that the initial head in the aquifer is h_0, and that the adjacent layers are impervious:

1. Show that for a single well, the steady-state piezometric head can be related to the rate of pumping through the following expression:

   $$h = h_w \pm \frac{Q}{2\pi\, L\, k}\, ln\left(\frac{2r}{d}\right)$$

   where h_w is the water elevation and d is the diameter of the well.

2. On the basis of the above, sketch the equipotential lines and the flow net for the steady flow between wells for the case of L = 20m, k = 10×10^{-6} cm/s and d = 0.4m.

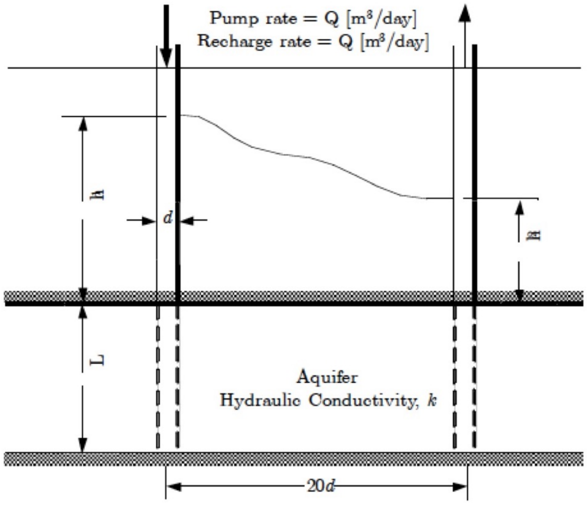


Figure 5.8 Piezometric heads due to pumping and recharge from a confined aquifer.

Solution:

1. Flow to a single well (in m^3/s):

$$\boldsymbol{q} = A \times \boldsymbol{v} = -k \times A \times \boldsymbol{\nabla} p$$

For a radial flow at a distance r from the center of a single well of length L, the area through which the flow happens is $A = 2\pi\, r\, L$ and therefore:

$$q(r) = -2\pi\, r\, L\, k\, \frac{dh}{dr}$$

In steady state, $q(r) = \pm Q$, (+Q if water is injected at a flow rate of Q, -Q if water is pumped at a flow rate of Q), and so:

$$\pm Q = 2\pi\, r\, L\, k\, \frac{dh}{dr}$$

We thus have:

$$dh = \pm \frac{Q}{2\pi\, L\, k} \frac{dr}{r}$$

Integrating between $r = d/2$ at the circumference of the well, and r:

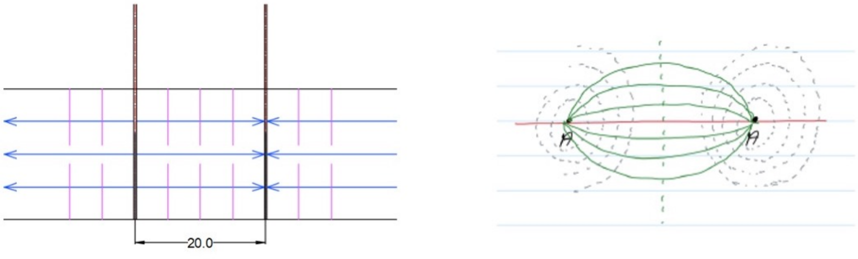
$$h(r) - h_w = \pm \frac{Q}{2\pi\, L\, k} \ln\left(\frac{2r}{d}\right)$$

which is the expected result.

2. For two wells (injection well B and extraction well A), the water head is obtained by superposition:

$$h = h_A + h_B = 2\, h_0 - \frac{Q}{2\pi\, L\, k} \ln\left(\frac{2r_B}{d}\right) + \frac{Q}{2\pi\, L\, k} \ln\left(\frac{2r_A}{d}\right) = 2\, h_0 + \frac{Q}{2\pi\, L\, k} \ln\left(\frac{r_A}{r_B}\right)$$

where r_A is the radial distance to extraction well A, and r_B is the radial distance to injection well B. The equipotential lines can be drawn from the knowledge of h. The flow lines are orthogonal to the equipotential lines, see Figure 5.9.



Jiaojun Liu's drawing (2020)

Luis Vergaray's drawing (2020)

Figure 5.9 Flow net for the problem with two injection wells (Problem 5.17).

5.18 Homework 4 - Problem 3

In the construction of a large excavation, the inflow of water to the excavation is prevented by installing an impervious slurry wall which extends to the level of impervious rock (see Figure 5.10). Outside the excavation, the heads in the soil are to be reduced by pumping from a series of wells. Assuming a steady rate of pumping Q_w = 600 m^3/hr from a single well, calculate the flow field in the soil assuming that the flow into the well is horizontal and that the soil is homogeneous and isotropic. The piezometric head in the far-field is h_0 = 100 m. What is the steady state head recorded at the observation well? Sketch the flow field for the case when the horizontal transmissivity of the aquifer is T = 4,000 m^2/day. *Treat the well as a line source in the last part of this problem.*

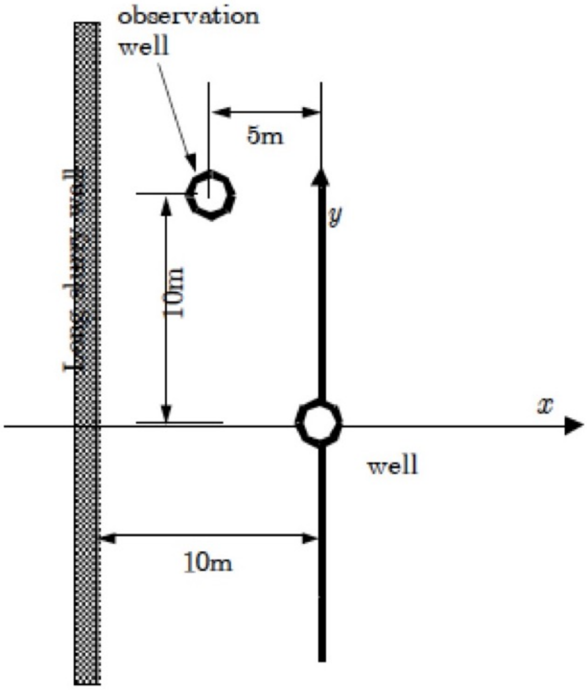


Figure 5.10 Horizontal flow field for steady-state pumping.

Solution: The slurry wall acts like an axis of symmetry. The system is equivalent to two puping wells that are 20 meters apart, wit no slurry wall. First consider a single well. In steady state, we have:

$$\boldsymbol{Q} = k\boldsymbol{i}A = 2\pi\, k\, r\, h\, \frac{dh}{dr}$$

After rearranging and integrating on both sides:

$$\frac{Q}{2\pi\, k} \int_{r_w}^{r} \frac{dr}{r} = \int_{h_w}^{h} d\, dh$$

$$h^2 - h_w^2 = \frac{Q}{\pi\, k} \ln\left(\frac{r}{r_w}\right)$$

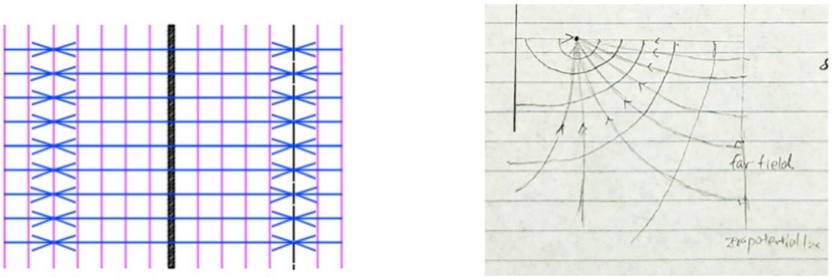
where r_w is the radius of the well, r is the distance to the well, h_w is the water head in the well and h is the water head at the observation point (x,y), given as:

$$h = \sqrt{\frac{Q}{\pi\, k} \ln\left(\frac{r}{r_w}\right) + h_w^2}$$

For one well, the pressure head drop at point (x,y) is $\Delta h = h_0 - h$. For two well A and B (20 meters apart): $\Delta h = \Delta h_A + \Delta h_B$. Then:

$$\Delta h = 2h_0 - \sqrt{\frac{Q_A}{\pi\, k} \ln\left(\frac{r_A}{r_w}\right) + h_A^2} - \sqrt{\frac{Q_B}{\pi\, k} \ln\left(\frac{r_B}{r_w}\right) + h_B^2}$$

with $Q_A = Q_B$, $h_A = h_B$, but $r_A \neq r_B$. If A is the well on the same side as the observation well O, then $r_A^2 = (x_A - x_O)^2 + (y_A - y_O)^2 = 25 + 100 = 125m^2$. If B is the well on the other side as the observation well, then $r_B^2 = (x_B - x_O)^2 + (y_B - y_O)^2 = 15^2 + 100 = 325m^2$. Assuming a certain value for $h_A = h_B$ and for r_w, one can then calculate the pressure head drop at the observation well, and then the pressure head at the observation well is $h_0 + \Delta h$. We plot the equipotential lines from the knowledge of h, where h is calculated with an assumed value of $h_A = h_B$ and an assumed value of r_w. The flow lines are orthogonal to the equipotential lines, see Figure 5.11.



Jiaojun Liu's drawing (2020)

Haozhou He's drawing (2020)

Figure 5.11 Flow net for the problem with an injection well and a slurry wall (Problem 5.18).

CHAPTER 6

FINITE ELEMENT METHOD FOR PORO-ELASTICITY

PROBLEMS

6.1 Consider plane wall of thickness L, initially at a uniform temperature T_0, which has both surfaces suddenly exposed to a fluid at temperature T_∞. The governing differential equation is:

$$k\frac{\partial^2 T}{\partial x^2} = \rho c_0 \frac{\partial T}{\partial t}$$

The initial condition is $T(x,0) = T_0$ and we consider two sets of boundary conditions:

Set 1: $T(0,t) = T_\infty, \quad T(L,t) = T_\infty$

Set 2: $T(0,t) = T_\infty, \quad \left[k\frac{\partial T}{\partial x} + \beta(T - T_\infty\right]_{x=L} = 0$

Approximate the solution with two linear Finite Elements. Solve for the unknown temperatures and heat fluxes.

Solution: See the solution in the next 4 pages of notes.

Step 1 : normalization:
$$\frac{\partial T}{\partial t} - \frac{k}{\rho c_0}\frac{\partial^2 T}{\partial x^2} = 0$$

$$\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial (x/L)^2} \times \frac{1}{L^2} = 0$$

$$\frac{L^2}{\alpha}\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial (x/L)^2} = 0$$

$$\frac{\partial T}{\partial \left(\frac{\alpha}{L^2}t\right)} - \frac{\partial^2 T}{\partial \left(\frac{x}{L}\right)^2} = 0$$

$$\frac{\partial T}{\partial t'} - \frac{\partial^2 T}{\partial (x')^2} = 0 \, , \quad \begin{cases} t' = \frac{\alpha}{L^2} t \\ x' = \frac{x}{L} \end{cases}$$

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty} \Rightarrow \begin{cases} \frac{\partial \theta}{\partial t'} - \frac{\partial^2 \theta}{\partial x'^2} = 0 \\ \theta(x, 0) = 1 \end{cases}$$

Step 2 : FE model

Parabolic equation $\Rightarrow$ $\theta(x,t) = U(x)\, e^{-\lambda t}$
$$-\lambda U(x) - \frac{d^2 U}{dx^2} = 0$$

$$\left([K^e] - \lambda [M^e]\right)\{U^e\} = \{Q^e\}$$

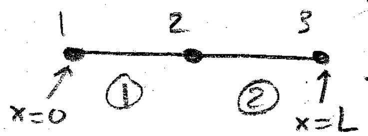
$$Q_i^e = \pm \left.\frac{dU}{dx}\right|_{x = x_i}$$

$$K_{ij}^e = \int_{x_a}^{x_b} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx \, , \quad M_{ij}^e = \int_{x_a}^{x_b} \phi_i(x)\phi_j(x)\,dx$$

For 1 linear element:

$$K^e_{ij} = \frac{1}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [M^e] = \frac{h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Step 3: Assembling: For a uniform mesh of two linear elements:



elt \ dof	1	2
1	1	2
2	2	3

$$\left(2 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\lambda}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

Step 4: Boundary Conditions

① $T(0,t) = T(L,t) = T_\infty$

$\theta(0,t') = \theta(1,t) = 0$

$U(0) = U(1) = 0$

$\Rightarrow \begin{cases} U_1 = U_3 = 0 \\ Q_2 = 0 \quad \text{(no concentrated heat flux)} \end{cases}$

② $T(0,t) = T_\infty, \quad k\dfrac{\partial T}{\partial x} = -\beta(T - T_\infty)$ at $x = L$

$\theta(0,t') = 0, \quad \left.\dfrac{\partial \theta}{\partial x'}\right|_{x'=1} = -\dfrac{\beta L}{k}\,\theta(1,t')$

$U(0) = 0, \quad \left.\dfrac{dU}{dx'}\right|_{x'=1} = -\dfrac{\beta L}{k}\,U(1)$

$\Rightarrow \begin{cases} U_1 = 0, \quad Q_3 = -\dfrac{\beta L}{k}\,U_3 \\ Q_2 = 0 \quad \text{(no concentrated heat flux)} \end{cases}$

Step 5: Condensation / Resolution

① $\left(4 - \frac{\lambda}{3}\right) U_2 = 0$

Only one solution for the eigenvalue:
supposing $U_2 \neq 0$ (if $U_2 = 0$, we have $U_1 = U_2 = U_3 = 0$
thus $U(x) = 0$, nonsense) : $\lambda = 12$.

Associated mode shape:

- if $x \in]0; \frac{L}{2}[$: $U(x') = U_2\, \phi_2^{(1)}(x') = U_2 \frac{x'}{1/2} = 2U_2 x'$
- if $x \in]\frac{L}{2}; L[$: $U(x') = U_2\, \phi_1^{(2)}(x') = U_2 \left(1 - \frac{x' - 1/2}{1/2}\right) = U_2(2 - 2x')$

$\theta(x', t) = U(x')\, e^{-12t}$

$$\theta(x', 0) = 1 \Rightarrow \begin{cases} U_2\, \phi_2^{(1)}(x') = 1 & \text{on }]0, \frac{L}{2}[\\ U_2\, \phi_1^{(2)}(x') = 1 & \text{on }]\frac{L}{2}, L[\end{cases}$$

To satisfy the initial condition at the nodes where no b.c. is applied:
$U_2\, \phi_2^{(1)}(1/2) = U_2\, \phi_1^{(2)}(1/2) = 1 \Rightarrow U_2 = 1$.

$$\Rightarrow U_2 = 1 \quad \text{and} \quad \theta(x', t) = \begin{cases} 2x'\, e^{-12t} & \text{if } x \in]0, \frac{L}{2}[\\ 2(1-x')\, e^{-12t} & \text{if } x \in]\frac{L}{2}, L[\end{cases}$$

NB: exact solution of the type:

$$\theta(x', t') = C_1 \sin \alpha_1 x'\, e^{-\lambda_1 t}$$

with $\lambda_1 = 12$, and $C_1 \sin \alpha_1 x' = \begin{cases} U_2\, \phi_2^{(1)}(x') & \text{on }]0, \frac{L}{2}[\\ U_2\, \phi_1^{(2)}(x') & \text{on }]\frac{L}{2}, L[\end{cases}$

$\Rightarrow$ Number of eigenvalues determined by the FEM = number of degrees of freedom (after condensation)

② $U_1=0,\ Q_2=0,\ Q_3=-\frac{\beta L}{k}U_3$:

$$\left(\begin{bmatrix}4 & -2\\ -2 & 2\end{bmatrix}-\frac{\lambda}{12}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}U_2\\ U_3\end{Bmatrix}=\begin{Bmatrix}0\\ -\frac{\beta L}{k}U_3\end{Bmatrix}$$

$$\left(\begin{bmatrix}4 & -2\\ -2 & 2+\frac{\beta L}{k}\end{bmatrix}-\frac{\lambda}{12}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}U_2\\ U_3\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}\quad (*)$$

$\Rightarrow$ 2 degrees of freedom (U_2, U_3), 2 eigenvalues:

$$\lambda_1 \simeq 4.49\ ,\ \lambda_2 \simeq 36.7$$

$\Rightarrow$ 2 eigenvectors (obtained after substituting λ for λ_1 and λ_2 in $(*)$):

For λ_1: $\begin{Bmatrix}U_2\\ U_3\end{Bmatrix}_1=\begin{Bmatrix}1\\ 1.05\end{Bmatrix}$ (if U_2 normalized)

For λ_2: $\begin{Bmatrix}U_2\\ U_3\end{Bmatrix}_2=\begin{Bmatrix}1\\ -1.63\end{Bmatrix}$ (if U_2 normalized)

NB1: Each mode shape satisfies bc

NB2: Initial condition satisfied if $\theta_h(x,0)=1$ at nodes (ie $T=T_0$)

Noting i the number of the mode shape:

$$\begin{cases}\left.\sum_{i=1}^{2} e^{-\lambda_i t}\, U_2^{(i)}\, \phi_2^{(1)}(x)\right|_{t=0}=1 \text{ on }]0;\tfrac{L}{2}[\quad \left(\begin{matrix}\text{bc imposes } \theta_h(0,0)=0\\ \text{at } x=0\end{matrix}\right)\\ \left.\sum_{i=1}^{2} e^{-\lambda_i t}\, U_2^{(i)}\left(\phi_1^{(2)}(x)+\frac{U_3^{(i)}}{U_2^{(i)}}\phi_2^{(2)}(x)\right)\right|_{t=0}=1 \text{ on }]\tfrac{L}{2};L[\end{cases}$$

Using the initial condition at $x=\frac{L}{2}$ (element 1) and $x=L$ (element 2):

$$\Rightarrow \begin{cases}U_2^{(1)}+U_2^{(2)}=1\\ U_3^{(1)}+U_3^{(2)}=1\end{cases}\Rightarrow\begin{cases}U_2^{(1)}+U_2^{(2)}=1\\ 1.05\,U_2^{(1)}-1.63\,U_2^{(2)}=1\end{cases}$$

$$\Rightarrow\begin{cases}-2.68\,U_2^{(2)}=-0.05\\ U_2^{(1)}=1-U_2^{(2)}\end{cases}$$

$$\Rightarrow U_2^{(2)}\simeq 0.019\ ,\ U_2^{(1)}\simeq 0.981$$

$$\Rightarrow U_3^{(2)}=-0.031\ ,\ U_3^{(1)}=1.030$$

6.2 Consider a uniform beam of rectangular cross section (B×H), fixed at $x = 0$ and free at $x = L$. We use the Euler-Bernouilli beam theory. Neglecting the rotary inertia term, the governing equation for beam deflection is:

$$\rho A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right] = q(x,t)$$

The boundary conditions are:

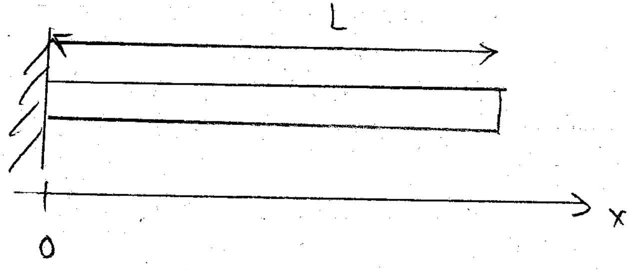
$$W(0) = 0, \quad \frac{dW}{dx} = 0, \quad \left[EI\frac{d^2W}{dx^2}\right]_{x=L} = 0, \quad \left[EI\frac{d^3W}{dx^3}\right]_{x=L} = 0$$

Determine the first two flexural frequencies of the beam by using the minimum number of Euler-Bernouilli beam elements.

Solution: See the solution in the next 2 pages of notes.

Example (6.1.2. p 305) :

Cantilever Beam modeled one Euler - Bernouilli Beam Element :



• x=0: Fixed.

$$\begin{cases} w(0,t)=0 \Rightarrow W(0)=0 \\ \theta(0,t)=0 \Rightarrow \left.\frac{dW}{dx}\right|_{x=0}=0 \end{cases}$$

• x = L: free end:

$$\begin{cases} V(L,t)=0 \\ M(L,t)=0 \end{cases}$$

Ignoring the effects of rotary inertia ρI, and with constant E, I :

$$\begin{cases} EI\, d^3W/dx\big|_{x=L}=0 \\ EI\, d^2W/dx\big|_{x=L}=0 \end{cases}$$

With 1 cubic EB element, applying the b.c. leads to:

$$\left([K^e]-\omega^2[M^e]\right)\begin{Bmatrix}0\\0\\\Delta_3=W_2\\\Delta_4=\theta_2\end{Bmatrix}=\begin{Bmatrix}Q_1=-V_1\\Q_2=-M_1\\0\\0\end{Bmatrix}$$

Neglecting rotary inertia (ρI), we obtain, after condensation:

$$\underbrace{\left(\overbrace{\frac{2EI}{L^3}\begin{bmatrix}6&3L\\3L&2L^2\end{bmatrix}}^{[K'^e]}-\omega^2\overbrace{\frac{\rho AL}{420}\begin{bmatrix}156&22L\\22L&4L^2\end{bmatrix}}^{[M'^e]}\right)}_{[C^e]}\begin{Bmatrix}\Delta_3\\\Delta_4\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Non trivial solution for $\det[C^e]=\det\left([K'^e]-\omega^2[M'^e]\right)=0$

$\det[C^e]=0 \Longleftrightarrow 15120-1224\,\omega^2+\omega^4=0$ (see p. 307)

$\Longrightarrow \omega_1^{\ }, \omega_2$: 2 natural vibration frequencies.
(take the positive roots)

$$\omega_1=\frac{3.53}{L^2}\sqrt{\frac{EI}{\rho A}} \qquad \omega_2=\frac{34.81}{L^2}\sqrt{\frac{EI}{\rho A}}$$

Eigenvectors (mode shapes):

1] $$\left([K'_e] - \omega_1^2 [M'_e]\right) \begin{Bmatrix} \Delta_3^{(1)} \\ \Delta_4^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow -0.7259L \, \Delta_3^{(1)} = \Delta_4^{(1)}$$ (see p. 3c7)

2] $$\left([K'_e] - \omega_2^2 [M'_e]\right) \begin{Bmatrix} \Delta_3^{(2)} \\ \Delta_4^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

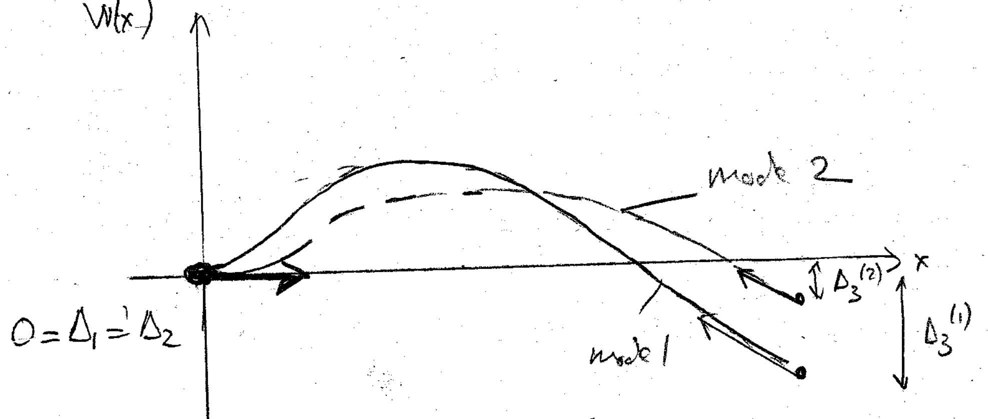
$$\Rightarrow -0.1312L \, \Delta_3^{(2)} = \Delta_4^{(2)}$$ (see p. 307)

$$\Rightarrow \quad \begin{Bmatrix} \Delta_3^{(1)} \\ \Delta_4^{(1)} \end{Bmatrix} = \begin{Bmatrix} -0.7259L \\ 1 \end{Bmatrix} \alpha_1$$

$$\begin{Bmatrix} \Delta_3^{(2)} \\ \Delta_4^{(2)} \end{Bmatrix} = \begin{Bmatrix} -0.1312L \\ 1 \end{Bmatrix} \alpha_2$$

α_1, α_2 : constants determined by the initial conditions (see example with a parabolic equation)

$$\Rightarrow \text{Mode shapes: } \begin{cases} w^{(1)}(x) = \Delta_3^{(1)} \phi_3(x) + \Delta_4^{(1)} \phi_4(x) \\ w^{(2)}(x) = \Delta_3^{(2)} \phi_3(x) + \Delta_4^{(2)} \phi_4(x) \end{cases}$$



6.3 Consider a uniform bar of cross-sectional area A, modulus of elasticity E, mass density m, and length L. The axial displacement under the action of time-dependent axial forces is governed by the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad a = \left(\frac{E}{m}\right)^{1/2}$$

Determine the transient response [i.e., find u(x, t)] of the bar when the end x = 0 is fixed and the end x = L is subjected to a force P_0. Assume zero initial conditions. Use one linear element to approximate the spatial variation of the solution, and solve the resulting ordinary differential equation in time exactly to obtain:

$$u_2(x,t) = \frac{P_0 L}{AE}\frac{x}{L}(1 - cos\alpha t), \quad \alpha = \sqrt{3}\frac{a}{L}$$

Solution: After writing the variational formulation and discretizing in space with Lagrange linear polynomial, we obtain the following matrix equation:

$$\frac{EA}{h}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} + \frac{mAh}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

The boundary conditions are $U_1 = 0$ (fixed end at $x = 0$) and $Q_2 = P_0$ (force imposed at $x = L$). The condensed equation is:

$$\ddot{U}_2(t) + \alpha^2 U_2(t) = \frac{3P_0}{mAh}$$

With the following initial conditions: $U_2(0) = 0$, $\dot{U}_2(0) = 0$, in which $\alpha = \sqrt{3E/(mh^2)}$. The solution is of the form:

$$U_2(t) \; = \; A\, cos\alpha t \; + B\, sin\alpha t \; + C$$

Using the initial conditions and the governing equation, we obtain A+C = 0, B = 0, and C = P_0h/EA. The final solution is:

$$u(x,t) \; = \; \sum_{i=1}^{2} U_i(t)\psi_i(x) = (Acos\alpha t \; + C)\, \psi_2(x) \; = \; \frac{P_0 h}{EA}(1 - cos\alpha t)\frac{x}{h}$$

6.4 Consider a simply supported beam (of Young's modulus E, mass density ρ, area of cross section A, second moment of area about the axis of bending I, and length L) with an elastic support at the center of the beam (see Figure 6.1). Determine the fundamental natural frequency using the minimum number of Euler-Bernoulli beam elements. It is reminded that in Euler-Bernouilli beam theory, if the rotary inertia term is neglected, the governing equation for beam deflection is:

$$\rho A \frac{\partial^2 w}{\partial t^2} \; + \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right] \; = q(x,t)$$

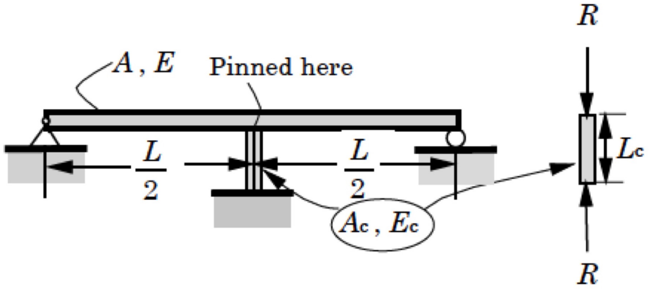


Figure 6.1 Beam vibration problem

Solution: Under the assumption of space and time separation, the solution of the hyperbolic equation of beam vibration can be written in the form: $w(x,t) = W(x)e^{i\omega t}$. After simplifying the equation by dividing by the time exponent, writing the variational formulation and discretizing in space, we obtain the following elementary equations:

$$\left([K^e] - \omega^2[M^e]\right)\{W^e\} = \{F^e\} + \{Q^e\}$$

In which:

$$\begin{aligned}
K_{ij}^e &= \int_{x_e}^{x_{e+1}} EI\frac{d^2\Psi_i^e}{dx^2}\frac{d^2\Psi_j^e}{dx^2}dx \\
M_{ij}^e &= \int_{x_e}^{x_{e+1}} \rho A\Psi_i^e(x)\Psi_j^e(x)dx \\
F_i^e &= \int_{x_e}^{x_{e+1}} \Psi_i^e(x)Q(x)dx \\
Q_i^e &= EI\left[\frac{d\Psi_i^e}{dx}\frac{d^3w}{dx^3}\right]_{x_e}^{x_{e+1}} + EI\left[\Psi_i^e(x)\frac{d^2w}{dx^2}\right]_{x_e}^{x_{e+1}}
\end{aligned}$$

In which Ψ_i denotes a Hermite interpolation function (for an Euler-Bernouilli beam element). In order to determine the natural frequencies of the beam, we actually seek to solve the equation for ω, in the absence of external action applied (i.e., free vibrations):

$$\left([K] - \omega^2[M]\right)\{W\} = \{0\}$$

The problem is symmetric, so we model only half the beam with the FEM. For the left half of the beam (between $x = 0$ and $x = L/2$), only one Euler-Bernouilli beam element is needed. There are four degrees of freedom: the deflection at nodes 1 and 2 (W_1^e and W_3^e respectively) and the angle of the deflection slope at nodes 1 and 2 (W_2^e and W_4^e respectively). The corresponding secondary variables are the shear force at nodes 1 and 2 (Q_1^e and Q_3^e respectively) and the bending moment at nodes 1 and 2 (Q_2^e and Q_4^e respectively). At node 1, the deflection is zero (W_1^e=W_1=0) and the bending moment is zero (Q_2^e=Q_2=0). At node 2, the deflection depends on the elastic properties of the support ($W_3^e = W_3 = -Q_3L_c/(E_cA_c)$; with $Q_3 = Q_3^{(e=1)} + Q_1^{(e=2)} = 2Q_3^{(e=1)}$), and the slope angle is zero (W_4^e=W_4=0). The 4×4 system of elementary equations can be condensed into a 2×2 system of equations. After computing the integrals with Hermite interpolation polynomials, we get:

$$\left(\frac{2EI}{h^3}\begin{bmatrix} 2h^2 & 3h \\ 3h & 6 \end{bmatrix} - \omega^2\frac{\rho Ah}{420}\begin{bmatrix} 4h^2 & -13h \\ -13h & 156 \end{bmatrix}\right)\begin{Bmatrix} W_2 \\ W_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -cW_3 \end{Bmatrix}$$

In which: $h = L/2$, $c = (E_cA_c)/(2L_c)$. After rearranging:

$$\left(\begin{bmatrix} 2h^2 & 3h \\ 3h & 6+c \end{bmatrix} - \lambda\begin{bmatrix} 4h^2 & -13h \\ -13h & 156 \end{bmatrix}\right)\begin{Bmatrix} W_2 \\ W_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

In which $\lambda = \omega^2\frac{\rho Ah^4}{840EI}$. We find the natural frequencies (ω) by solving the following equation for λ:

$$\det\left(\begin{bmatrix} 2h^2 & 3h \\ 3h & 6+c \end{bmatrix} - \lambda\begin{bmatrix} 4h^2 & -13h \\ -13h & 156 \end{bmatrix}\right) = 0$$

The characteristic polynomial is:

$$455\lambda^2 - 2(129+c)\lambda + (3+2c) = 0$$

The two roots are positive and define eigenfrequencies:

$$\lambda = \frac{129+c}{910} \pm \frac{1}{910}\sqrt{(129+c)^2 - 455 \times (3+2c)}$$

The fundamental frequency is the smallest root:

$$\omega_1 = \frac{129+c}{910} - \frac{1}{910}\sqrt{(129+c)^2 - 455 \times (3+2c)}$$

6.5 Consider the transient heat conduction problem governed by the following equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1$$

with boundary conditions:

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = 0$$

and initial condition:

$$u(x,0) = 1$$

where u is the non-dimensionalized temperature. Discuss the stability of the FEM model for one linear element and for two linear elements.

Solution: See the solution in the next 2 pages of notes.

f) Example: problem 6.2.1. p 328

Heat Transfer Problem:
$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0, & 0<x<1 \\ u(0,t)=0 \\ \dfrac{\partial u}{\partial x}(1,t)=0 \\ u(x,0)=1 \end{cases}$$

① Space discretization: 1 linear element:
$$u(x,t) \simeq u_1(t)\,\varphi_1(x) + u_2(t)\,\varphi_2(x)$$

Weak Form with Ritz Method:
$$\sum_{j=1}^{2} \underbrace{\left[\int_0^1 \varphi_i(x)\,\varphi_j(x)\,dx\right]}_{M_{ij}} \dot{u}_j(t) + \sum_{i=1}^{2}\underbrace{\left[\int_0^1 \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx\right]}_{K_{ij}} u_j(t)$$

$$= \underbrace{\left[\varphi_i(x)\,\frac{\partial u}{\partial x}\right]_0^1}_{Q_i}$$

After some computation:
$$\underbrace{\frac{1}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}}_{[M]}\begin{Bmatrix}\dot{u}_1\\ \dot{u}_2\end{Bmatrix} + \underbrace{\begin{bmatrix}1&-1\\-1&1\end{bmatrix}}_{[K]}\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \begin{Bmatrix}Q_1\\Q_2\end{Bmatrix} \qquad (*)$$

② Time discretization: α-family of approximation
$$(1-\alpha)\,\dot{u}_s + \alpha\,\dot{u}_{s+1} = \frac{u_{s+1}-u_s}{\Delta t}$$

Writing $(*)$ at $t=t_s$ and at $t=t_{s+1}$ gives:

$$\begin{bmatrix} \frac{1}{3}+\alpha\Delta t & \frac{1}{6}-\alpha\Delta t \\ \frac{1}{6}-\alpha\Delta t & \frac{1}{3}+\alpha\Delta t \end{bmatrix}\begin{Bmatrix} u_1(t_{s+1}) \\ u_2(t_{s+1}) \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}-(1-\alpha)\Delta t & \frac{1}{6}+(1-\alpha)\Delta t \\ \frac{1}{6}+(1-\alpha)\Delta t & \frac{1}{3}-(1-\alpha)\Delta t \end{bmatrix}\begin{Bmatrix} u_1(t_s) \\ u_2(t_s) \end{Bmatrix} + \Delta t \begin{Bmatrix} \alpha Q_{1_{s+1}}+(1-\alpha)Q_{1s} \\ \alpha Q_{2_{s+1}}+(1-\alpha)Q_{2s} \end{Bmatrix}$$

③ Time Marching Scheme

Boundary Conditions $\begin{cases} u_1(t_s) = u_1(t_{s+1}) = 0 \\ Q_2(t_s) = Q_2(t_{s+1}) = 0 \end{cases}$

$$\Rightarrow \boxed{\left(\frac{1}{3}+\alpha\Delta t\right) u_2(t_{s+1}) = \left(\frac{1}{3}-(1-\alpha)\Delta t\right) u_2(t_s)}$$

⇒ possible to solve by reccurence if we know $u(t_0)$.

Initial condition: $u(x,0) \simeq \underbrace{u_1(t_0)}_{0}\,\varphi_1(x) + u_2(t_0)\,\varphi_2(x) = 1$

At $x = x_2$: $u(x_2, 0) = u_2(t_0) = 1 \Rightarrow \boxed{u_2(t_0) = 1}$

④ Stability: if $\alpha < \frac{1}{2}$, the approximation scheme is conditionally stable ⇒ necessary to find Δt_{crit}.

$$([K] - \lambda[M])\{u\} = \underbrace{\{Q\}}_{\text{mixed b.c. only}}$$

$u_1 = 0$, $Q_2 = 0$ therefore:

$K_{22}\,u_2 - \lambda M_{22}\,u_2 = 0 \Rightarrow u_2 - \frac{\lambda}{3}u_2 = 0 \Rightarrow \lambda = 3$ for a non-trivial solution.

IF $\alpha = 0$ (Forward difference scheme):

$$\Delta t < \Delta t_{crit} = \frac{2}{(1-2\alpha)\lambda_{max}} = \frac{2}{3} \Rightarrow \boxed{\Delta t_{crit} \simeq 0.667s}$$

6.6 We wish to determine the transverse motion of a beam clamped at both ends and subjected to an initial deflection, by using Euler-Bernouilli theory. The governing equation is:

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0, \quad 0 < x < 1$$

with the following boundary conditions:

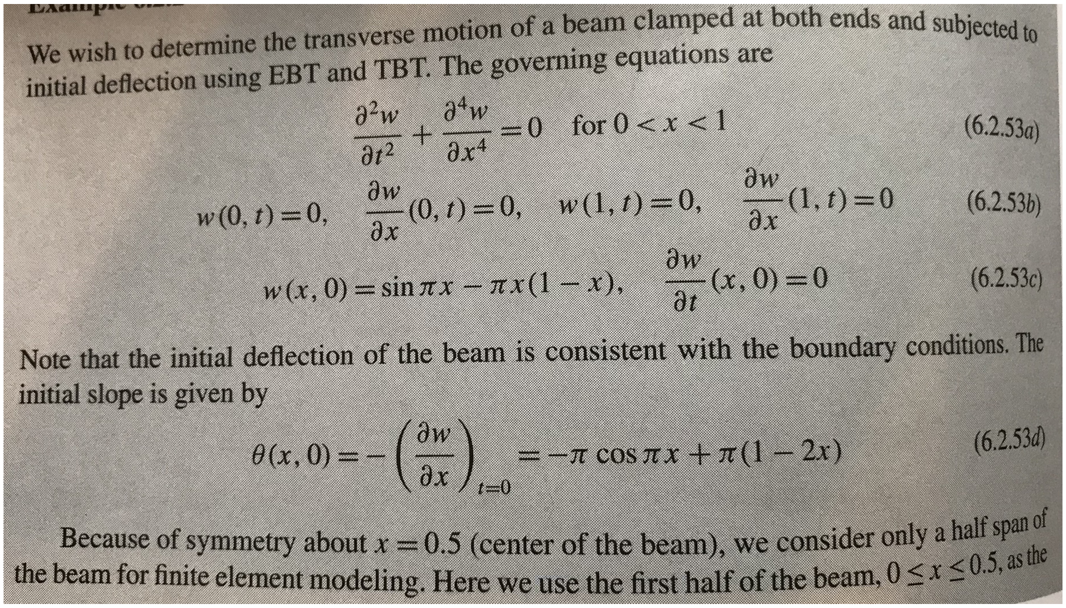
$$w(0,t) = 0, \quad \frac{\partial w}{\partial x}(0,t) = 0, \quad w(1,t) = 0, \quad \frac{\partial w}{\partial x}(1,t) = 0$$

and the following initial conditions:

$$w(x,0) = sin\pi x \ - \pi x(1-x), \quad \frac{\partial w}{\partial t}(x,0) = 0$$

Establish a stability criterion with the lowest number of Euler-Bernouilli beam elements possible.

Solution: See the following 5 figures.



We wish to determine the transverse motion of a beam clamped at both ends and subjected to initial deflection using EBT and TBT. The governing equations are

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0 \quad \text{for } 0 < x < 1 \tag{6.2.53a}$$

$$w(0,t) = 0, \quad \frac{\partial w}{\partial x}(0,t) = 0, \quad w(1,t) = 0, \quad \frac{\partial w}{\partial x}(1,t) = 0 \tag{6.2.53b}$$

$$w(x,0) = \sin \pi x - \pi x(1-x), \quad \frac{\partial w}{\partial t}(x,0) = 0 \tag{6.2.53c}$$

Note that the initial deflection of the beam is consistent with the boundary conditions. The initial slope is given by

$$\theta(x,0) = -\left(\frac{\partial w}{\partial x}\right)_{t=0} = -\pi \cos \pi x + \pi(1-2x) \tag{6.2.53d}$$

Because of symmetry about $x = 0.5$ (center of the beam), we consider only a half span of the beam for finite element modeling. Here we use the first half of the beam, $0 \leq x \leq 0.5$, as the computational domain. The boundary condition at $x = 0.5$ is $\theta(0.5, t) = -(\partial w/\partial x)(0.5, t) = 0$. The semidiscretized finite element model of a typical element is

$$\frac{h_e}{420}\begin{bmatrix} 156 & -22h_e & 54 & 13h_e \\ -22h_e & 4h_e^2 & -13h_e & -3h_e^2 \\ 54 & -13h_e & 156 & 22h_e \\ 13h_e & -3h_e^2 & 22h_e & 4h_e^2 \end{bmatrix} \begin{Bmatrix} \ddot{\Delta}_1^e \\ \ddot{\Delta}_2^e \\ \ddot{\Delta}_3^e \\ \ddot{\Delta}_4^e \end{Bmatrix} + \frac{2}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix} \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

We begin with a one-element mesh with the Euler–Bernoulli beam element. The semidiscretized model is

$$\frac{h}{420}\begin{bmatrix} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{bmatrix} \begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{Bmatrix} + \frac{2}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3h \\ -3h & h^2 & 3h & 2h^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix}$$

The boundary conditions for the one-element model translate into

$$U_1 = 0, \quad U_2 = 0, \quad U_4 = 0, \quad Q_3^1 = 0 \quad \text{for all } t > 0$$

The initial conditions can be computed from (6.2.53*c*) and (6.2.53*d*)

$$\left.\begin{aligned} &U_1 = 0, \quad U_2 = 0, \quad U_3 = 0.2146, \quad U_4 = 0 \\ &\dot{U}_1 = 0, \quad \dot{U}_2 = 0, \quad \dot{U}_3 = 0 \qquad\qquad \dot{U}_4 = 0 \end{aligned}\right\} \quad \text{for } t = 0$$

The condensed equation of the time marching scheme for this case takes the form

$$(K_{33} + a_3 M_{33})(U_3)_{s+1} = (\hat{F}_3)_{s,s+1} \equiv M_{33}(a_3 U_3 + a_4 \dot{U}_3 + a_5 \ddot{U}_3)_s, \quad s = 0, 1, \ldots$$

where a_3, a_4, and a_5 are defined in (6.2.40). The second derivative $\ddot{U}_3$ for time $t = 0$ (i.e., when $s = 0$) is computed from the equation of motion:

$$(\ddot{U}_3)_0 = -\frac{K_{33}(U_3)_0}{M_{33}} = -\left(\frac{12}{h^3} \times 0.2146\right)\frac{420}{156h} = -110.932$$

For $\gamma < \frac{1}{2}$, we must compute the critical time step Δt_{cri}, which depends on the square of maximum natural frequency of the beam [see Eq. (6.2.31)]. For the present model, $\omega_{\max}$ is computed from the eigenvalue problem

$$(K_{33} - \omega^2 M_{33})U_3 = 0 \quad \text{or } \omega^2 = K_{33}/M_{33} = 516.923$$

Hence, the critical time step for $\alpha = 0.5$ and $\gamma = \frac{1}{3}$ (i.e., the linear acceleration scheme) is

$$\Delta t_{\text{cri}} = \sqrt{12}/\omega_{\max} = 0.15236$$

Although there is no restriction on time integration schemes with $\alpha = 0.5$ and $\gamma > 0.5$, the critical time step provides an estimate of the time step to be used to obtain transient solution.

Figure 6.2.5 shows plots of $w(0.5, t)$ versus time for the scheme with $\alpha = 0.5$ and $\gamma = \frac{1}{3}$. Three different time steps, $\Delta t = 0.175, 0.150$, and 0.05, are used to illustrate the accuracy. For $\Delta t = 0.175 > \Delta t_{\text{cri}}$, the solution is unstable, whereas for $\Delta t < \Delta t_{\text{cri}}$, it is stable but inaccurate. The period of the solution is given by

$$T = 2\pi/\omega = 0.27635$$

For two- and four-element meshes of Euler–Bernoulli elements, the critical time steps are (details are not presented here)

$$(\Delta t_{\text{cri}})_2 = 0.00897, \quad (\Delta t_{\text{cri}})_4 = 0.00135$$

where the subscripts refer to the number of elements in the mesh. The transverse deflection obtained with the one and two Euler–Bernoulli elements ($\Delta t = 0.005$) in half beam for a complete period (0, 0.28) are shown in Fig. 6.2.6.

The problem can also be analyzed using the Timoshenko beam element (RIE). In writing the governing equations [see (6.1.43*a*) and (6.1.43*b*)] of the TBT as they apply to the present problem, we first identify the coefficients GAK_s, EI, ρA, and ρI consistent with those in the differential equation (6.2.53*a*). We have $EI = 1.0$ and $\rho A = 1.0$. Then GAK_s can be computed as

$$GAK_s = \frac{E}{2(1+\nu)} BHK_s = \frac{EI}{2(1+\nu)} \frac{12}{H^2} \frac{5}{6} = \frac{4}{H^2} EI \tag{6.2.54}$$

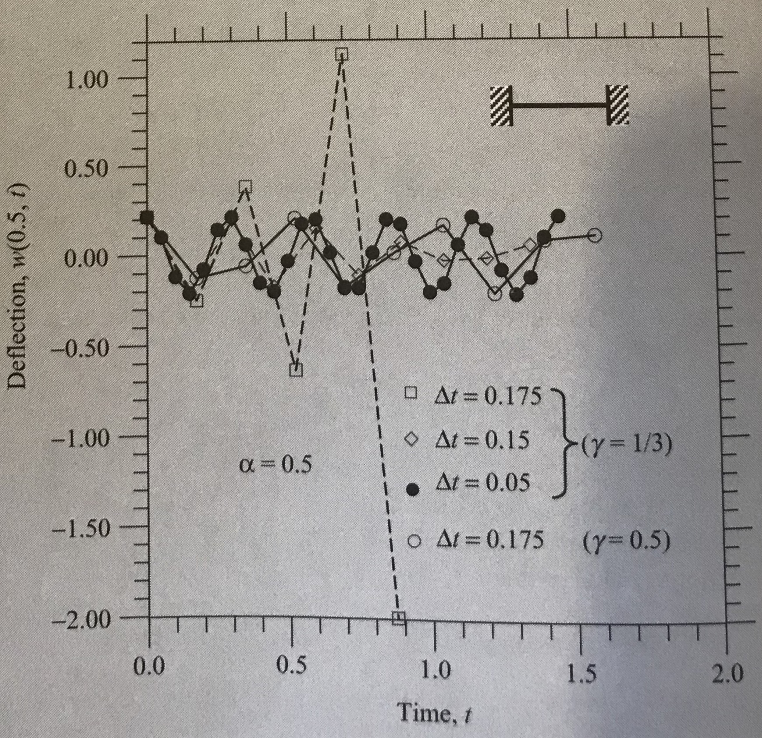


Figure 6.2.5 Center deflection $w(0.5, t)$ versus time t for a clamped beam.

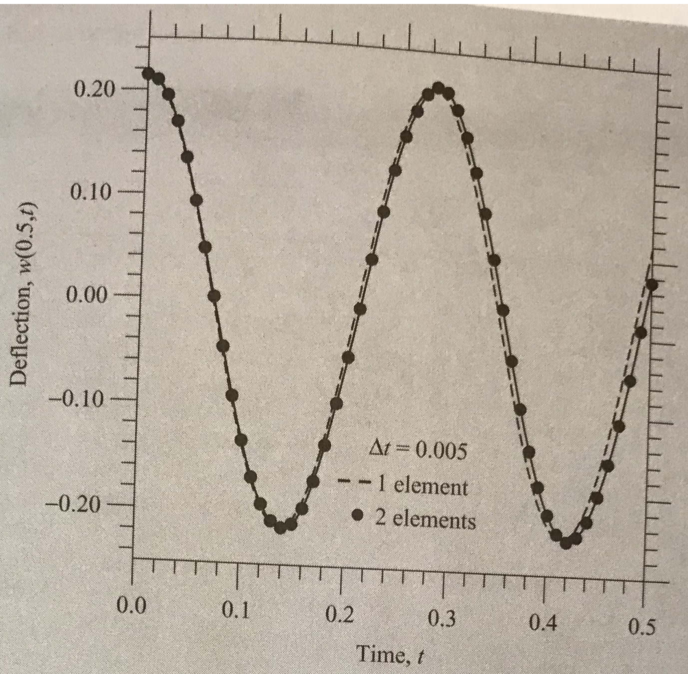


Figure 6.2.6 Center transverse deflection versus time for a clamped beam subjected to an initial transverse deflection ($\Delta t = 0.005$, $\alpha = 0.5$, and $\gamma = 0.5$).

where B is the width and H the height of the beam, and $I = \frac{1}{12}BH^3$, $\nu = 0.25$, and $K_s = \frac{5}{6}$ are used in arriving at the last expression. Similarly,

$$\rho I = \rho \frac{1}{12} BH^3 = \frac{1}{12} \rho A H^2 \tag{6.2.55}$$

Thus, the governing equations of the TBT for the problem at hand are

$$\frac{\partial^2 w}{\partial t^2} - \frac{4}{H^2} \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} + \Psi \right) = 0 \tag{6.2.56a}$$

$$\frac{H^2}{12} \frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} + \frac{4}{H^2} \left(\frac{\partial w}{\partial x} + \Psi \right) = 0 \tag{6.2.56b}$$

The values of $w(0.5, t)$ as obtained using the Euler–Bernoulli and Timoshenko elements (both elements include the rotary inertia) for various numbers of elements are presented in Table 6.2.2. The time step is taken to be $\Delta t = 0.005$, which is smaller than the critical time step of the two-element mesh of the Euler–Bernoulli beam element when $\gamma = \frac{1}{3}$. Plots of $w(0.5, t)$ obtained with two and four linear Timoshenko beam elements, two quadratic Timoshenko beam elements for $L/H = 100$ (since $L = 1.0$, we take $H = 0.01$; for $L/H = 100$ the shear deformation effect is negligible) along with the two-element solution of the Euler–Bernoulli beam element are shown in Fig. 6.2.7. The two linear element mesh of TBT beam elements predicts transient response that differs significantly from the EBT solution, and the TBT solution converges to the EBT solution as the number of elements or the degree of approximation is increased. Should we use conditionally stable schemes, it can be shown that the Timoshenko beam element requires larger Δt_{cri} than the Euler–Bernoulli beam element. This is because, as L/H is decreased, the $\omega_{\max}$ predicted by the TBT is smaller than that predicted by the EBT.

Table 6.2.2 Effect of mesh on the transient response of a beam clamped at both ends ($\Delta t = 0.005$).

	EBT elements					TBT elements*		
	$\alpha = 0.5, \gamma = 1/3$		$\alpha = 0.5, \gamma = 0.5$			$\alpha = 0.5, \gamma = 0.5$		
Time t	1	2	1	2	4	2L	4L	2Q
0.00	0.2146	0.2146	0.2146	0.2146	0.2146	0.2146	0.2146	0.2146
0.01	0.2091	0.2098	0.2091	0.2098	0.2098	0.2124	0.2099	0.2116
0.02	0.1928	0.1951	0.1928	0.1951	0.1951	0.1938	0.1936	0.1951
0.03	0.1666	0.1696	0.1667	0.1696	0.1698	0.1550	0.1686	0.1690
0.04	0.1319	0.1346	0.1320	0.1348	0.1350	0.1145	0.1374	0.1427
0.05	0.0904	0.0930	0.0905	0.0931	0.0935	0.0695	0.0980	0.1067
0.06	0.0442	0.0481	0.0443	0.0482	0.0483	0.0093	0.0073	0.0657
0.07	−0.0043	0.0014	−0.0041	0.0014	0.0018	−0.0467	−0.0403	0.0234
0.08	−0.0525	−0.0462	−0.0523	−0.0459	−0.0455	−0.0917	−0.0844	−0.0267
0.09	−0.0980	−0.0926	−0.0978	−0.0923	−0.0916	−0.1422	−0.1254	−0.0706
0.10	−0.1385	−0.1345	−0.1383	−0.1342	−0.1336	−0.1833	−0.1588	−0.1100
0.11	−0.1719	−0.1685	−0.1717	−0.1685	−0.1682	−0.2010	−0.1875	−0.1461
0.12	−0.1964	−0.1933	−0.1963	−0.1933	−0.1932	−0.2154	−0.2069	−0.1717
0.13	−0.2108	−0.2088	−0.2108	−0.2088	−0.2087	−0.2205	−0.2063	−0.1969
0.14	−0.2144	−0.2153	−0.2144	−0.2150	−0.2148	−0.1986	−0.2138	−0.2110
0.15	−0.2070	−0.2113	−0.2071	−0.2112	−0.2111	−0.1696	−0.2134	−0.2146

* 2L = two linear elements; 4L = four linear elements; 2Q = two quadratic elements.

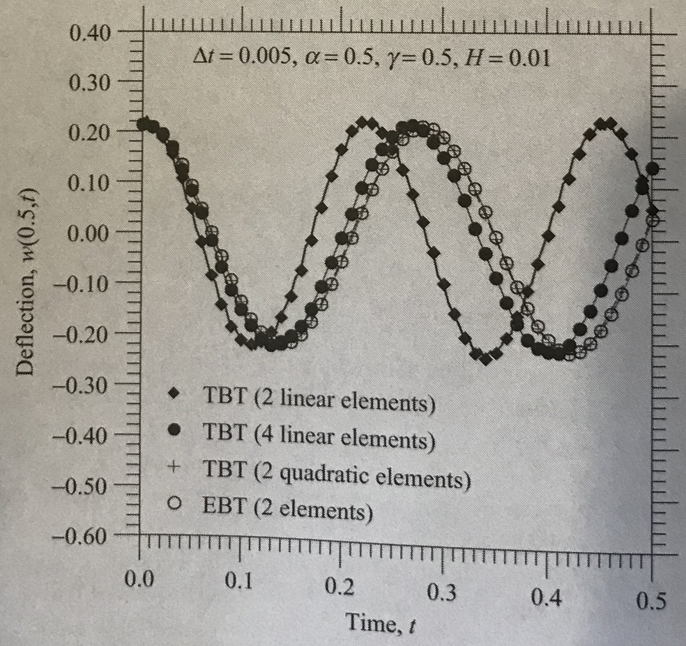


Figure 6.2.7 Transient response of a beam clamped at both ends, according to the TBT and EBT ($EI = 1$, $\rho A = 1$, $H = 0.01$, $\Delta t = 0.005$, $\alpha = 0.5$, and $\gamma = 0.5$).

6.7 Establish a stability criterion for the 1D consolidation problem. Assume that there is drainage at the top and and that the bottom boundary is impermeable.

Solution:

The 1D consolidation equation is:

$$\frac{\partial p_w}{\partial t} = c_v \frac{\partial^2 p_w}{\partial z^2}, \quad c_v = \frac{k_w}{\mu_w (S + \alpha^2 m_v)}$$

For an element $]z_a,\, z_b[$, the weak formulation of the problem is as follows:

$$\forall w(z) \sim \delta_z p_w,$$

$$\int_{z_a}^{z_b} w(z)\dot{p}_w dz + \int_{z_a}^{z_b} c_v \frac{dw(z)}{dz}\frac{\partial p_w}{\partial z} dz = \left[w(z) \frac{1}{\rho_w (S + \alpha^2 m_v)} \underbrace{k_w \frac{\rho_w}{\mu_w} \frac{\partial p_w}{\partial z}}_{-\widehat{q}(z)} \right]_{z_a}^{z_b}$$

Using Ritz method, with linear element:

$$\forall i = 1, 2,$$

$$\sum_{j=1}^{2} \left(\int_{z_a}^{z_b} \Psi_i(z)\Psi_j(z) dz \right) \dot{p}_j + \sum_{j=1}^{2} \left(\int_{z_a}^{z_b} c_v \frac{d\Psi_i}{dz}\frac{d\Psi_j}{dz} dz \right) p_j = - \left[\Psi_i(z) \frac{\widehat{q}(z)}{\rho_w (S + \alpha^2 m_v)}) \right]_{z_a}^{z_b}$$

In a matrix form:

$$[H^e]\{p\} + [S^e]\{\dot{p}\} = \{Q^e\}$$

with:

$$H^e_{ij} = \int_{z_a}^{z_b} c_v \frac{d\Psi_i}{dz}\frac{d\Psi_j}{dz} dz$$

$$S^e_{ij} = \int_{z_a}^{z_b} \Psi_i(z)\Psi_j(z) dz$$

$$Q^e_i = - \left[\Psi_i(z) \frac{\widehat{q}(z)}{\rho_w (S + \alpha^2 m_v)}) \right]_{z_a}^{z_b}$$

The primary variable is equal to zero at the first node, at $z = h$. Therefore, the first row and the first column of the assembled stiffness matrix will vanish after condensation. According to the boundary conditions, the seconday variable is zero at the last node, at $z = 0$. According to the continuity conditions, the seconday variable is zero at the intermediate nodes. So that after space discretization and condensation, the Finite Element matrix equation is:

$$[H]\{p\} + [S]\{\dot{p}\} = \{0\}$$

The governing equation is parabolic, therefore we use an alpha-family of approximation:

$$\begin{aligned} (p_w)_{n+\theta} &= (1-\theta) \times (p_w)_n + \theta \times (p_w)_{n+1} \\ (\dot{p}_w)_{n+\theta} &= \frac{(p_w)_{n+1} - (p_w)_n}{\Delta t} \end{aligned}$$

Therefore, at time $t_{n+\theta}$, we have:

$$\Delta t[H]\{p\}_{n+\theta} + \Delta t[S]\{\dot{p}\}_{n+\theta} = \{0\}$$

$$(\theta \Delta t[H] + [S])\,\{p\}_{n+1} = ((\theta - 1)\Delta t[H] + [S])\,\{p\}_n$$

$[S]$ is invertible, therefore:

$$\big([Id] + \theta \Delta t[S]^{-1}[H]\big)\,\{p\}_{n+1} = \big([Id] + (\theta - 1)\Delta t[S]^{-1}[H]\big)\,\{p\}_n \tag{6.1}$$

We introduce λ_j and $\{X_j\}$, the eigenvalues and eigenvectors of the operator $[S]^{-1}[H]$, respectively. We have:

$$\forall j, \quad [S]^{-1}[H]\{X_j\} = \lambda_j\{X_j\}$$

Multiplying both sides of Equation 6.1 by $\{X_j\}$, we thus get:

$$(\{X_j\} + \theta \Delta t \lambda_j \{X_j\})\,\{p\}_{n+1} = (\{X_j\} + (\theta - 1)\Delta t \lambda_j \{X_j\})\,\{p\}_n \tag{6.2}$$

Now multiplying both sides of Equation 6.2 by $\{X_j\}^T/||\{X_j\}||^2$, we get:

$$(1 + \theta \Delta t \lambda_j)\,\{p\}_{n+1} = (1 + (\theta - 1)\Delta t \lambda_j)\,\{p\}_n \tag{6.3}$$

From Equation 6.3, we see that the FEM model is stable if and only if:

$$|\frac{1 + (\theta - 1)\Delta t \mu_j}{1 + \theta \Delta t \mu_j}| < 1 \tag{6.4}$$

For linear elements:

$$\Psi_1^e(z) = 1 - \frac{z}{h_e}, \quad \Psi_2^e(z) = \frac{z}{h_e}$$

$$[H^e] = \frac{c_v}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[S^e] = \frac{h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

We find:

$$[S^e]^{-1}[H^e] = \frac{6c_v}{(h_e)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

We solve the following characteristic equation to cal the eigenvalues of $[S^e]^{-1}[H^e]$:

$$\det\big([Id] - \mu_j[S^e]^{-1}[H^e]\big) = 0$$

$$\left(\frac{6c_v}{(h_e)^2} - \mu_j\right)^2 - \left(-\frac{-6c_v}{(h_e)^2}\right)^2 = 0$$

The eigenvalues are:

$$\mu_1 = \frac{12c_v}{(h_e)^2}, \quad \mu_2 = 0$$

$\mu_2 = 0$ leads to the trivial solution for $\{X_j\}$, so for the non trivial solution, the stability criterion is:

$$|\frac{(h_e)^2 + 12c_v(\theta - 1)\Delta t}{(h_e)^2 + 12c_v \theta \Delta t \mu_j}| < 1 \tag{6.5}$$

If $(h_e)^2 + 12c_v(\theta - 1)\Delta t > 0$, then the FEM model is unconditionnally stable. Otherwise, the stability criterion is:

$$\frac{(h_e)^2}{6c_v\Delta t} > 1 - 2\theta \tag{6.6}$$

Equation 6.6 is lways satisfied if $\theta \geq 0.5$. The FEM model is conditionally stable otherwise. One can see that the stability of the FEM depends on the time of the time step relative to the element size.

6.8 Dual boundary conditions occur in flow problems in which a pressurized fluid reservoir is in contact with the domain under study:

- If the pore pressure at the boundary is more tensile (i.e. lower) than a prescribed value p_*, then one needs to apply a fluid flow $\overline{q}_n$ at the boundary.
- If the pore pressure at the boundary is more compressive (i.e. greater) than the prescribed value p_*, then one needs to impose the pore pressure p_* at all nodes of the boundary.

Figure 6.2 on the left hand-side shows the example of ponding, for which such boundary conditions are necessary. Explain how you would use the dual boundary condition for a problem of rainfall (Figure 6.2 on the right hand-side) and for a problem of tunnel excavation in a water saturated rock mass (Figure 6.3).

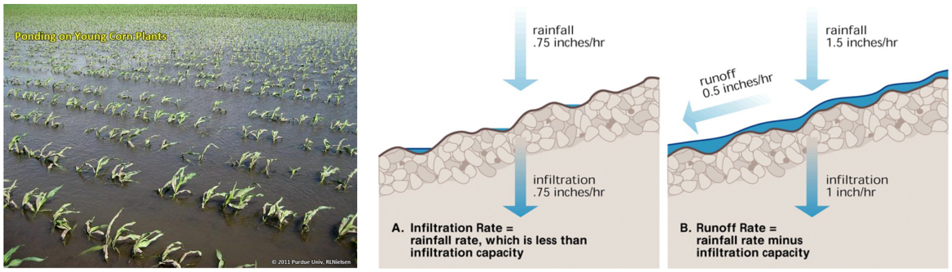


Figure 6.2 Problems in which a dual boundary condition is needed: ponding (left) and rainfall (right).

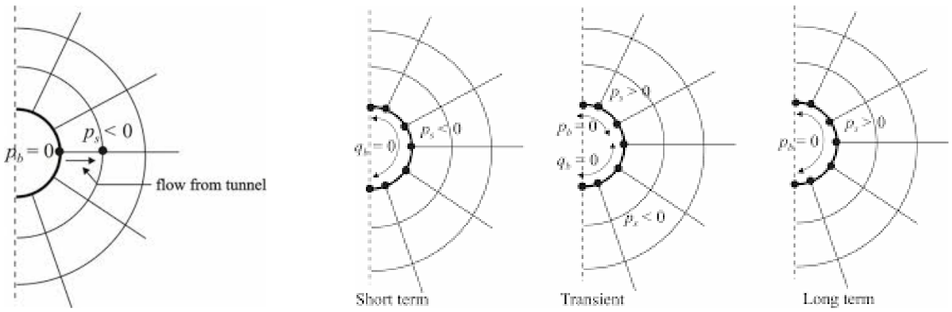


Figure 6.3 Problem in which a dual boundary condition is needed: tunnel excavation.

Solution:
For the rainfall problem:

- If the soil is of sufficient permeability and/or the rainfall intensity is small, the soil can adsorb the rainfall water so a flow boundary condition should be applied. Initially, a water pressure is imposed at the boundary with $p_i < p_*$, so it is necessary to impose a rainfall flow $\overline{q}_n$.
- If the soil is less permeable and/or the the rainfall intensity is high, the soil will not be able to absorb the rainfall water so ponding happens at the surface and a water pressure boundary condition needs to be applied. Pore pressure builds up in the soil; when water pressure at the boundary exceeds the given limit ($p_i > p_*$), then it is necessary to impose a pore pressure p_* at the boundary.

For the tunnel problem:

- Just after excavation, $p = 0$ (permeable tunnel wall). Fluid pressure in the rock mass is tensile ($p < 0$) therefore a water flow condition is the most appropriate in the short term: $p < p_* = 0$, impose $\overline{q}_n = 0$ (no flow from tunnel to rock mass).
- In the long term, pore pressure in the rock mass becomes more compressive. When $p \geq p_* = 0$ at the boundary, it is more appropriate to impose the pore water boundary condition $p_* = 0$ at the tunnel wall (water drainage).
- At intermediate times between the short and long terms, some nodes are subjected to a flow b.c. while others are subjected to a pressure b.c.

6.9 Consider a triaxial compression test performed on a water-saturated soil specimen, as shown in Figure 6.4. The experiment is undrained, and axis-symmetry is assumed.The soil is assumed to be linear elastic. Biot's hydro-mechanical constitutive relationships hold.

- Write the strong formulation of the problem (introduce as many constitutive parameters as necessary).
- Write the weak formulation of the problem (in cylindrical coordinates).
- The specimen is modeled with three rectangular elements, as shown in Figure 6.4. Calculate the elementary stiffness of each element, assuming that the displacement field is interpolated with quadratic polynomials, and the pore pressure field is interpolated with linear polynomials.
- Assemble the elementary equations established in question 2 above.
- Discretize the assembled equations in time. Provide the conditions in which the time marching scheme is stable.
- Introduce the boundary conditions in the Finite Element equations, condense the system of equations if possible, and solve it for the primary variables.
- Post-process the secondary variables.

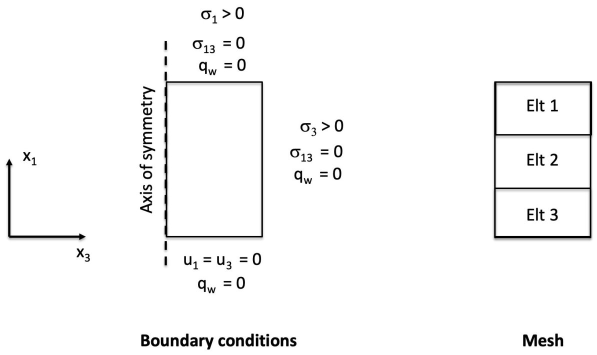


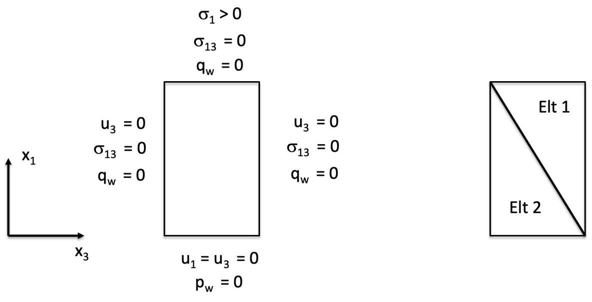
Figure 6.4 Finite Element model of undrained triaxial compression test performed on water-saturated soil

Solution:

6.10 Consider an oedometer test performed on a water-saturated soil specimen, as shown in Figure 6.5. The specimen is drained at the bottom and it is studied in plane strain. The soil is assumed to be linear elastic. Biot's hydro-mechanical constitutive relationships hold.

- Write the strong formulation of the problem (introduce as many constitutive parameters as necessary).
- Write the weak formulation of the problem.
- The specimen is modeled with two triangular elements, as shown in Figure 6.5. Calculate the elementary stiffness of each element, assuming that the displacement field is interpolated with quadratic polynomials, and the pore pressure field is interpolated with linear polynomials.
- Assemble the elementary equations established in question 2 above.
- Discretize the assembled equations in time. Provide the conditions in which the time marching scheme is stable.
- Introduce the boundary conditions in the Finite Element equations, condense the system of equations if possible, and solve it for the primary variables.
- Post-process the secondary variables.

Solution:



Boundary conditions **Mesh**

Figure 6.5 Finite Element model of drained oedometric test performed on water-saturated soil

6.11 Consider an oedometer test performed on a partially saturated soil specimen, as shown in Figure 6.6. The specimen is drained at the bottom and it is studied in plane strain. The soil is assumed to be linear elastic.

- Write the strong formulation of the problem (introduce as many constitutive parameters as necessary).
- Write the weak formulation of the problem.
- The specimen is modeled with two triangular elements, as shown in Figure 6.6. Calculate the elementary stiffness of each element, assuming that the displacement field is interpolated with quadratic polynomials, and the pore pressure fields is interpolated with linear polynomials.
- Assemble the elementary equations established in question 2 above.
- Discretize the assembled equations in time. Provide the conditions in which the time marching scheme is stable.
- Introduce the boundary conditions in the Finite Element equations, condense the system of equations if possible, and solve it for the primary variables.
- Post-process the secondary variables.

Solution:

6.12 We use the FEM to compare two one-dimensional consolidation experiments (see the column of soil in Figure 6.7):

1. $S_{w0} = 0.92$ (homogeneous initial partial saturation) with a capillary pressure - water saturation curve independent of temperature;
2. $S_{w0} = 0.92$ (homogeneous initial partial saturation), with a capillary pressure - water saturation curve that depends on temperature.

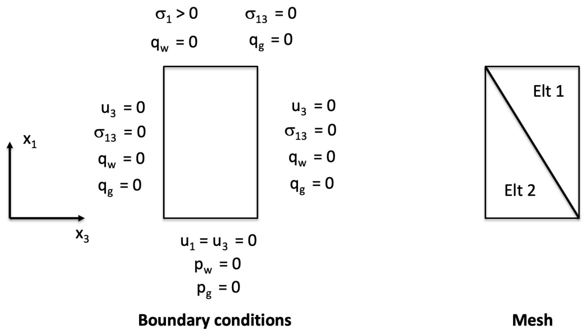


Figure 6.6 Finite Element model of drained oedometric test performed on partially saturated soil

A Brooks & Corey relationship is assumed between the capillary pressure, the saturation degree and the relative permeability. The solid grains and the water are assumed to be incompressible. The boundary conditions are the following: no lateral displacement or heat flux on lateral boundaries; at the top: uniform stress, T=343.15 K, $p_g = p_{atm}$ and p_c chosen to ensure $S_w = 0.92$; no vertical displacement and no heat flux on the bottom boundary. The same results were obtained with 9 and 18 8-noded isoparametric elements with a 3x3 Gaussian integration scheme. The time step was 0.01 days for the first 100 steps, and then the time step was multiplied by 10 every 100 time steps, until 10^7 days elapsed. Comment on the results obtained in Figures 6.8-6.11 (in particular, explain the difference between the profiles obtained in the saturated and unsaturated cases).

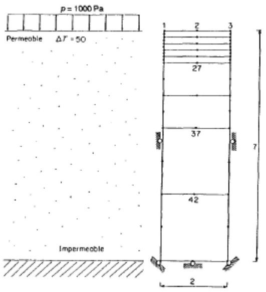


Figure 6.7 Unsaturated soil consolidation under non isothermal conditions

Solution:

Figure 6.8: Resulting temperature profiles are similar in both cases and are similar to the temperature profile obtained in saturated conditions. This is due to the identical averaged thermal conductivity and relatively high thermal capacity assumed in the saturated case.

Figure 6.9: In saturated cases, samples exhibit an initial period of no-deformation, followed by settlements, followed by heave. The latter phase is due to the thermal dilation of the solid and liquid components of the soil sample. Unsaturated samples exhibit immediate settlements due to gas expulsion, a first period of no settlement, followed by heave. The absence of settlements between the no-deformation and heave phases suggests that the remaining gas, subjected to a high temperature and confined in a finite volume, reached the

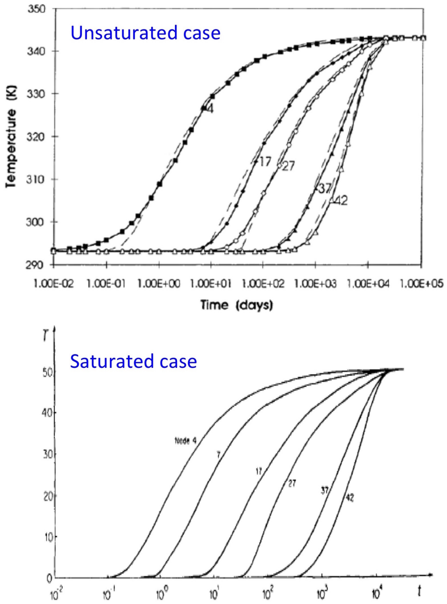
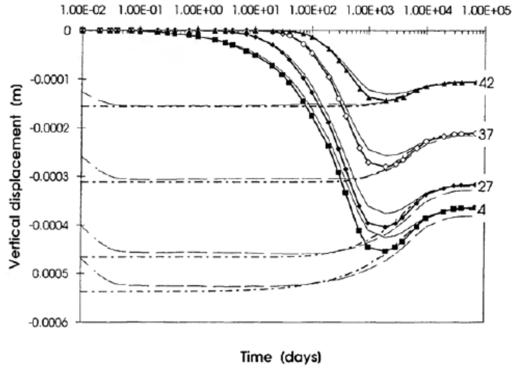
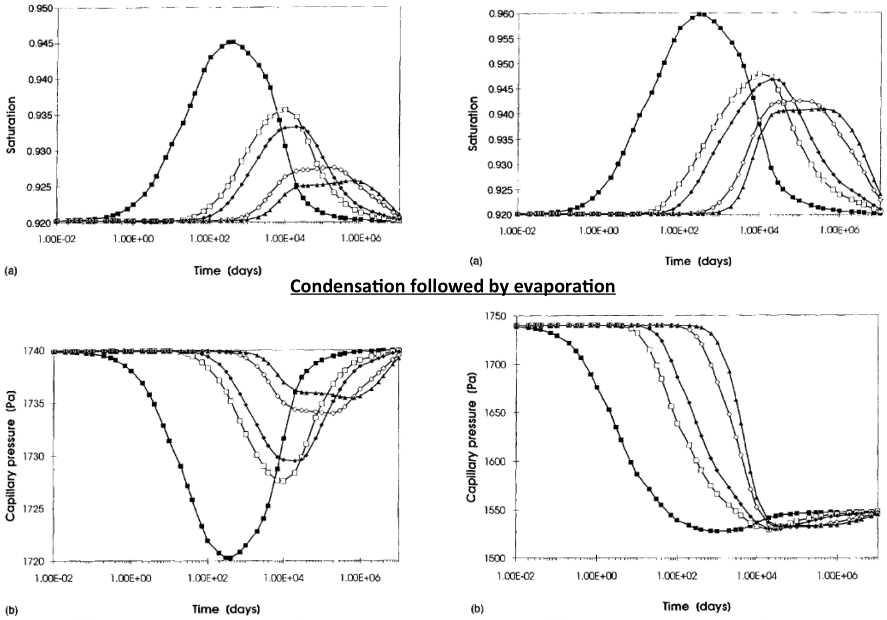


Figure 6.8 Unsaturated consolidation: temperature profiles



Comparison between the saturated case (solid lines) and the second partially saturated case (chain dots): heavy lines are the present solution; light lines are the solution of Schrefler *et al.* [22]. The numbers on the curves are the nodal points selected within the mesh

Figure 6.9 Unsaturated consolidation: displacement profiles, comparison between saturated and unsaturated cases



Condensation followed by evaporation

Capillary pressure – saturation curve independent from temperature

Capillary pressure – saturation curve dependent on temperature

Figure 6.10 Unsaturated consolidation: water retention curves

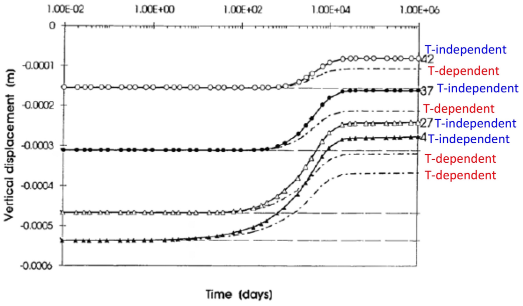


Figure 6.11 Unsaturated consolidation: displacement profiles, comparison between the two unsaturated models (WRC that depends/does not depend on temperature)

pressure of condensation ($pV = nRT$). We note that if phase changes are not accounted for (dash lines), immediate settlements are followed by a brief consolidation process before stabilizing in the phase of no-deformation followed by heave. That means that if condensation is not accounted for in the model, the increase of pore pressure induces a liquid outflow.

Figure 6.10: These two last observations are confirmed by the curves that show the time evolution of the degree of saturation and of the capillary pressure. In both models, the degree of saturation increases during the no-deformation phase, i.e. at constant volume. Thus the liquid contents increases, indicating condensation. Heave correlates with a decrease of the degree of saturation: fluids occupy more space and the gas fraction increases, indicating evaporation. In the first model, the capillary pressure solely depends on the degree of saturation, thus the capillary pressure decreases during the no-deformation phase and increases during the heave phase. In the second model, the capillary pressure depends also on temperature. As temperature increases, the vapor content of the air-vapor mixture increases and the contact angle between the wetting phase (liquid) and the non-wetting phase (gas mixture) decreases. According to Laplace Law, that corresponds to a decrease of capillary pressure. As a result, in the second model, the capillary pressure decreases during the no-deformation phase (condensation) and remains constant during the heave phase (evaporation).

Figure 6.11: There are more pronounced phase changes in the capillary pressure- saturation model that depends on temperature. The evolution of capillary pressure follows that of temperature. In the first model, the capillary pressure is higher, thus the compressibility of the sample is lower, and therefore the predicted final settlements are lower than with the second model.

6.13 Homework 4 - Problem 4

Consider the problem of determining the temperature distribution of a solid cylinder, initially at a uniform temperature T_0 and cooled in a medium of zero temperature (i.e., T_∞ = 0). The governing equation of the problem is:

$$\rho\, c \frac{\partial T}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\, k \frac{\partial T}{\partial r} \right) = 0, \quad 0 < r < R$$

The boundary conditions are:

$$\frac{\partial T}{\partial r}(0,t) = 0, \quad \left(r\, k\, \frac{\partial T}{\partial r} + \beta\, T \right)_{r=R} = 0$$

The initial conditions are $T(r,t) = T_0$. Determine the pressure distribution $T(r,t)$ using one linear finite element. Take $R = 2.5$cm, $T_0 = 130^o$C, $k = 215$W/(m^{o}C), $\beta = 525$W/(m^{o}C), $\rho = 2700$kg/m^2, and $c = 0.9$kJ/(kg^{o}C). What is the heat loss at the surface?

Solution: The FE model is given by:

$$[M^e]\{\dot{u}^e\} + [K^e]\{u^e\} = \{Q^e\}$$

Where:

$$K_{ij}^e = 2\pi \int_{r_A}^{r_B} rk \frac{d\psi_i}{dr} \frac{d\psi_j}{dr} dr$$
$$M_{ij}^e = 2\pi \int r_A{}^{r_B} \rho c r \psi_i(r) \psi_j(r) dr$$

The matrices $[K^e]$ and $[M^e]$ for a linear element can be expressed by developping the expressions of the linear interpolation polynomials ψ_i. For $[M^e]$ in particular:

$$[M^e] = \frac{2\pi\rho ch}{12} \begin{bmatrix} h + 4r_A & h + 2r_A \\ h + 2r_A & 3h + 4r_A \end{bmatrix}$$

The boundary conditions are: $Q_1^1 = 0$ and $Q_2^1 = -2\pi\beta U_2$. The one element mesh (h = R) gives the equations ($r_A = 0$ since we have only one element):

$$\pi k \begin{bmatrix} +1 & -1 \\ -1 & 1 + 2\beta \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} + \frac{2\pi\rho ch}{12} \begin{bmatrix} h & h \\ h & 3h \end{bmatrix} \begin{Bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

From there, use the α-family of approximation to establish the time marching scheme. The degrees of freedom can be found by recurrence, knowing the initial temperature state of the rod. The heat loss at the surface is $Q_2 = -\beta T_2$, in which T_2 was calculated previously as part of the degrees of freedom.

6.14 Homework 4 - Problem 5

We consider a cylindrical soil sample, of radius a, enclosed between two stiff horizontal plates. The soil sample is supposed to be surrounded by a drainage layer around it, and an impermeable membrane surrounding the drainage layer, so that the radial pressure can be transmitted to the sample, and drainage occurs to the outer boundary. The sample is subjected to a uniform radial pressure of magnitude q at the drained outer boundary. The problem is axis-symmetric. The momentum balance equation in the radial direction is:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

The fluid mass conservation equation is:

$$\alpha \frac{\partial \epsilon}{\partial t} + S \frac{\partial p}{\partial t} = \frac{k}{\gamma_f} \left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} \right)$$

in which ϵ is the volumetric strain, p is the fluid pore pressure, α is Biot's coefficient, k is the coefficient of permeability, γ_f is the volumetric weight of the pore fluid and S is the storativity, defined as:

$$S = \frac{n}{K_f} + \frac{\alpha - n}{K_s}$$

in which n is the porosity, K_f is the bulk modulus of the fluid, and K_s is the bulk modulus of the solid phase. We assume the following stress-strain constitutive relationship:

$$\begin{aligned}\sigma_{rr} - \alpha\, p &= -\left(K - \frac{2}{3}G\right)\epsilon - 2G\frac{\partial u}{\partial r}\\ \sigma_{\theta\theta} - \alpha\, p &= -\left(K - \frac{2}{3}G\right)\epsilon - 2G\frac{u}{r}\end{aligned}$$

in which u is the radial displacement field in the solid phase, K is the bulk modulus of the porous medium, and G is the shear modulus of the porous medium. Moreover we assume that fluid flow is governed by Darcy's law:

$$\mathbf{q_f} = -\frac{k}{\mu_f}\nabla p$$

in which $\mathbf{q_f}$ is the flux per unit area (m/s), and μ_f is the fluid dynamic viscosity (Pa.s).

1. Show that the equation of equilibrium can be expressed in terms of the volume strain as:
$$\left(K + \frac{4}{3}G\right)\frac{\partial \epsilon}{\partial r} = \alpha\frac{\partial p}{\partial r}$$
2. Write the weak formulation of the problem.
3. Discretize the weak form in space, and provide the expression of the matrix and vector coefficients of the following Finite Element equation:
$$\begin{bmatrix} K_e & -Q \\ 0 & H \end{bmatrix}\begin{Bmatrix} u \\ p \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ Q^T & S \end{bmatrix}\begin{Bmatrix} \dot{u} \\ \dot{p} \end{Bmatrix} = \begin{Bmatrix} f_u \\ f_p \end{Bmatrix}$$
4. Discretize the equation above in time and explain how to ensure the stability of the numerical scheme (do not solve the associated eigenvalue problems).

Solution:

1. We combine the momentum balance equation:
$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$
and the stress/strain constitutive relationship:
$$\begin{aligned}\sigma_{rr} - \alpha\, p &= -\left(K - \frac{2}{3}G\right)\epsilon - 2G\frac{\partial u}{\partial r}\\ \sigma_{\theta\theta} - \alpha\, p &= -\left(K - \frac{2}{3}G\right)\epsilon - 2G\frac{u}{r}\end{aligned}$$

We obtain:

$$\alpha\frac{\partial p}{\partial r} - \left(K - \frac{2}{3}G\right)\frac{\partial \epsilon}{\partial r} - 2G\frac{\partial^2 u}{\partial r^2} - 2G\frac{1}{r}\frac{\partial u}{\partial r} + 2G\frac{u}{r^2} = 0 \tag{6.7}$$

Moreover the volumetric strain in cylindrical coordinates is:

$$\epsilon = \frac{\partial u}{\partial r} + \frac{u}{r}$$

which yields:

$$\frac{\partial \epsilon}{\partial r} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{u}{r^2} \tag{6.8}$$

The combination of equations 6.7 and 6.8 provides:

$$\alpha\frac{\partial p}{\partial r} = \left(K - \frac{2}{3}G\right)\frac{\partial \epsilon}{\partial r} + 2G\frac{\partial \epsilon}{\partial r}$$

And we conclude that the equation of equilibrium can be rewritten as:

$$\alpha\frac{\partial p}{\partial r} = \left(K + \frac{4}{3}G\right)\frac{\partial \epsilon}{\partial r}$$

2. We consider a weight function $\mathbf{w} \sim \Delta\mathbf{u}$ (vector) and a weight function $w* \sim \Delta p$ (scalar). The mechanical equation of equilibrium, combined with the stress-strain relationship, provides the following equation:

$$\mathbf{div}\left(\mathbf{D_e} : \epsilon\right) + \alpha\mathbf{div}\left(p\delta\right) = 0$$

in which $\mathbf{D_e}$ is the fourth-order elasticity tensor, ϵ is the second-order deformation tensor and δ second-order identity tensor. The first equation of the weak formulation is obtained by multiplying equation 2 by $\mathbf{w}$, integrating over the domain of study Ω_t and performing an integration by parts to balance the orders of the differential operators between $\mathbf{w}$ and $\mathbf{u}$. We obtain:

$$\forall\mathbf{w} \sim \Delta\mathbf{u}, \quad \int_{\Omega_t} \nabla\mathbf{w} : \left(\mathbf{D_e} : \nabla\mathbf{u}\right) dV + \int_{\Omega_t} \alpha\nabla\mathbf{w} : \delta p dV = \int_{\Gamma_f} \mathbf{w} \cdot \widehat{\mathbf{t}} dS$$

in which it is assumed that an essential boundary condition is imposed over the entire boundary of the domain except on the portion Γ_f. To obtain the second equation of the weak formulation, we multiply the fluid mass balance equation by $w*$, integrate over the domain of study Ω_t and perform integration by parts as needed. We obtain:

$$\forall w* \sim \Delta p, \quad \int_{\Omega_t} \tfrac{k}{\gamma_f}\nabla(w*) \cdot \nabla p \, dV - \int_{\Omega_t} \alpha \, (w*)\delta : \nabla(\dot{\mathbf{u}}) dV$$
$$- \int_{\Omega_t} \left(\tfrac{\alpha - n}{K_s} + \tfrac{n}{K_f}\right) (w*)\dot{p} \, dV = - \int_{\Gamma_q} (w*) \tfrac{\mu_f}{\gamma_f} q_f \, dS$$

in which it is assumed that an essential boundary condition is imposed over the entire boundary of the domain except on the portion Γ_q.

3. After space discretization, the elementary FE equation is noted:

$$\begin{bmatrix} K_e & -Q \\ 0 & H \end{bmatrix} \begin{Bmatrix} u \\ p \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ Q^T & S \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{p} \end{Bmatrix} = \begin{Bmatrix} f_u \\ f_p \end{Bmatrix}$$

According to the weak formulation obtained in question 2, with the standard notations of the course, the coefficients of the matrices and vectors are the following:

$$[K_e] \;=\; \int_{\Omega_e} [B]^T [D_e][B] dV, \quad [B] = [L][N_u]$$

in which $[D_e]$ is the elasticity matrix, $[L]$ is the differential operator (relating displacements to strains) and $[N_u]$ is the matrix of displacement interpolation functions.

$$[Q] \;=\; -\int_{\Omega_e} [B]^T \alpha \{m\}^T \{N_p\} dV, \quad \{m\}^T = \{1\,1\,1\,0\,0\,0\}$$

in which $\{N_p\}$ is the vector of pore pressure interpolation functions.

$$[H] \;=\; \int_{\Omega_e} [\nabla\{N_p\}]^T \frac{k}{\gamma_f} [Id][\nabla\{N_p\}] dV$$

$$[S] \;=\; -\int_{\Omega_e} \{N_p\}^T \left(\frac{\alpha - n}{K_s} + \frac{n}{K_f} \right) \{N_p\} dV$$

$$\{f_u\} \;=\; \int_{\Gamma_e} [N_u]^T \{\widehat{t}\} dS$$

$$\{f_p\} \;=\; -\frac{\mu_f}{\gamma_f} \int_{\Gamma_e} \{N_p\}^T \{q_f\} dS$$

4. For the parabolic equation of interest, we use a Finite Difference scheme for time discretization. The time marching scheme is given by:

$$\begin{aligned} &\left(\theta \Delta t \begin{bmatrix} K_e & -Q \\ 0 & H \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ Q^T & S \end{bmatrix} \right) \left\{ \begin{matrix} u \\ p \end{matrix} \right\}_{n+1} = \\ &\left(-(1-\theta) \Delta t \begin{bmatrix} K_e & -Q \\ 0 & H \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ Q^T & S \end{bmatrix} \right) \left\{ \begin{matrix} u \\ p \end{matrix} \right\}_{n} + \Delta t \left\{ \begin{matrix} f_u \\ f_p \end{matrix} \right\}_{n+\theta} \end{aligned}$$

which we note:

$$(\theta \Delta t\, [C] \,+ [B]) \left\{ \begin{matrix} u \\ p \end{matrix} \right\}_{n+1} = (-(1-\theta)\Delta t\, [C] \,+ [B]) \left\{ \begin{matrix} u \\ p \end{matrix} \right\}_{n} + \Delta t \left\{ \begin{matrix} f_u \\ f_p \end{matrix} \right\}_{n+\theta}$$

Noting μ_j the complex eigenvalues of the matrix $[B]^{-1}[C]$, the stability condition is expressed as:

$$\forall j, \quad |\frac{1-(1-\theta)\Delta t \mu_j}{1+\theta \Delta t \mu_j}| < 1$$

Noting $\mu_j^R = Re(\mu_j)$ and $\mu_j^I = Im(\mu_j)$, the inequality 4 is rewritten as:

$$\forall j, \quad -2\mu_j^R < (2\theta - 1) \left[(\mu_j^R)^2 + (\mu_j^I)^2 \right] \Delta t$$

The stability criteria are the following:

- If $\theta > 1/2$, the numerical scheme is unconditionnally stable if $\mu_j^R > 0$, and conditionnally stable if $\mu_j^R \leq 0$. In the latter case, the time step needs to satisfy:

$$\Delta t > -\frac{2}{2\theta - 1} \frac{\mu_j^R}{(\mu_j^R)^2 + (\mu_j^I)^2}$$

- If $\theta \leq 1/2$, the time step needs to satisfy:

$$\Delta t < \frac{2}{1 - 2\theta} \frac{\mu_j^R}{(\mu_j^R)^2 + (\mu_j^I)^2}$$

CHAPTER 7

FUNDAMENTAL PRINCIPLES OF PLASTICITY

PROBLEMS

7.1 Derive the stress-strain relationship for the Drucker-Prager elastic-perfectly plastic model, described by the following equations:

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^p_{ij}$$

$$\dot{\sigma}_{ij} = D^e_{ijkl}\dot{\epsilon}^e_{kl}$$

$$f(\sigma_{ij}) = \sqrt{J_2} - \alpha\, I_1 - k$$

$$g(\sigma_{ij}) = \sqrt{J_2} - \beta\, I_1$$

in which k, α and β are material constants, with $\alpha \neq \beta$.

Solution:

$$\dot{\sigma}_{ij} = D^e_{ijkl}\dot{\epsilon}^e_{kl}$$

$$\dot{\sigma}_{ij} = D^e_{ijkl}\left(\dot{\epsilon}_{kl} - \dot{\epsilon}^p_{kl}\right)$$

$$\dot{\sigma}_{ij} = D^e_{ijkl}\left(\dot{\epsilon}_{kl} - \lambda\frac{\partial g}{\partial \sigma_{kl}}\right) \tag{7.1}$$

To find the plastic multiplier, use the consistency condition:

$$df = 0 \Rightarrow \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = 0$$

$$\frac{\partial f}{\partial \sigma_{ij}} \left(D^e_{ijkl} \dot{\epsilon}_{kl} - D^e_{ijkl} \dot{\epsilon}^p_{kl} \right) = 0$$

$$\frac{\partial f}{\partial \sigma_{ij}} \left(D^e_{ijkl} \dot{\epsilon}_{kl} - \lambda \, D^e_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} \right) = 0$$

We get:

$$\lambda = \frac{\frac{\partial f}{\partial \sigma_{ij}} D^e_{ijkl} \dot{\epsilon}_{kl}}{\frac{\partial f}{\partial \sigma_{ij}} D^e_{ijkl} \frac{\partial g}{\partial \sigma_{kl}}}$$

Introducing the plastic mulitplier in equation 7.1:

$$\dot{\sigma}_{ij} = \left[D^e_{ijkl} - D^e_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} \frac{\frac{\partial f}{\partial \sigma_{ij}} D^e_{ijkl}}{\frac{\partial f}{\partial \sigma_{ij}} D^e_{ijkl} \frac{\partial g}{\partial \sigma_{kl}}} \right] \dot{\epsilon}_{kl} \tag{7.2}$$

To express the incremental stress/strain relationship explicitly, we need to calculate $\frac{\partial f}{\partial \sigma_{kl}}$ and $\frac{\partial g}{\partial \sigma_{kl}}$. From the expressions given in the problem:

$$\begin{aligned} \frac{\partial f}{\partial \sigma_{kl}} &= \frac{1}{2\sqrt{J_2}} \frac{\partial J_2}{\partial \sigma_{kl}} - \alpha \frac{\partial I_1}{\partial \sigma_{kl}} \\ \frac{\partial g}{\partial \sigma_{kl}} &= \frac{1}{2\sqrt{J_2}} \frac{\partial J_2}{\partial \sigma_{kl}} - \beta \frac{\partial I_1}{\partial \sigma_{kl}} \end{aligned}$$

To express equation 7.2 explicitly, we need to calculate $\frac{\partial I_1}{\partial \sigma_{ij}}$ and $\frac{\partial J_2}{\partial \sigma_{ij}}$. The details are provided below.

$$\begin{aligned} \frac{\partial I_1}{\partial \sigma_{ij}} &= \frac{\partial \sigma_{kl}}{\partial \sigma_{ij}} \delta_{kl} \\ &= \delta_{ij} \end{aligned}$$

In tensor notation:

$$\frac{\partial I_1}{\partial \boldsymbol{\sigma}} = \boldsymbol{\delta}$$

To calculate $\frac{\partial J_2}{\partial \sigma_{ij}}$, we start by evaluating $\frac{\partial I_2}{\partial \sigma_{ij}}$. We switch to tensor notation:

$$\begin{aligned}
\frac{\partial I_2}{\partial \boldsymbol{\sigma}} &= \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\sigma}}\left[(I_1)^2 - Tr(\boldsymbol{\sigma}^2)\right] \\
&= \frac{\partial I_1}{\partial \boldsymbol{\sigma}} - \frac{1}{2}\frac{\partial \boldsymbol{\sigma}^2}{\partial \boldsymbol{\sigma}} : \boldsymbol{\delta} \\
&= I_1\boldsymbol{\delta} - \boldsymbol{\sigma}^T \cdot \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\sigma}} : \boldsymbol{\delta} \\
&= I_1\boldsymbol{\delta} - \boldsymbol{\sigma}^T \cdot \boldsymbol{\Delta} : \boldsymbol{\delta} \\
&= I_1\boldsymbol{\delta} - \boldsymbol{\sigma}^T \cdot \boldsymbol{\delta} \\
&= I_1\boldsymbol{\delta} - \boldsymbol{\sigma}^T \\
&= I_1\boldsymbol{\delta} - \boldsymbol{\sigma}
\end{aligned}$$

in which $\boldsymbol{\Delta}$ is the fourth-order identity tensor. Then, we can calculate $\frac{\partial J_2}{\partial \sigma_{ij}}$:

$$\begin{aligned}
\frac{\partial J_2}{\partial \boldsymbol{\sigma}} &= \frac{\partial J_2}{\partial \boldsymbol{s}} : \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\sigma}} \\
&= \left(-\frac{\partial J_1}{\partial \boldsymbol{\sigma}} + \boldsymbol{s}\right) : \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\sigma}} \\
&= \boldsymbol{s} : \left(\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\sigma}} - \frac{1}{3}\frac{\partial I_1}{\partial \boldsymbol{\sigma}} \otimes \boldsymbol{\delta}\right) \\
&= \boldsymbol{s} : \left(\boldsymbol{\Delta} - \frac{1}{3}\boldsymbol{\delta} \otimes \boldsymbol{\delta}\right) \\
&= \boldsymbol{s} : \boldsymbol{\Delta} \\
&= \boldsymbol{s}
\end{aligned}$$

where $\boldsymbol{s}$ is the deviatoric stress, for which the first invariant J_1 is zero.

7.2 Derive the stress-strain relationship for the mixed hardening elastic-plastic model, described by the following equations:

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^p_{ij}$$

$$\dot{\sigma}_{ij} = D^e_{ijkl}\dot{\epsilon}^e_{kl}$$

$$f(\sigma_{ij}, \alpha_{ij}, \beta) = f_1(\sigma_{ij} - \alpha_{ij}) - R(\beta) = 0$$

$$\beta = \int \left[\sigma_{ij}\dot{\epsilon}^p_{ij}\right]$$

$$\dot{\alpha}_{ij} = d\mu\ (\sigma_{ij} - \alpha_{ij})$$

$$g(\sigma_{ij}, \alpha_{ij}, \beta) = f(\sigma_{ij}, \alpha_{ij}, \beta)$$

Solution: The procedure is similar to the previous problem. To start, replace the elastic strain increment by its decomposition into total and plastic strain increments. Then, use the consistency condition to get the expression of the plastic multiplier. Details follow:

$$\dot{\sigma}_{ij} = D^{e}_{ijkl}\left(\dot{\epsilon}_{kl} - \lambda\frac{\partial g}{\partial \sigma_{kl}}\right)$$

Consistency equation:

$$df = 0 \Rightarrow \frac{\partial f}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{\alpha}}\dot{\boldsymbol{\alpha}} + \frac{\partial f}{\partial \beta}\dot{\beta} = 0$$

$$\frac{\partial f}{\partial \boldsymbol{\sigma}}\boldsymbol{D_e} : \dot{\boldsymbol{\epsilon}} - \frac{\partial f}{\partial \boldsymbol{\sigma}}\boldsymbol{D_e} : \dot{\boldsymbol{\epsilon}}^{p} + d\mu\frac{\partial f}{\partial \boldsymbol{\alpha}}\left(\boldsymbol{\sigma} - \boldsymbol{\alpha}\right) + \frac{\partial f}{\partial \beta}\boldsymbol{\sigma} : \dot{\epsilon}^{p} = 0$$

$$\frac{\partial f}{\partial \boldsymbol{\sigma}}\boldsymbol{D_e} : \dot{\boldsymbol{\epsilon}} - \lambda\frac{\partial f}{\partial \boldsymbol{\sigma}}\boldsymbol{D_e} : \frac{\partial g}{\partial \boldsymbol{\sigma}} + d\mu\frac{\partial f}{\partial \boldsymbol{\alpha}}\left(\boldsymbol{\sigma} - \boldsymbol{\alpha}\right) + \lambda\frac{\partial f}{\partial \beta}\boldsymbol{\sigma} : \frac{\partial g}{\partial \boldsymbol{\sigma}} = 0$$

From there, we get the expression of λ:

$$\lambda = \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}}\boldsymbol{D_e} : \dot{\boldsymbol{\epsilon}}}{\frac{\partial f}{\partial \boldsymbol{\sigma}}\boldsymbol{D_e} : \frac{\partial g}{\partial \boldsymbol{\sigma}} - \frac{\partial f}{\partial \beta}\boldsymbol{\sigma} : \frac{\partial g}{\partial \boldsymbol{\sigma}}} + \frac{d\mu\frac{\partial f}{\partial \boldsymbol{\alpha}}\left(\boldsymbol{\sigma} - \boldsymbol{\alpha}\right)}{\frac{\partial f}{\partial \boldsymbol{\sigma}}\boldsymbol{D_e} : \frac{\partial g}{\partial \boldsymbol{\sigma}} - \frac{\partial f}{\partial \beta}\boldsymbol{\sigma} : \frac{\partial g}{\partial \boldsymbol{\sigma}}}$$

The incremental stress-strain relationship becomes:

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{e} : \dot{\boldsymbol{\epsilon}} - \boldsymbol{D}^{e} : \frac{\partial g}{\partial \boldsymbol{\sigma}}\frac{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{D_e}}{\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{D_e} : \frac{\partial g}{\partial \boldsymbol{\sigma}} - \frac{\partial f}{\partial \beta}\boldsymbol{\sigma} : \frac{\partial g}{\partial \boldsymbol{\sigma}}\right)} : \dot{\boldsymbol{\epsilon}} - \boldsymbol{D}^{e} : \frac{\partial g}{\partial \boldsymbol{\sigma}}\frac{d\mu\frac{\partial f}{\partial \boldsymbol{\alpha}} : \left(\boldsymbol{\sigma} - \boldsymbol{\alpha}\right)}{\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{D_e} : \frac{\partial g}{\partial \boldsymbol{\sigma}} - \frac{\partial f}{\partial \beta}\boldsymbol{\sigma} : \frac{\partial g}{\partial \boldsymbol{\sigma}}\right)}$$

CHAPTER 8

PERFECT PLASTICITY IN GEOMECHANICS

PROBLEMS

8.1 A material element is subjected to proportional loading. The principal stresses are given by (2σ, σ, 0) where σ is an increasing stress value.

1. Find the magnitude of σ where the material begins to yield, according to Tresca's criterion.

2. Adopting the associated flow rule, find also the plastic strain rate $\dot{\epsilon}^p_{ij}$ at onset of yielding expressed in terms of the plastic multiplier $\dot{\lambda}$.

3. If the effective plastic strain rate $\dot{\epsilon}^p_{eff}$ is defined as:

$$\dot{\epsilon}^p_{eff} = \sqrt{\frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij}},$$

   how is $\dot{\epsilon}^p_{eff}$ related to $\dot{\lambda}$?

4. Now suppose that the principal stresses are given by (σ, σ, 0). What problem is encountered is you should determine $\dot{\epsilon}^p_{ij}$?

Solution:

1. According to Tresca's criterion, when the material begins to yield when:

$$\sigma_1 - \sigma_3 - 2\,S_u = 0$$

In the test described in the problem:

$$2\sigma - 0 - 2\,S_u = 0$$

In other words:

$$\sigma = S_u$$

2. Adopting an associate flow rule:

$$\dot{\boldsymbol{\epsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\frac{\partial(\sigma_1 - \sigma_3 - 2S_u)}{\partial \boldsymbol{\sigma}}$$

In the principal stress base:

$$\dot{\boldsymbol{\epsilon}}^p = \begin{bmatrix} \dot{\lambda} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\dot{\lambda} \end{bmatrix}_{(\boldsymbol{e_1},\boldsymbol{e_2},\boldsymbol{e_3})}$$

3. We have:

$$\dot{\boldsymbol{\epsilon}}^p : \dot{\boldsymbol{\epsilon}}^p = 2\dot{\lambda}^2$$

So the effective plastic strain is calculated as:

$$\dot{\epsilon}^p_{eff} = \frac{2}{\sqrt{3}}\dot{\lambda}$$

4. The problem if $\sigma_1 = \sigma_2 = \sigma$, the problem is that the state of stress is at a corner point of the yield surface in the deviatoric plane. Therefore, there is no unique orientation for the plastic strain flow at this point. The plastic strain increment cannot be determined by an associate flow rule.

8.2 The stress at a point is given by:

$$[\sigma_{ij}] = \begin{bmatrix} 30 & 45 & 60 \\ 45 & 20 & 50 \\ 60 & 50 & 10 \end{bmatrix} \text{ MPa}$$

Determine the stress invariants I_1, J_2, J_3 and the Lode angle θ.

Solution:

We use MATLAB to calculate the three invariants. The code is the following:

```
A=[30,45,60;45,20,50;60,50,10]
B=[10,45,60;45,0,50;60,50,-10]
J2=0.5*trace(B*B)
J3=trace(B*B*B)/3
alpha=(3(3/2))*J3/(2*J2(3/2))
theta=asin(alpha)/3
thetadeg=theta*180/pi()
```

The results are the following:

$$I_1 = Tr(\boldsymbol{\sigma}) = 60\,\text{MPa}$$

$$J_2 = \frac{1}{2}Tr(\boldsymbol{s}^2) = 8,225\,\text{MPa}$$

$$J_3 = \frac{1}{3}Tr(\boldsymbol{s}^3) = 265,250\,\text{MPa}$$

The Lode angle is calculated by using the following relationship:

$$\sin(3\theta_l) = \frac{3\sqrt{3}J_3}{2J_2^{3/2}}$$

We get:

$$\theta_l \simeq 22.5^o$$

8.3 A material is to be loaded to a stress state

$$[\sigma_{ij}] = \begin{bmatrix} 50 & -30 & 0 \\ -30 & 90 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

What should be the minimum uniaxial yield stress of the material so that it does not fail, according to (a) Tresca criterion; (b) von Mises criterion? What do the theories predict when the yield stress of the material is 80MPa?

Solution:

We first calculate the principal stresses in the specimen:

$$\sigma_{I,II} = \frac{\sigma_{11}+\sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + (\sigma_{12})^2}, \quad \sigma_{III} = 0$$

We find:

$$\sigma_I = 106\,\text{MPa}, \quad \sigma_{II} = 34\,\text{MPa}, \quad \sigma_{III} = 0$$

Tresca's criterion is:

$$\sigma_I - \sigma_{III} = 2\,S_u$$

In uniaxial tension:

$$\sigma_I = 2\,S_u$$

So the uniaxial yield strenght is $\sigma_Y = 2\,S_u$. For the present loading case, the material will not fail according to Tresca's criterion if:

$$\sigma_I - \sigma_{III} \leq 2\,S_u = \sigma_Y$$

with $\sigma_{III} = 0$ and $\sigma_1 = 106\,\text{MPa}$. So according to Tresca's criterion, there is no failure if:

$$\sigma_Y \geq \sigma_I = 106\,\text{MPa}$$

Von Mises criterion is:

$$(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2 - 6k^2 = 0$$

In uniaxial tension:

$$2(\sigma_I)^2 - 6k^2 = 0$$

In other words, the uniaxial tension stress according to von Mises criterion is:

$$\sigma_Y = \sqrt{3}k$$

For the present loading case, von Mises criterion is:

$$2(\sigma_I)^2 + 2(\sigma_{II})^2 - 2\sigma_I\sigma_{II} - 2(\sigma_Y)^2 = 0$$

So the material fails in tension if:

$$(\sigma_Y)^2 = (\sigma_I)^2 + (\sigma_{II})^2 - \sigma_I\sigma_{II}$$

With the values of the principal stresses calculated above, we find that the material will not fail according to the von Mises criterion if:

$$\sigma_Y \geq 94\,\text{MPa}$$

So if $\sigma_Y = 80\,\text{MPa}$, we conclude that both theories predict that the material will fail because both theories predict that the uniaxial yield stress of the material should be more than 80 MPa for the material not to fail.

8.4 A material element is subjected to proportional loading. The principal stresses are given by (2σ, σ, 0) where σ is an increasing stress value.

1. Find the magnitude of σ where the material begins to yield, according to von Mises's criterion.

2. Adopting the associated flow rule, find also the plastic strain rate $\dot{\epsilon}^p_{ij}$ at onset of yielding expressed in terms of the plastic multiplier $\dot{\lambda}$.

3. If the effective plastic strain rate $\dot{\epsilon}^p_{eff}$ is defined as:

$$\dot{\epsilon}^p_{eff} = \sqrt{\frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij}},$$

how is $\dot{\epsilon}^p_{eff}$ related to $\dot{\lambda}$?

4. Repeat the three questions above when the principal stresses are given by (σ, σ, 0).

Solution:

1. According to von Mises criterion, first yield happens when:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 6k^2 = 0$$

With the loading conidtions described in the problem:

$$(2\sigma - \sigma)^2 + (\sigma)^2 + (2\sigma)^2 - 6k^2 = 0$$

So that we get:

$$\sigma = k$$

2. With an associate flow rule:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\frac{\partial}{\partial \boldsymbol{\sigma}}\left((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 6k^2\right)$$

In the principal stress base:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = \dot{\lambda}\begin{bmatrix} 4\sigma_1 - 2\sigma_2 - 2\sigma_3 & 0 & 0 \\ 0 & 4\sigma_2 - 2\sigma_1 - 2\sigma_3 & 0 \\ 0 & 0 & 4\sigma_3 - 2\sigma_1 - 2\sigma_2 \end{bmatrix}_{(\boldsymbol{e_1},\boldsymbol{e_2},\boldsymbol{e_3})}$$

With the loading conditions described in the problem:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = 6\dot{\lambda}\sigma\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{(\boldsymbol{e_1},\boldsymbol{e_2},\boldsymbol{e_3})}$$

At yield, $\sigma = k$ so that:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = 6\dot{\lambda}k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{(\boldsymbol{e_1},\boldsymbol{e_2},\boldsymbol{e_3})}$$

3. From the previous question, we have:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} : \dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = 72\dot{\lambda}^2 k^2$$

So that the effective plastic strain is:

$$\dot{\epsilon}^p_{eff} = 4\sqrt{3}\dot{\lambda}k$$

4. For the loading (σ, σ, 0), the yield according to von Mises criterion happens when:

$$(\sigma - \sigma)^2 + (\sigma)^2 + (\sigma)^2 - 6k^2 = 0$$

So that:

$$\sigma = \sqrt{3}k$$

Then, the plastic strain becomes:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = 2\dot{\lambda}\sigma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})}$$

At yield, $\sigma = \sqrt{3}k$ so that:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = 2\sqrt{3}\dot{\lambda}k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})}$$

The effective plastic strain rate is:

$$\dot{\epsilon}^p_{eff} = 4\sqrt{3}\dot{\lambda}k$$

8.5 Prandtl-Reuss equations are obtained by combining Hooke's law with a flow rule that assumes that the plastic strain increments are proportional to the principal deviatoric stresses s_i:

$$\frac{\dot{\epsilon}^p_1}{s_1} = \frac{\dot{\epsilon}^p_2}{s_2} = \frac{\dot{\epsilon}^p_3}{s_3} = \dot{\lambda} \geq 0$$

In Cartesian coordinates, it is common to express the flow rule above in the following alternate form:

$$\frac{\dot{\epsilon}^p_{xx} - \dot{\epsilon}^p_{yy}}{s_{xx} - s_{yy}} = \frac{\dot{\epsilon}^p_{xx} - \dot{\epsilon}^p_{yy}}{\sigma_{xx} - \sigma_{yy}} = \frac{\dot{\epsilon}^p_{yy} - \dot{\epsilon}^p_{zz}}{s_{yy} - s_{zz}} = ... = \dot{\lambda} \geq 0$$

In terms of actual stresses, one can show that:

$$\begin{aligned} \dot{\epsilon}^p_{xx} &= \frac{2}{3}\dot{\lambda}\left[\sigma_{xx} - \frac{1}{2}(\sigma_{yy} + \sigma_{zz})\right], \quad \dot{\epsilon}^p_{yy} = \frac{2}{3}\dot{\lambda}\left[\sigma_{yy} - \frac{1}{2}(\sigma_{zz} + \sigma_{xx})\right] \\ \dot{\epsilon}^p_{zz} &= \frac{2}{3}\dot{\lambda}\left[\sigma_{zz} - \frac{1}{2}(\sigma_{xx} + \sigma_{yy})\right], \quad \dot{\epsilon}^p_{xy} = \dot{\lambda}\sigma_{xy}, \quad \dot{\epsilon}^p_{yz} = \dot{\lambda}\sigma_{yz}, \quad \dot{\epsilon}^p_{zx} = \dot{\lambda}\sigma_{zx} \end{aligned}$$

The Prandtl-Reuss equations are the full elastic-plastic stress-strain relations that are obtained by combining the previous equations with Hooke's law:

$$\begin{aligned}
\dot{\epsilon}_{xx} &= \frac{1}{E}\left[\dot{\sigma}_{xx} - \nu\left(\dot{\sigma}_{yy} + \dot{\sigma}_{zz}\right)\right] + \frac{2}{3}\dot{\lambda}\left[\sigma_{xx} - \frac{1}{2}\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
\dot{\epsilon}_{yy} &= \frac{1}{E}\left[\dot{\sigma}_{yy} - \nu\left(\dot{\sigma}_{zz} + \dot{\sigma}_{xx}\right)\right] + \frac{2}{3}\dot{\lambda}\left[\sigma_{yy} - \frac{1}{2}\left(\sigma_{zz} + \sigma_{xx}\right)\right] \\
\dot{\epsilon}_{zz} &= \frac{1}{E}\left[\dot{\sigma}_{zz} - \nu\left(\dot{\sigma}_{xx} + \dot{\sigma}_{yy}\right)\right] + \frac{2}{3}\dot{\lambda}\left[\sigma_{zz} - \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
\dot{\epsilon}_{xy} &= \frac{1+\nu}{E}\dot{\sigma}_{xy} + \dot{\lambda}\sigma_{xy}, \quad \dot{\epsilon}_{yz} = \frac{1+\nu}{E}\dot{\sigma}_{yz} + \dot{\lambda}\sigma_{yz}, \quad \dot{\epsilon}_{zx} = \frac{1+\nu}{E}\dot{\sigma}_{zx} + \dot{\lambda}\sigma_{zx}
\end{aligned}$$

$$\dot{\epsilon}_{ij} = \frac{1+\nu}{E}\dot{\sigma}_{ij} - \frac{\nu}{E}\delta_{ij}\dot{\sigma}_{kk} + \dot{\lambda}s_{ij}$$

Consider the uniaxial straining of a perfectly plastic isotropic von Mises material specimen. There is only one non-zero strain, ϵ_{xx}. One only need to consider two stresses, σ_{xx}, σ_{yy}, since $\sigma_{zz} = \sigma_{yy}$ by isotropy.

1. Write down the two relevant Prandtl-Reuss equations.
2. Evaluate the stresses and strains at first yield.
3. For plastic flow, show that $\dot{\sigma}_{xx} = \dot{\sigma}_{yy}$ and that:

$$\frac{\dot{\sigma}_{xx}}{\dot{\epsilon}_{xx}} = \frac{E}{3(1-2\nu)}$$

Solution:

1. The two relevant Prandtl-Reuss equations are the following:

$$\begin{aligned}
\dot{\epsilon}_{xx} &= \frac{1}{E}\left(\dot{\sigma}_{xx} - 2\nu\dot{\sigma}_{yy}\right) + \frac{2}{3}\dot{\lambda}\left(\sigma_{xx} - \sigma_{yy}\right) \\
0 &= \frac{1}{E}\left[\dot{\sigma}_{yy} - \nu\left(\dot{\sigma}_{yy} + \dot{\sigma}_{xx}\right)\right] + \frac{2}{3}\dot{\lambda}\left[\sigma_{yy} - \frac{1}{2}\left(\sigma_{yy} + \sigma_{xx}\right)\right]
\end{aligned}$$

2. According to von Mises criterion, at first yield, we have:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 6k^2 = 0$$

$$(\sigma_{xx} - \sigma_{yy})^2 - 3k^2 = 0$$

And therefore:

$$\sigma_{xx} - \sigma_{yy} = \sqrt{3}k \tag{8.1}$$

At first yield, the stress/strain relationships are still elastic so that the Prandtl-Reuss equations become:

$$\begin{aligned}\dot{\epsilon}_{xx} &= \frac{1}{E}\left(\dot{\sigma}_{xx} - 2\nu\dot{\sigma}_{yy}\right) \\ 0 &= \frac{1}{E}\left[\dot{\sigma}_{yy} - \nu\left(\dot{\sigma}_{yy} + \dot{\sigma}_{xx}\right)\right]\end{aligned}$$

Noting that the experiment starts at zero strain and zero stress, integrating the above two equations between the initial time and the time when yield first occcurs provides:

$$\begin{aligned}\epsilon_{xx} &= \frac{1}{E}\left(\sigma_{xx} - 2\nu\sigma_{yy}\right) \\ 0 &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{yy} + \sigma_{xx}\right)\right]\end{aligned} \tag{8.2}$$

Using the second of equations 8.2 along with equation 8.1:

$$\sigma_{xx} - \frac{\nu}{1-\nu}\sigma_{xx} = \sqrt{3}k$$

We conclude that, at yield:

$$\sigma_{xx} = \frac{1-\nu}{1-2\nu}\sqrt{3}k$$

Then, using the second of the Prandtl-Reuss equations 8.2, we get:

$$\sigma_{yy} = \frac{\nu}{1-2\nu}\sqrt{3}k$$

We have σ_{zz} by isotropy, and the other three stress components are zero (no shear). Accoring to the loading conditions imposed in the problem, the only non-zero strain is ϵ_{xx}. We use the first of Equations 8.2 to get:

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - 2\nu\sigma_{yy}\right)$$

in which we used the fact that the experiment starts at zero strain. Introducing the expressions of the stress components found above:

$$\epsilon_{xx} = \frac{(1+\nu)\sqrt{3}k}{E}$$

3. At yield, the consistency conditions impose $f = \dot{f} = 0$, therefore:

$$\sigma_{xx} - \sigma_{yy} = \sqrt{3}k \tag{8.3}$$

$$\dot{\sigma}_{xx} = \dot{\sigma}_{yy} \tag{8.4}$$

Using the first of Prandtl-Reuss equations:

$$\dot{\epsilon}_{xx} = \frac{1}{E}\left(\dot{\sigma}_{xx} - 2\nu\dot{\sigma}_{yy}\right) + \frac{2}{3}\dot{\lambda}\left(\sigma_{xx} - \sigma_{yy}\right)$$

Since $\dot{\sigma}_{xx} = \dot{\sigma}_{yy}$ and $\sigma_{xx} - \sigma_{yy} = \sqrt{3}k$:

$$\dot{\epsilon}_{xx} = \frac{1-2\nu}{E}\dot{\sigma}_{xx} + \frac{2}{\sqrt{3}}\dot{\lambda}k \tag{8.5}$$

Now using the second of Prandtl-Reuss equations, using $\dot{\sigma}_{xx} = \dot{\sigma}_{yy}$ and $\sigma_{xx} - \sigma_{yy} = \sqrt{3}k$:

$$0 = \frac{1-2\nu}{E}\dot{\sigma}_{xx} - \frac{k}{\sqrt{3}}\dot{\lambda} \tag{8.6}$$

Now combning equations 8.5 and 8.6:

$$\dot{\epsilon}_{xx} = \frac{1-2\nu}{E}\dot{\sigma}_{xx} + \frac{2(1-2\nu)}{E}\dot{\sigma}_{xx}$$

And we finally get:

$$\frac{\dot{\sigma}_{xx}}{\dot{\epsilon}_{xx}} = \frac{E}{3(1-2\nu)}$$

8.6 Mohr-Coulomb's yield criterion:

1. Show that the magnitude of the hydrostatic stress vector is $\rho = \sqrt{3}c\cot\Phi$ for the Mohr-Coulomb yield criterion when the deviatoric stress is zero.
2. Show that, for a Mohr-Coulomb material, $\sin\Phi = (r-1)/(r+1)$ where $r = f_{Yc}/f_{Yt}$ is the compressive to tensile strength ratio.
3. A sample of concrete is subjected to a stress $\sigma_{11} = \sigma_{22} = -p$, $\sigma_{33} = -Ap$ where the constant $A > 1$. Using the Mohr-Coulomb criterion and the result of the previous question, show that the material will not fail provided $A < \frac{1}{r}\left(1 + \frac{f_{Yc}}{p}\right)$.

Solution:

1. The Moh-Coulomb criterion is:

$$\sigma_1 - \sigma_3 - (\sigma_1 + \sigma_3)\sin\Phi - 2c\cos\Phi = 0$$

In the absence of deviatoric stress, $\sigma_1 - \sigma_3 = 0$ and:

$$(\sigma_1 + \sigma_3)\sin\Phi = -2c\cos\Phi$$

For hydrostatic stress: $\sigma_1 = \sigma_2 = \sigma_3 = p$ (where p is typically a compression, otherwise yield cannot happen) So we have:

$$p = -c\,atan\Phi$$

The norm of the hydrostatic vector is:

$$\rho = \frac{|I_1|}{\sqrt{3}} = \sqrt{3}|p|$$

So we have:

$$\rho = \frac{c\sqrt{3}}{\tan \Phi}$$

2. Mohr's circles for the uniaxial tension test and the uniaxial compression test are represented in Figure 8.1.

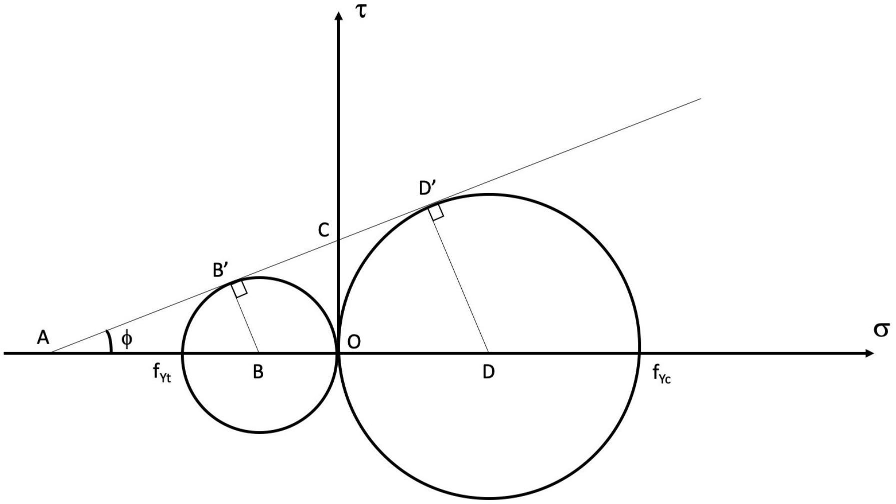


Figure 8.1 Mohr-Coulomb failure envelope.

We have:

$$\sin \Phi = \frac{BB'}{AB} = \frac{DD'}{AD}$$

BB' is the radius of the left-hand side circle, which is $f_{Yt}/2$. DD' is the radisu of the right-hand side circle, which is $f_{Yc}/2$. BD is the sum of the two radii. So we have:

$$\sin \Phi = \frac{f_{Yt}}{2AB} = \frac{f_{Yc}}{2AB + f_{Yt} + f_{Yc}}$$

Re-arranging:

$$AB = \frac{f_{Yc} + f_{Yt}}{f_{Yc} - f_{Yt}} \frac{f_{Yt}}{2}$$

So we have:

$$\sin \Phi = \frac{f_{Yc} - f_{Yt}}{f_{Yc} + f_{Yt}} = \frac{r-1}{r+1}$$

3. Using Mohr-Coulomb criterion, the material will not fail if:

$$\sigma_1 - \sigma_3 - (\sigma_1 + \sigma_3) \sin \Phi - 2c \cos \Phi \leq 0$$

$$-p(1-A) + p(1+A) \sin \Phi - 2c \cos \Phi \leq 0$$

$$Ap(1+\sin \Phi) - p(1-\sin \Phi) - 2c \cos \Phi \leq 0$$

$$A \leq \frac{1-\sin \Phi}{1+\sin \Phi} + 2\frac{c}{p}\frac{\cos \Phi}{1+\sin \Phi}$$

From the previous question:

$$1 + \sin\Phi = \frac{2r}{r+1}, \quad 1 - \sin\Phi = \frac{2}{r+1}, \quad \frac{1-\sin\Phi}{1+\sin\Phi} = \frac{1}{r}$$

So we have:

$$A \leq \frac{1}{r} + \frac{c}{p}\frac{r+1}{r}\cos\Phi$$

From the geometry shown in Figure 8.1, we have:

$$\cos\Phi = \frac{AO}{AC}, \quad \sin\Phi = \frac{c}{AC}$$

So that:

$$c\cos\Phi = \sin\Phi\,(AB + f_{Yt}/2)$$

Using the expression of AB from the previous question and re-arranging:

$$c\cos\Phi = \sin\Phi\frac{f_{Yt}f_{Yc}}{f_{Yc} - f_{Yt}}$$

And finally, the criterion becomes:

$$A \leq \frac{1}{r} + \frac{1}{p}\frac{r+1}{r}\sin\Phi\frac{f_{Yt}f_{Yc}}{f_{Yc} - f_{Yt}}$$

$$A \leq \frac{1}{r} + \frac{1}{p}\frac{r+1}{r}\frac{r-1}{r+1}\frac{f_{Yc}}{r-1}$$

$$A \leq \frac{1}{r} + \frac{1}{p}\frac{1}{r}f_{Yc}$$

Finally:

$$A \leq \frac{1}{r}\left(1 + \frac{f_{Yc}}{p}\right)$$

8.7 A material element is subjected to proportional loading. The principal stresses are given by $(2\sigma, \sigma, 0)$ where σ is an increasing stress value.

1. Find the magnitude of σ where the material begins to yield, according to Mohr-Coulomb's criterion.
2. Adopting the associated flow rule, find also the plastic strain rate $\dot{\epsilon}^p_{ij}$ at onset of yielding expressed in terms of the plastic multiplier $\dot{\lambda}$.
3. If the effective plastic strain rate $\dot{\epsilon}^p_{eff}$ is defined as:

$$\dot{\epsilon}^p_{eff} = \sqrt{\frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij}},$$

how is $\dot{\epsilon}^p_{eff}$ related to $\dot{\lambda}$?

4. Now suppose that the principal stresses are given by (σ, σ, 0). What problem is encountered is you should determine $\dot{\epsilon}^p_{ij}$?

Solution:

1. The Mohr-Coulomb criterion is:

$$\sigma_1 - \sigma_3 - (\sigma_1 + \sigma_3)\sin\Phi - 2c\cos\Phi = 0$$

With the loading conditions described in the problem:

$$\sigma(1 - \sin\Phi) = c\cos\Phi$$

$$\sigma = \frac{c\cos\Phi}{1 - \sin\Phi}$$

2. With an associate flow rule, the plastic strain rate is:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\frac{\partial(\sigma_1 - \sigma_3 - (\sigma_1 + \sigma_3)\sin\Phi - 2c\cos\Phi)}{\partial \boldsymbol{\sigma}}$$

In the principal stress base:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = \dot{\lambda}\begin{bmatrix} 1 - \sin\Phi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(1 + \sin\Phi) \end{bmatrix}_{(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})}$$

3. From the previous question:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} : \dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = \dot{\lambda}^2\left[(1 - \sin\Phi)^2 + (1 + \sin\Phi)^2\right]$$

From the previous problem:

$$1 + \sin\Phi = \frac{2r}{r+1}, \quad 1 - \sin\Phi = \frac{2}{r+1}$$

So the increment of effective plastic strain is:

$$\dot{\epsilon}^p_{eff} = \dot{\lambda}\sqrt{\frac{2}{3}\left[\left(\frac{2r}{r+1}\right)^2 + \left(\frac{2}{r+1}\right)^2\right]}$$

$$\dot{\epsilon}^p_{eff} = \frac{2}{r+1}\dot{\lambda}\sqrt{\frac{2}{3}\left(r^2 + 1\right)}$$

4. If the loading is $\sigma_1 = \sigma_2 = \sigma$, then the state of stress lies at a corner of the yield surface in the deviatoric plane, and so there is no unique plastic flow direction. The plastic strain tensor cannot be calculated with an associate flow rule.

8.8 Drucker-Prager's yield criterion:

1. Show that the magnitude of the hydrostatic stress vector is $\rho = |\rho| = k/\sqrt{3}\alpha$ for the Drucker-Prager yield criterion when the deviatoric stress is zero.
2. Given the yield stresses σ_t and σ_c in uniaxial tension and compression, respectively, find the yield stress in shear resulting from the following yield criteria: (a) Tresca; (b) von Mises; (c) Mohr-Coulomb; (d) Drucker-Prager.

Solution:

1. Drucker-Prager criterion is:

$$\sqrt{J_2} - \alpha\, I_1 - k = 0$$

If the deviatoric stress is zero, then $J_2 = 0$ and the criterion reduces to:

$$I_1 = -\frac{k}{\alpha}$$

The magnitude of the hydrostatic stress vector is:

$$\rho = \frac{|I_1|}{\sqrt{3}}$$

So, we have:

$$\rho = \frac{k}{\sqrt{3}\alpha}$$

2. Shear yield stress

(a) Tresca's criterion is:

$$\sigma_1 - \sigma_3 - 2\, S_u = 0$$

In uniaxial compression, the material fails when $\sigma_1 = \sigma_c$ and $\sigma_3 = 0$; we then have: $\sigma_c - 0 - 2\, S_u = 0$. In uniaxial tension, the material fails when $\sigma_3 = \sigma_t < 0$ (soil mechanics sign convention) and $\sigma_1 = 0$; we then have: $0 - \sigma_t - 2\, S_u = 0$. So we have:

$$S_u = \frac{\sigma_c}{2} = -\frac{\sigma_t}{2}$$

Using Mohr's circles, one can wee that the maximum shear stress is $\tau_{max} = (\sigma_1 - \sigma_3)/2$. The material fails when:

$$\tau_{max} = S_u = \tau_Y$$

We conclude that for Tresca's criterion:

$$\tau_Y = \frac{\sigma_c}{2} = -\frac{\sigma_t}{2}$$

(b) Von Mises criterion is:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 6k^2 = 0$$

In uniaxial compression, the material fails when $\sigma_1 = \sigma_c$ and $\sigma_2 = \sigma_3 = 0$; we then have:

$$2(\sigma_c)^2 - 6k^2 = 0$$

so we have:

$$\sigma_c = \sqrt{3}k$$

In uniaxial tension, the material fails when $\sigma_3 = \sigma_t$ and $\sigma_1 = \sigma_2 = 0$; we then have:

$$2(\sigma_t)^2 - 6k^2 = 0$$

so we have:

$$\sigma_t = \sqrt{3}k$$

In pure shear, $\sigma_2 = 0$, $\sigma_1 = -\sigma_3 = \sigma$ and $\tau_{max} = \sigma$.

$$(\sigma - 0)^2 + (0 - \sigma)^2 + (\sigma + \sigma)^2 - 6k^2 = 0$$

In other words:

$$\tau_{max} = k = \tau_Y$$

So we have:

$$\tau_Y = \frac{\sigma_c}{\sqrt{3}k} = \frac{\sigma_t}{\sqrt{3}k}$$

(c) The Mohr-Coulomb criterion is:

$$\sigma_1 - \sigma_3 - (\sigma_1 + \sigma_3)\sin\Phi - 2c\cos\Phi = 0$$

In uniaxial compression, $\sigma_1 = \sigma_c$, $\sigma_2 = \sigma_3 = 0$ and the material fails when:

$$(1 - \sin\Phi)\sigma_c - 2c\cos\Phi = 0$$

In uniaxial tension, $\sigma_3 = \sigma_t$, $\sigma_1 = \sigma_2 = 0$ and the material fails when:

$$-(1 + \sin\Phi)\sigma_t - 2c\cos\Phi = 0$$

In pure shear: $\sigma_1 = -\sigma_3$, $\sigma_2 = 0$ and $\tau = \sigma_1$, and the material fails when:

$$2\tau - 2c\cos\Phi = 0$$

So we have:

$$\tau_Y = c\cos\Phi = (1 - \sin\Phi)\frac{\sigma_c}{2}$$

From Problem 9.6, we have:

$$\sin\Phi = \frac{r-1}{r+1}, \quad r = \frac{\sigma_c}{\sigma_t}$$

As a result:

$$\tau_Y = (1 - \frac{r-1}{r+1})\frac{\sigma_c}{2} = \frac{1}{r+1}\sigma_c$$

And finally:

$$\tau_Y = \frac{\sigma_c\sigma_t}{\sigma_c + \sigma_t}$$

(d) Drucker-Prager's criterion is:

$$\sqrt{J_2} - \alpha\, I_1 - k = 0$$

When using principal stresses:

$$\sqrt{\frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} - \alpha\, I_1 - k = 0$$

In pure shear, $\sigma_1 = -\sigma_3$, $\sigma_2 = 0$, and $\tau = \sigma_1$. The material thus fails in pure shear when:

$$|\tau| - k = 0$$

Thus:

$$\tau_Y = k \tag{8.7}$$

In uniaxial compression: $\sigma_1 = \sigma_c$, $\sigma_2 = \sigma_3 = 0$, and the material fails when:

$$\frac{\sigma_c}{\sqrt{3}} - \alpha\, \sigma_c - k = 0 \tag{8.8}$$

In uniaxial tension: $\sigma_3 = \sigma_t < 0$, $\sigma_1 = \sigma_2 = 0$, and the material fails when:

$$-\frac{\sigma_t}{\sqrt{3}} - \alpha\, \sigma_t - k = 0 \tag{8.9}$$

We subtract Equation 8.9 from Equation 8.8 to find α:

$$\alpha = \frac{\sigma_t + \sigma_c}{\sqrt{3}(\sigma_c - \sigma_t)} \tag{8.10}$$

From Equations 8.7, 8.8 and **??**, we have:

$$\tau_Y = \left(1 - \frac{\sigma_t + \sigma_c}{\sigma_c - \sigma_t}\right)\frac{\sigma_c}{\sqrt{3}}$$

$$\tau_Y = \frac{2\sigma_t\sigma_c}{\sqrt{3}(\sigma_t - \sigma_c)}$$

8.9 A material element is subjected to proportional loading. The principal stresses are given by $(2\sigma, \sigma, 0)$ where σ is an increasing stress value.

1. Find the magnitude of σ where the material begins to yield, according to Drucker-Prager's criterion.
2. Adopting the associated flow rule, find also the plastic strain rate $\dot{\epsilon}^p_{ij}$ at onset of yielding expressed in terms of the plastic multiplier $\dot{\lambda}$.
3. If the effective plastic strain rate $\dot{\epsilon}^p_{eff}$ is defined as:

$$\dot{\epsilon}^p_{eff} = \sqrt{\frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij}},$$

how is $\dot{\epsilon}^p_{eff}$ related to $\dot{\lambda}$?

4. Repeat the three questions above when the principal stresses are given by $(\sigma, \sigma, 0)$.

Solution:

1. Drucker-prager criterion is:

$$\sqrt{J_2} - \alpha\, I_1 - k = 0$$

With the loading conditions described in the problem:

$$\sqrt{J_2} = \sqrt{\frac{1}{2} s_{ij} s_{ij}} = \sigma, \quad I_1 = 3\sigma$$

So the material yields when:

$$\sigma = \frac{k}{1 - 3\alpha}$$

2. Using an associated flow rule:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\lambda} \frac{\partial}{\partial \boldsymbol{\sigma}} \left(\sqrt{J_2} - \alpha\, I_1 - k \right) = \dot{\lambda} \left(\frac{\boldsymbol{s}}{2\sqrt{J_2}} - \alpha \boldsymbol{\delta} \right)$$

in which $\boldsymbol{\delta}$ is the second-order identity tensor. With the loading conditions given in the problem:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = \dot{\lambda} \begin{bmatrix} 1/2 - \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -1/2 - \alpha \end{bmatrix}$$

3. From the previous question, we have:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} : \dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = \dot{\lambda}^2 \left[(1/2 - \alpha)^2 + \alpha^2 + (1/2 + \alpha)^2 \right] = \dot{\lambda}^2 \left(3\alpha^2 + 1/2 \right)$$

From there, we get the effective plastic strain:

$$\dot{\epsilon}^p_{eff} = \dot{\lambda} \sqrt{\left(2\alpha^2 + \frac{1}{3} \right)}$$

4. For $\sigma_1 = \sigma_2 = \sigma$, $\sigma_3 = 0$, we have $J_2 = \sigma^2/3$ and so the material fails when:

$$\frac{\sigma}{\sqrt{3}} - 2\alpha\sigma - k = 0$$

$$\sigma = \frac{k}{\frac{1}{\sqrt{3}} - 2\alpha}$$

Using an associated flow rule, the plastic strain rate is:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = \dot{\lambda} \frac{\partial}{\partial \boldsymbol{\sigma}} = \dot{\lambda} \left(\frac{\boldsymbol{s}}{2\sqrt{J_2}} - \alpha \boldsymbol{\delta} \right)$$

and with the given loading conditions:

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}} = \dot{\lambda}\begin{bmatrix} \frac{1}{2\sqrt{3}} - \alpha & 0 & 0 \\ 0 & \frac{1}{2\sqrt{3}} - \alpha & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} - \alpha \end{bmatrix}$$

After some calculations, one finds:

$$\dot{\epsilon}^{p}_{eff} = \dot{\lambda}\sqrt{\left(2\alpha^2 + \frac{1}{3}\right)}$$

Note: we find the same effective plastic strain as in the previous loading case.

8.10 Conventional triaxial compression tests (σ_1, $\sigma_2 = \sigma_3$) were conducted on cylindrical rock specimens. The test results are reported in the following table.

σ_1 (MPa)	$\sigma_2 = \sigma_3$ (MPa)
48.3	1.7
53.7	2.8
56.7	3.4
70.8	6.9
70.9	6.9
94.8	13.8
94.7	13.8
94.9	13.8
115.8	20.7
115.9	20.7

Direct tension tests gace a tensile strength of $\sigma_t = 3.4$ MPa. We want to model the rock strength. Use plots to estimate the failure criterion parameters and discuss the applicability of the (a) Linear Mohr-Coulomb criterion; (b) Non-linear Hoek-Brown criterion.

Solution:

We recall the Mohr-Coulomb criterion:

$$\sigma_1 - \sigma_3 - (\sigma_1 + \sigma_3)\sin\Phi - 2c\,\cos\Phi = 0$$

and the Hoek-Brown criterion:

$$\sigma_1 - \sigma_3 - \sqrt{m\,Y\,\sigma_3 + s\,Y^2} = 0$$

The problem is solved by plotting the states of stress measured experimentally in the (σ_1, σ_3) plane, and by fitting each of the two criteria above to the data. The material parameters requested are obtained when the best fit is found between the experimental datapoints and the curves predicted by the models.

8.11 Homework 5 - Problem 1

In order to test whether the Von Mises or Tresca criteria best modelled the real behaviour of metals, Taylor & Quinney (1931), in a series of classic experiments, subjected a number of thin-walled cylinders made of copper and steel to combined tension and torsion, as shown in Figure 8.2. The cylinder wall is in a state of plane stress, with $\sigma_{11} = \sigma$, $\sigma_{12} = \tau$ and all other stress components zero.

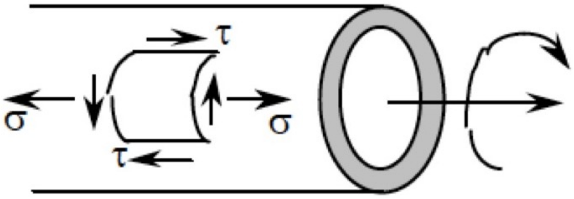


Figure 8.2 Taylor and Quinney's experimental set-up.

1. Show that the principal stress corresponding to the stress-state described above are zero and:

$$\frac{1}{2}\sigma \pm \sqrt{\frac{1}{4}\sigma^2 + \tau^2}$$

2. Show that the Mises condition reduces to:

$$\sigma^2 + 3\tau^2 = 3k^2 \text{ or } \left(\frac{\sigma}{Y}\right)^2 + \left(\frac{\tau}{Y/\sqrt{3}}\right)^2 = 1$$

   in which Y is the yield stress in tension.

3. Suppose that, in the Taylor and Quinney tension-torsion tests, one has $\sigma = Y/2$ and $\tau = \sqrt{3}Y/4$. Plot this stress state in the 2D principal stress state (Use Question 1 to evaluate the principal stresses.) Keeping now the normal stress at $\sigma = Y/2$, what value can the shear stress be increased to before the material yields, according to the von Mises criterion?

Solution:

1. In plane stress:

$$\sigma_{I,III} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + (\sigma_{12})^2}$$

We have $\sigma_{11} = \sigma$, $\sigma_{22} = 0$ and $\sigma_{12} = \tau$ so that:

$$\sigma_{I,III} = \frac{\sigma}{2} \pm \sqrt{\frac{\sigma^2}{4} + \tau^2}$$

Because the material element is in plane stress, the median principal stress σ_{II} is the out of plane normal stress σ_{33}, which is 0.

2. The Von Mises criterion is:

$$\sqrt{J_2} - k = 0$$

With the expressions of the principal stresses found in the previous question:

$$\left(\frac{\sigma}{2} + \sqrt{\frac{\sigma^2}{4} + \tau^2}\right)^2 + \left(\frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} + \tau^2}\right)^2 + \left(2\sqrt{\frac{\sigma^2}{4} + \tau^2}\right)^2 = 6k^2$$

After re-arranging:

$$\sigma^2 + 3\tau^2 = 3k^2$$

Dividing each side by the yield stress in tension $Y = \sqrt{3}k$:

$$\left(\frac{\sigma}{Y}\right)^2 + \left(\frac{\tau}{Y/\sqrt{3}}\right)^2 = 1$$

3. We first calculate the non-zero principal stresses for the given loading conditions $\sigma = Y/2$ and $\tau = \sqrt{3}Y/4$:

$$\sigma_I = \frac{Y}{4} + \sqrt{\frac{Y^2}{16} + \frac{3Y^2}{16}} = \frac{Y}{4} + \frac{Y}{2} = \frac{3Y}{4}$$

$$\sigma_{III} = \frac{Y}{4} - \sqrt{\frac{Y^2}{16} + \frac{3Y^2}{16}} = \frac{Y}{4} - \frac{Y}{2} = -\frac{Y}{4}$$

We thus have $\sigma_I = -3\sigma_{III}$. From there, it is possible to plot σ_{III} as a function of σ_I. If the normal stress is kept at $Y/2$, then the maximum shear stress that can be reached before failure is such that the von Mises criterion is reached under that shear stress:

$$\sigma^2 + 3\tau_{max}^2 = 3k^2 = Y^2$$

Therefore:

$$3\tau_{max}^2 = 3k^2 = Y^2 - \frac{Y^2}{4} = \frac{3Y^2}{4}$$

So that:

$$\tau_{max} = \frac{Y}{2}$$

We check that $\tau_{max} > \frac{\sqrt{3}Y}{4}$.

8.12 Homework 5 - Problem 2

Consider the combined tension-torsion of a thin-walled cylindrical tube. The tube is made of a perfectly plastic Von Mises metal and Y is the uniaxial yield strength in tension. The only stresses are $\sigma = \sigma_{xx}$ and $\sigma_{xy} = \tau$ and the Prandtl-Reuss equations reduce to:

$$\begin{aligned}
\dot\epsilon_{xx} &= \frac{1}{E}\dot\sigma_{xx} + \frac{2}{3}\dot\lambda\sigma_{xx} \\
\dot\epsilon_{yy} &= \dot\epsilon_{zz} = -\frac{\nu}{E}\dot\sigma_{xx} - \frac{1}{3}\dot\lambda\sigma_{xx} \\
\dot\epsilon_{xy} &= \frac{1+\nu}{E}\dot\sigma_{xy} + \dot\lambda\sigma_{xy}
\end{aligned}$$

The axial strain is increased from zero until yielding occurs (with $\epsilon_{xy} = 0$). From first yield, the axial strain is held constant and the shear strain is increased up to its final value of $(1+\nu)Y/\sqrt{3}E$.

1. Write down the yield criterion in terms of σ and τ only and sketch the yield locus in σ-τ space.
2. Evaluate the stresses and strains at first yield.
3. Evaluate $\dot\lambda$ in terms of σ, $\dot\sigma$.
4. Relate σ, $\dot\sigma$ to τ, $\dot\tau$ and hence derive a differential equation for shear strain in terms of τ only.
5. Solve the differential equation and evaluate any constant of integration.
6. Evaluate the shear stress when ϵ_{xy} reaches its final value of $(1+\nu)Y/\sqrt{3}E$. Taking $\nu = 1/2$, put in the form $\tau = \alpha Y$.

Solution:

1. The problem is in plane stress so the answer to this question is the same as that of questions 1 and 2 in the previous problem:

   $$\sigma^2 + 3\tau^2 = 3k^2$$

   Or, dividing each side by the yield stress in tension $Y = \sqrt{3}k$:

   $$\left(\frac{\sigma}{Y}\right)^2 + \left(\frac{\tau}{Y/\sqrt{3}}\right)^2 = 1$$

   The equation above is that of an ellipse of long axis Y in the σ direction and of short axis $Y/\sqrt{3}$ in the τ direction.

2. At first yield, there is no plastic flow yet, so $\dot\lambda = 0$, so the Prandtl-Reuss equations reduce to:

   $$\begin{aligned}
   \dot\epsilon_{xx} &= \frac{1}{E}\dot\sigma_{xx} \\
   \dot\epsilon_{yy} &= \dot\epsilon_{zz} = -\frac{\nu}{E}\dot\sigma_{xx} \\
   \dot\epsilon_{xy} &= \frac{1+\nu}{E}\dot\sigma_{xy}
   \end{aligned}$$

Before yield, the axial strain is increased while maintaining $\epsilon_{xy} = 0$. At the same time, the specimen is free of stress on the lateral faces. The axial strain is increased until the yield is reached. Therefore, at first yield, remembering that $Y = \sqrt{3}k$ in tension from previous problems:

$$\begin{aligned}
\sigma_{xx} &= E\,\epsilon^Y_{xx} = \sqrt{3}\,k \\
\epsilon_{xx} &= \epsilon^Y_{xx} = \frac{\sqrt{3}\,k}{E} \\
\epsilon_{yy} &= \epsilon_{zz} = -\nu\,\epsilon^Y_{xx} = -\nu\,\frac{\sqrt{3}\,k}{E} \\
\sigma_{yy} &= \sigma_{zz} = 0 \\
\epsilon_{xy} &= 0 \\
\sigma_{xy} &= 0
\end{aligned}$$

in which ϵ^Y_{xx} is the axial yield strain. Since the specimen is in plane stress, $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$ and in the absence of shear stress in the z-direction, $\epsilon_{xz} = \epsilon_{yz} = 0$.

3. We now use the first of the three Prandtl-Reuss equations given in the problem to find an expression for the plastic multiplier:

$$\dot{\epsilon}_{xx} = \frac{1}{E}\dot{\sigma}_{xx} + \frac{2}{3}\dot{\lambda}\sigma_{xx}$$

After the first yield, ϵ_{xx} is maintained constant while torsion is applied. As a result, after first yield:

$$0 = \frac{1}{E}\dot{\sigma}_{xx} + \frac{2}{3}\dot{\lambda}\sigma_{xx}$$

in which we are given $\sigma_{xx} = \sigma$. Therefore:

$$\dot{\lambda} = -\frac{3}{2E}\frac{\dot{\sigma}}{\sigma}$$

4. In order to find the expression of the incremental shear strain, we use the third of the given Prandtl-Reuss equations:

$$\dot{\epsilon}_{xy} = \frac{1+\nu}{E}\dot{\tau} + \dot{\lambda}\tau$$

Introducing the expression of the plastic multiplier found at the previous question:

$$\dot{\epsilon}_{xy} = \frac{1+\nu}{E}\dot{\tau} - \frac{3}{2E}\frac{\dot{\sigma}}{\sigma}\tau \tag{8.11}$$

During yield, the consistency conditions impose $f = \dot{f} = 0$, so that:

$$\begin{aligned}
&\sigma^2 + 3\tau^2 = Y^2 \qquad (8.12)\\
&\sigma\dot{\sigma} + 3\tau\dot{\tau} = 0
\end{aligned}$$

in which we used the result of question 1. Now using the second of Equations 8.12:

$$\frac{\dot{\sigma}}{\sigma}\tau + \frac{3\tau^2\dot{\tau}}{\sigma^2} = 0$$

Using the above expression in Equation 8.11:

$$\dot{\epsilon}_{xy} = \frac{1+\nu}{E}\dot{\tau} + \frac{9}{2E}\frac{\tau^2\dot{\tau}}{\sigma^2}$$

combining the above equation with the first of Equations 8.12:

$$\dot{\epsilon}_{xy} = \frac{1+\nu}{E}\dot{\tau} + \frac{9}{2E}\frac{\tau^2\dot{\tau}}{Y^2 - 3\tau^2}$$

which is the requested differential equation for the shear strain, in terms of τ only.

5. To solve the differential equation found in the previous question, we integrate between the time at first yield and the current time, and that at first yield, $\epsilon_{xy} = 0$ and $\tau = 0$. We use MATLAB to integrate the second term in the expression of ϵ_{xy}. The code is the following:

```
syms x c
fun=x*x/(c*c-3*x*x)
ff=int(fun,x)
```

The result is:

$$\epsilon_{xy} = \left(\frac{2(1+\nu)-3}{2E}\right)\tau + \frac{1}{2E}\left(\sqrt{3}Y\,atanh\left(\frac{\sqrt{3}\tau}{Y}\right)\right)$$

in which the inverse of the hyperbolic tangent is defined as:

$$atanh(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

6. Using the response to the previous question, when the shear strain reaches its final value, we have:

$$\frac{(1+\nu)Y}{\sqrt{3}E} = \left(\frac{2(1+\nu)-3}{2E}\right)\tau + \frac{1}{2E}\left(\sqrt{3}Y\,atanh\left(\frac{\sqrt{3}\tau}{Y}\right)\right)$$

Now using $\nu = 1/2$:

$$1 = atanh\left(\frac{\sqrt{3}\tau}{Y}\right)$$

Using MATLAB for the calculation of the hyperbolic tangent:

$$0.7616 = \frac{\sqrt{3}\tau}{Y}$$

And finally:

$$\tau = \frac{0.7616\,Y}{\sqrt{3}} \simeq 0.4397\,Y$$

8.13 Homework 5 - Problem 3

The shear strength of a soil is most frequently characterized by a frictional failure criterion. Two commonly used failure criteria are:
Mohr-Coulomb:

$$f = \left(\frac{\sigma_1 - \sigma_3}{2}\right) - \left(\frac{\sigma_1 + \sigma_3}{2}\right) \sin\Phi = 0$$

Extended Von Mises:

$$f = J_2 - h^2\sigma^2 = 0$$

where σ_1 and σ_3 are the major and minor principal stresses, Φ is the friction angle, $J_2 = s_{ij}s_{ij}/2$ is the second invariant of the deviatoric stress tensor, $\sigma = \sigma_{kk}/3$ is the mean (octahedral) stress and h is a constant. In conventional practice, the friction angle $\Phi = \Phi_{TC}$ is reported from measurements of the shear strength in triaxial compression type tests (in which $\sigma_{xx} = \sigma_{zz} < \sigma_{yy}$).

1. Derive an expression of the frictional parameter h in terms of this friction angle Φ_{TC}.
2. Assuming that failure of the soil is best described by the extended Von Mises criterion, what is the frictional angle mobilized in a triaxial extension test? *Here, calculate* Φ_{TE} *for the case when* $\sigma_{xx} = \sigma_{zz} > \sigma_{yy}$.
3. Plot values of Φ_{TE} as a function of the measured friction angle Φ_{TC} for the extended Von Mises criterion.

Solution:

1. During a triaxial compression test, the yield is reached when:

$$\left(\frac{\sigma_{yy} - \sigma_{xx}}{2}\right) - \left(\frac{\sigma_{yy} + \sigma_{xx}}{2}\right) \sin\Phi_{TC} = 0 \tag{8.13}$$

with the Mohr-Coulomb criterion, and when:

$$\frac{1}{6}\left[2\left(\sigma_{yy} - \sigma_{xx}\right)^2\right] - h^2\left(\sigma_{yy} + 2\sigma_{xx}\right) = 0 \tag{8.14}$$

with the extended Von Mises criterion. From Equation 8.13:

$$\sin\Phi_{TC} = \frac{\sigma_{yy} - \sigma_{xx}}{\sigma_{yy} + \sigma_{xx}} \tag{8.15}$$

From Equation 8.14:

$$h^2 = \frac{\left(\sigma_{yy} - \sigma_{xx}\right)^2}{3\left(\sigma_{yy} + 2\sigma_{xx}\right)^2} \tag{8.16}$$

From Equation 8.15:

$$\frac{\sigma_{xx}}{\sigma_{yy}} = \frac{1 - \sin\Phi_{TC}}{1 + \sin\Phi_{TC}} \tag{8.17}$$

Combining equations 8.15 and 8.16:

$$h^2 = \frac{(\sin\Phi_{TC})^2}{3}\frac{\left(\sigma_{yy} + \sigma_{xx}\right)^2}{\left(\sigma_{yy} + 2\sigma_{xx}\right)^2} = \frac{(\sin\Phi)^2}{3}\frac{\left(1 + \sigma_{xx}/\sigma_{yy}\right)^2}{\left(1 + 2\sigma_{xx}/\sigma_{yy}\right)^2} \tag{8.18}$$

Combining equations 8.17 and 8.19 and re-arranging:

$$h = \frac{2\sin\Phi_{TC}}{\sqrt{3}(3-\sin\Phi_{TC})} \tag{8.19}$$

2. During a triaxial extension test, the yield is reached when:

$$\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right) - \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right)\sin\Phi_{TE} = 0 \tag{8.20}$$

with the Mohr-Coulomb criterion, and when:

$$\frac{1}{6}\left[2\left(\sigma_{xx}-\sigma_{yy}\right)^2\right] - h^2\left(\sigma_{yy}+2\sigma_{xx}\right) = 0 \tag{8.21}$$

with the extended Von Mises criterion. From Equation 8.20:

$$\sin\Phi_{TE} = \frac{\sigma_{xx}-\sigma_{yy}}{\sigma_{yy}+\sigma_{xx}} \tag{8.22}$$

From Equation 8.21:

$$h^2 = \frac{\left(\sigma_{yy}-\sigma_{xx}\right)^2}{3\left(\sigma_{yy}+2\sigma_{xx}\right)^2} \tag{8.23}$$

From Equation 8.22:

$$\frac{\sigma_{xx}}{\sigma_{yy}} = \frac{1+\sin\Phi_{TE}}{1-\sin\Phi_{TE}} \tag{8.24}$$

Combining equations 8.22 and 8.23:

$$h^2 = \frac{(\sin\Phi_{TE})^2}{3}\frac{\left(\sigma_{yy}+\sigma_{xx}\right)^2}{\left(\sigma_{yy}+2\sigma_{xx}\right)^2} = \frac{(\sin\Phi)^2}{3}\frac{\left(1+\sigma_{xx}/\sigma_{yy}\right)^2}{\left(1+2\sigma_{xx}/\sigma_{yy}\right)^2} \tag{8.25}$$

Combining equations 8.24 and 8.26 and re-arranging:

$$h = \frac{2\sin\Phi_{TE}}{\sqrt{3}(3+\sin\Phi_{TE})} \tag{8.26}$$

3. Here, we combine equations 8.19 and 8.26 to relate Φ_{TC} to Φ_{TE}:

$$\frac{\sin\Phi_{TC}}{\sin\Phi_{TE}} \times \frac{3+\sin\Phi_{TE}}{3-\sin\Phi_{TC}} = 1$$

After re-arranging:

$$\sin\Phi_{TE} = \frac{3\sin\Phi_{TC}}{3-2\sin\Phi_{TC}}$$

From there, it is possible to plot Φ_{TE} as a function of Φ_{TC}.

8.14 Homework 5 - Problem 4

The objective of this problem is to derive the incremental constitutive equations of an elasto-plastic rock, for a plastic potential similar to Drucker-Prager's. Assume that in the elastic domain, the rock has a linear isotropic behavior. The following notations are adopted:

$$\begin{aligned} s_{ij} &= 2G\, e^e_{ij} \\ p &= K\, \epsilon^e_{kk} \end{aligned}$$

In which: e_{ij} stands for deviatoric strain, s_{ij} and p are the deviatoric and mean stress, respectively, and G and K are the shear and bulk modulus, respectively. Drucker-Prager plastic criterion writes:

$$F = \sqrt{\frac{1}{2} s_{ij} s_{ji}} - f(q - p)$$

In which q is a constant and f is a friction parameter, assumed to depend on deviatoric plastic deformation g^p. Drucker-Prager plastic potential writes:

$$Q = \sqrt{\frac{1}{2} s_{ij} s_{ji}} + dp$$

In which d is a dilatancy parameter, assumed to depend on deviatoric plastic deformation g^p.

1. Recall the decomposition of stress in volumetric and deviatoric parts. Apply this decomposition to the elastic and plastic parts of the strain tensor.

2. Determine the derivatives of the yield function and plastic potential. Write the consistency condition to derive an equation relating the increment of stress to the rate of deviatoric plastic deformation $\dot{g}^p = \sqrt{2\dot{e}^p_{ij}\dot{e}^p_{ji}}$.

3. Write the plastic flow rule and show that the plastic multiplier writes $\dot{\lambda}_p = \dot{g}^p$. Derive an equation relating the rate of volumetric plastic deformation $\dot{\epsilon}^p_v$ to the rate of deviatoric plastic deformation $\dot{g}^p$.

4. Use elastic constitutive equations and the consistency condition to show that:

$$\dot{g}^p \;=\; \frac{<1>}{G + fdK + (q-p)h_t} \left(G \frac{s_{ij}}{\sqrt{\frac{1}{2} s_{ij} s_{ji}}} + K f \delta_{ij} \right) \dot{\epsilon}_{ij}$$

   In which $h_t = \frac{df}{dg^p}$, and δ_{ij} is the second-order identity tensor.
   Explain the meaning of the notation: $< 1 >$ (in terms of loading and unloading phases).

5. Derive an explicit relationship between the increment of stress and the rate of total deformation, i.e. provide the expression of L^e_{ijkl} and L^p_{ijkl} in the equation $\dot{\sigma}_{ij} = \left(L^e_{ijkl} - L^p_{ijkl} \right) \dot{\epsilon}_{kl}$.

Solution: See the following pages.

DEA GEOMATERIAUX
Comportement des sols et des roches

ELASTO-PLASTICITE

cours Ex 6

Le but de cet exercice est de donner une expression explicite des équations constitutives ***incrémentales*** *d'une roche ayant un comportement élasto-plastique avec écrouissage dans le cas d'un critère de plasticité et d'un potentiel plastique de type Drucker-Prager.*

On suppose que le tenseur d'élasticité est linéaire et isotrope. On rappelle les relations d'élasticité

$$s_{ij} = 2Ge^{e}_{ij}$$
$$p = K\varepsilon^{e}_{kk}$$

On rappelle également les expressions du critère de plasticité et du potentiel plastique de Drucker-Prager

$$F = T - f(q-p) \qquad Q = \sqrt{J_{2s}} + dp$$

← param. de dilatance

$Q = T + p \cdot d.$

écrouissage isotrope

On suppose que le paramètre q est une constante et que les paramètre de frottement et de dilatance f et d sont des fonctions de la déformation déviatorique plastique accumulée g^p.

1. Ecrire la décomposition des tenseurs des contraintes et des déformations en une partie sphérique et une partie déviatorique ainsi que la décomposition du tenseur de déformations en une partie élastique et une partie plastique.

2. Calculer le gradient de la surface de charge et du potentiel plastique et en déduire l'expression explicite de la condition de consistance. On notera $h_t = \dfrac{df}{dg^p}$.

3. Exprimer la règle d'écoulement plastique et montrer que le multiplicateur plastique ψ vérifie la relation $\dot{\psi} = \dot{g}^p$ (rappel: $\dot{g}^p = \sqrt{2\dot{e}^p_{ij}\dot{e}^p_{ij}}$) . En déduire la relation entre le taux de déformation volumique plastique $\dot{\varepsilon}^p_v$ et le taux de déformation déviatorique plastique $\dot{g}^p$.

4. En utilisant les relations élastiques et la condition de consistance montrer la relation suivante:

$$\dot{g}^{p} = \frac{<1>}{G + fdK + (q-p)h_t}\left(G\frac{s_{ij}}{T} + Kf\delta_{ij}\right)\dot{\varepsilon}_{ij}$$

Expliciter le sens de la notation de Mc Auley <1>.

5. En déduire l'expression explicite des relations incrémentales de comportement.

6. On suppose que l'état initial à partir duquel on considère un incrément de contrainte et de déformation correspond à un état de compression triaxiale axisymétrique. Donner l'expression du tenseur élastoplastique dans ce cas particulier.

Equations incrémentales de comportement élastoplastique pour un modèle de Drucker-Prager

1 Décomposition des tenseurs

$$\dot{\sigma}_{ij} = \dot{s}_{ij} + \dot{p}\delta_{ij}$$

$$\dot{\varepsilon}_{ij} = \dot{e}_{ij} + \dot{\varepsilon}_v\delta_{ij}/3$$

$$\dot{\varepsilon}_{ij} = \overset{.e}{\varepsilon}_{ij} + \overset{.p}{\varepsilon}_{ij}$$

$$\dot{e}_{ij} = \overset{.e}{e}_{ij} + \overset{.p}{e}_{ij}$$

$$\dot{\varepsilon}_v = \overset{.e}{\varepsilon}_v + \overset{.p}{\varepsilon}_v$$

2 Gradients de la surface de charge et du potentiel plastique et équation de consistance

Equation de la surface de charge:

$$F = T - f\left(g^p\right)(q - p) = 0 \tag{1}$$

Potentiel plastique

$$Q = T + d\left(g^p\right)p \tag{2}$$

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial T}{\partial \sigma_{ij}} + f\frac{\partial p}{\partial \sigma_{ij}}$$

$$p = \frac{1}{3}\sigma_{kk} = \frac{1}{3}\sigma_{ij}\delta_{ij};\ \frac{\partial p}{\partial \sigma_{ij}} = \frac{1}{3}\delta_{ij}$$

$$T = \sqrt{\frac{1}{2}s_{ij}s_{ji}};\ \frac{\partial T}{\partial \sigma_{ij}} = \frac{s_{ij}}{2T}$$

d'où

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{s_{ij}}{2T} + f\frac{\delta_{ij}}{3} \tag{3}$$

$$\frac{\partial Q}{\partial \sigma_{ij}} = \frac{s_{ij}}{2T} + d\frac{\delta_{ij}}{3} \tag{4}$$

Equation de consistance

$$\dot{F} = \frac{\partial F}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial F}{\partial g^p} \dot{g}^{p} = 0 \tag{5}$$

L'expression 3 du gradient de la surface de charge introduite dans l'expressio 5 de la relation de consistance conduit à

$$\left(\frac{s_{ij}}{2T} + f \frac{\delta_{ij}}{3} \right) \dot{\sigma}_{ij} - (q-p)\, h_t \dot{g}^{p} = 0$$

avec $h_t = \frac{df}{dg^p}$

3 Evaluation du multiplicateur plastique

La règle d'écoulement s'écrit $\dot{\varepsilon}^{p}_{ij} = \dot{\psi} \frac{\partial Q}{\partial \sigma_{ij}}; \dot{\psi} \geq 0$ ou en utilisant l'expression 4 du gradient du potentiel plastique $\dot{\varepsilon}^{p}_{ij} = \dot{\psi} \left(\frac{s_{ij}}{2T} + d \frac{\delta_{ij}}{3} \right)$. On peut alors exprimer la partie déviatorique et la partie volumique du tenseur de vitesses déformations plastiques sous la forme

$\dot{e}^{p}_{ij} = \dot{\psi} \frac{s_{ij}}{2T}; \dot{\varepsilon}^{p}_{v} = \dot{\psi} d$

$\dot{g}^{p} = \sqrt{2 \dot{e}^{p}_{ij} \dot{e}^{p}_{ji}} = \sqrt{2 \dot{\psi}^{2} \frac{s_{ij} s_{ji}}{4T^2}}$

on obtient

$$\dot{g}^{p} = \dot{\psi} \tag{6}$$

d'où la relation de dilatance

$$\dot{\varepsilon}^{p}_{v} = d \dot{g}^{p} \tag{7}$$

4 Evaluation de $\dot{g}^{p}$

La relation de consistance 5 permet d'écrire

$\dot{g}^{p} = \frac{1}{(q-p)h_t} \left(\frac{s_{ij}}{2T} + f \frac{\delta_{ij}}{3} \right) \dot{\sigma}_{ij}$

En utilisant les relations de comportement élastique

$\dot{g}^{p} = \frac{1}{(q-p)h_t} \left(\frac{s_{ij}}{2T} + f \frac{\delta_{ij}}{3} \right) \left(2G \dot{e}^{e}_{ij} + K \dot{\varepsilon}^{e}_{v} \delta_{ij} \right)$

$$= \frac{1}{(q-p)h_t}\left(G\frac{s_{ij}\dot{e}_{ij}^{e}}{T} + fK\dot{\varepsilon}_v^{e}\right) = \frac{1}{(q-p)h_t}\left(G\frac{s_{ij}}{T}\left(\dot{e}_{ij} - \dot{e}_{ij}^{p}\right) + fK\left(\dot{\varepsilon}_v - \dot{\varepsilon}_v^{p}\right)\right.$$

$$= \frac{1}{(q-p)h_t}\left(G\frac{s_{ij}}{T}\left(\dot{e}_{ij} - \frac{s_{ij}}{2T}\dot{g}^{p}\right) + fK\left(\dot{\varepsilon}_v - \dot{g}^{p}d\right)\right)$$

Soit $\kappa = K/G$, l'expression ci-dessus conduit à

$$\left(1 + f\kappa d + \frac{(q-p)h_t}{G}\right)\dot{g}^{p} = \left(\frac{s_{ij}}{T}\dot{e}_{ij} + \kappa f\dot{\varepsilon}_v\right) = \left(\frac{s_{ij}}{T} + \kappa f\delta_{ij}\right)\dot{\varepsilon}_{ij}$$

On note $H = \left(1 + f\kappa d + \frac{(q-p)h_t}{G}\right)$ (module plastique) et $b_{ij}^{F} = \frac{s_{ij}}{T} + \kappa f\delta_{ij}$ d'où

$$\dot{g}^{p} = \frac{<1>}{H}b_{ij}^{F}\dot{\varepsilon}_{ij} \tag{8}$$

avec

$$<1> = \begin{cases} 1 \text{ si } F = 0 \text{ et } b_{ij}^{F}\dot{\varepsilon}_{ij} > 0 \text{ (charge)} \\ 0 \text{ si } \begin{cases} F < 0 \text{ (domaine élastique)} \\ \text{ou } F = 0 \text{ et } b_{ij}^{F}\dot{\varepsilon}_{ij} \leq 0 \text{ (décharge élastique)} \end{cases} \end{cases} \tag{9}$$

5 Equations élasto-plastiques incrémentales

$$\dot{\sigma}_{ij} = 2G\dot{e}_{ij}^{e} + K\dot{\varepsilon}_v^{e}\delta_{ij} = \left(2G\dot{e}_{ij} + K\dot{\varepsilon}_v\delta_{ij}\right) - \dot{g}^{p}G\left(\frac{s_{ij}}{T} + \kappa d\delta_{ij}\right)$$

Soit $b_{ij}^{Q} = \frac{s_{ij}}{T} + \kappa d\delta_{ij}$

$$\dot{\sigma}_{ij} = G\left(\left(2\dot{e}_{ij} + \kappa\dot{\varepsilon}_v\delta_{ij}\right) - \frac{<1>}{H}b_{ij}^{Q}b_{kl}^{F}\dot{\varepsilon}_{kl}\right)$$

$$\dot{\sigma}_{ij} = \left(L_{ijkl}^{e} - L_{ijkl}^{p}\right)\dot{\varepsilon}_{kl} \tag{10}$$

avec

$$L_{ijkl}^{e} = G\left(2\delta_{ik}\delta_{jl} + \left(\kappa - \frac{2}{3}\right)\delta_{ij}\delta_{kl}\right) \tag{11}$$

$$L_{ijkl}^{p} = G\frac{<1>}{H}b_{ij}^{Q}b_{kl}^{F} \tag{12}$$

6 Etat initial: compression triaxiale

En compression triaxiale

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & \sigma_z \end{pmatrix} ; \underline{\underline{s}} = \begin{pmatrix} \frac{\sigma_r-\sigma_z}{3} & 0 & 0 \\ 0 & \frac{\sigma_r-\sigma_z}{3} & 0 \\ 0 & 0 & -2\frac{\sigma_r-\sigma_z}{3} \end{pmatrix} ; \; T = \frac{\sigma_r-\sigma_z}{\sqrt{3}}$$

$\sigma_r, \sigma_z <$

$$\underline{\underline{b}}^F = \begin{pmatrix} \frac{1}{\sqrt{3}}+\kappa f & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}}+\kappa f & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}}+\kappa f \end{pmatrix}$$

$\mathcal{K} = \frac{\kappa}{G}$

$$\underline{\underline{b}}^Q = \begin{pmatrix} \frac{1}{\sqrt{3}}+\kappa d & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}}+\kappa d & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}}+\kappa d \end{pmatrix}$$

On obtient les coefficients suivants pour le tenseur élasto-plastique

$$\begin{aligned}
L_{rrrr} &= G\left\{\left(\kappa+\frac{4}{3}\right)-\frac{<1>}{H}\left(\frac{1}{\sqrt{3}}+\kappa d\right)\left(\frac{1}{\sqrt{3}}+\kappa f\right)\right\} \qquad (13) \\
L_{rr\theta\theta} &= G\left\{\left(\kappa-\frac{2}{3}\right)-\frac{<1>}{H}\left(\frac{1}{\sqrt{3}}+\kappa d\right)\left(\frac{1}{\sqrt{3}}+\kappa f\right)\right\} \\
L_{rrzz} &= G\left\{\left(\kappa-\frac{2}{3}\right)-\frac{<1>}{H}\left(\frac{1}{\sqrt{3}}+\kappa d\right)\left(\frac{-2}{\sqrt{3}}+\kappa f\right)\right\} \\
L_{\theta\theta rr} &= G\left\{\left(\kappa-\frac{2}{3}\right)-\frac{<1>}{H}\left(\frac{1}{\sqrt{3}}+\kappa d\right)\left(\frac{1}{\sqrt{3}}+\kappa f\right)\right\} \\
L_{\theta\theta\theta\theta} &= G\left\{\left(\kappa+\frac{4}{3}\right)-\frac{<1>}{H}\left(\frac{1}{\sqrt{3}}+\kappa d\right)\left(\frac{1}{\sqrt{3}}+\kappa f\right)\right\} \\
L_{\theta\theta zz} &= G\left\{\left(\kappa-\frac{2}{3}\right)-\frac{<1>}{H}\left(\frac{1}{\sqrt{3}}+\kappa d\right)\left(\frac{-2}{\sqrt{3}}+\kappa f\right)\right\} \\
L_{zzrr} &= G\left\{\left(\kappa-\frac{2}{3}\right)-\frac{<1>}{H}\left(\frac{-2}{\sqrt{3}}+\kappa d\right)\left(\frac{1}{\sqrt{3}}+\kappa f\right)\right\} \\
L_{zz\theta\theta} &= G\left\{\left(\kappa-\frac{2}{3}\right)-\frac{<1>}{H}\left(\frac{-2}{\sqrt{3}}+\kappa d\right)\left(\frac{1}{\sqrt{3}}+\kappa f\right)\right\} \\
L_{zzzz} &= G\left\{\left(\kappa+\frac{4}{3}\right)-\frac{<1>}{H}\left(\frac{-2}{\sqrt{3}}+\kappa d\right)\left(\frac{-2}{\sqrt{3}}+\kappa f\right)\right\} \\
L_{r\theta r\theta} &= L_{rzrz} = L_{\theta z\theta z} = 2G
\end{aligned}$$