Elasticity Theory for Rock Mechanics

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[These notes are from the former Engineering Rock Mechanics MS Course at Imperial College. These notes cover some of the same material as does Chapter 5 of the 4th edition of *Fundamentals of Rock Mechanics*, but pre-date that edition, and differ greatly in notation and other details.]

Section 1 - Review of the Equations of Elasticity

The basic governing equations of elasticity were presented last term, in various bits and pieces, in the Engineering Rock Mechanics Course. We will now review these equations, and collect them together. These equations are the cornerstone of almost all analyses of the mechanical behaviour of rock masses. This is because each of the more sophisticated constitutive models used in rock mechanics, such as plasticity, viscoelasticity, thermoelasticity, and poroelasticity, take elasticity as their starting point. We will first present the equations using Cartesian (x,y,z) notation, and then rewrite them in the more concise Cartesian tensor notation. Occasionally, we will also make use of "direct" matrix/vector notation.

1.1 Displacement

Each point in an elastic medium is labelled with a position vector, (x,y,z), that locates a particular "infinitesimally small" piece of rock with respect to some co-ordinate system, before any deformation occurs. After loads are applied, this piece of rock will move to a new location, (x',y',z'). It is therefore *displaced* by some vector (u,v,w), where u is the displacement in the x-direction, *etc*. The new location is therefore related to the old location by

$$(x',y',z') = (x,y,z) - (u,v,w). \tag{1.1}$$

The peculiar presence of the minus sign in eq. (1.1), which does not appear in most books on elasticity or continuum mechanics, is needed in order to be consistent with the "compression = positive" convention that is commonly used in rock mechanics. According to this convention, each displacement component will be *positive* if the particle of rock moves in the *negative* direction. Another interpretation is that the displacement vector points from the new position to the old position. In general, each of the three displacement components will vary with all three co-ordinates, *i.e.*, $u = u(x,y,z)$, *etc*.

1.2 Strain

As explained in the rock mechanics course, the stresses in a rock mass are not related to the displacements directly, but rather to the derivatives of the displacements, *i.e.,* the *strains*.

There are many ways to quantify the strain, but most analytical and numerical studies of rock deformation utilise the *infinitesimal strain tensor* for this purpose. The nine components of the infinitesimal strain tensor are defined by

$$\varepsilon_{xx} = \left(\frac{\partial u}{\partial x} \right), \qquad \varepsilon_{yy} = \left(\frac{\partial v}{\partial y} \right), \qquad \varepsilon_{zz} = \left(\frac{\partial w}{\partial z} \right), \tag{1.2a}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \tag{1.2b}$$

$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \tag{1.2c}$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right). \tag{1.2d}$$

There are nine components of strain. However, they are defined in such a way that only six components are independent. The terms that have both suffixes the same are the *normal strains,* and measure the fractional shortening in the direction of the three co-ordinate axes. The other six strains are *shear strains;* they measure angular distortions. There are also three pairs of terms called the *rotations* that are defined by

$$\omega_{yz} = -\omega_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \right), \tag{1.3}$$

and similarly for ω_{xz}, *etc*. These terms represent local rigid-body rotations that take place without any distortion or stretching of the rock. In general, they are not as important as the strains, since they are not directly related to the stresses.

If the strains are greater than a few percent, the infinitesimal strain tensor as defined above is not sufficient to quantify the deformation. In these cases, more complicated definitions are needed, which involve *products* of the derivatives of the displacement, *i.e*., terms such as $(\partial u / \partial x)^2$, *etc*. However, in most rock mechanics problems, the strains are indeed small, and the infinitesimal strain tensor accurately quantifies the strain. In any event, analyses based on *finite strain tensors* (of which there are various different definitions) are exceedingly difficult to carry out, and so finite strain theory is rarely used in practice.

1.3 Tractions

Consider a very small element of surface area of rock, which may be an actual rock surface (such as the surface of an excavation), or a hypothetical surface in the interior of a rock mass. The net force acting on this surface, divided by the area of the surface, is called the *traction*. The traction is therefore a vector, and has three components. It is usually denoted by $\boldsymbol{t} = (t_x, t_y, t_z)$. In other words, t_x is the force component in the x-direction, per unit area, acting on the surface. By convention, the components of the traction vector are

represented by *positive* numbers if they act in the *negative* co-ordinate directions, and *vice versa*. Tractions are important in rock mechanics, because the tractions that act on the exposed surfaces of rock serve as *boundary conditions* when a problem in rock mechanics is formulated mathematically. However, aside from appearing in the boundary conditions, the tractions are not used in formulating and solving the equations. This is because it is more convenient to use the *stresses,* which are defined below.

1.4 Stresses

The traction vector, as defined above, depends on the surface of the rock to which one is referring. Any rock surface can always be identified uniquely by its *outward unit normal vector*, $\boldsymbol{n} = (n_x, n_y, n_z)$. This vector is *normal* to the surface, points *away* from the rock mass, and has *unit* length; from the Pythagorean theorem, this last condition means that $n_x^2 + n_y^2 + n_z^2 = 1$. Hence, at any point in the rock mass, the vector $\boldsymbol{t}$ must be expressible as a function of the vector $\boldsymbol{n}$. We proved in the engineering rock mechanics course that $\boldsymbol{t}$ and $\boldsymbol{n}$ are related through a 3×3 matrix, $\boldsymbol{\tau}$, that is known as the *stress tensor*. In matrix/vector notation, the relation is $\boldsymbol{t} = \boldsymbol{\tau n}$. In explicit notation the relation is

$$t_x = \tau_{xx} n_x + \tau_{xy} n_y + \tau_{xz} n_z, \tag{1.4a}$$

$$t_y = \tau_{yx} n_x + \tau_{yy} n_y + \tau_{yz} n_z, \tag{1.4b}$$

$$t_z = \tau_{zx} n_x + \tau_{zy} n_y + \tau_{zz} n_z. \tag{1.4c}$$

The nine components $(\tau_{xx}, \tau_{xy}, etc.)$ form the *stress tensor*. In the ERM course we showed that the reason the stresses form a "tensor" is that, when you change the co-ordinate system, the nine stress components transform in a certain manner. These transformations are very important to know and understand, but they will not be used very often in the remainder of these notes.

1.5 Conservation of Angular Momentum

All of the equations defined thus far are mainly definitions, and in themselves do not actually embody any physical laws. Note that, thus far we have introduced twenty-one "unknowns" - three displacements, nine stresses, and nine strains. Hence, we eventually will need twenty-one equations. The strain-displacement equations, (1.2), are nine such equations; twelve more are needed. Six of these additional equations are found by invoking the three main laws of classical mechanics: conservation of linear momentum, conservation of angular momentum, and conservation of energy. These laws will have the same form for any type of rock behaviour - elastic, plastic, viscoelastic, *etc*. Finally, the last six equations are found by invoking a specific *constitutive law* to relate the stresses to the strains.

The law of conservation of angular momentum supplies three of the needed equations. As we showed in the ERM course, the law of conservation of angular momentum, when applied to a rock, tells us that the stress matrix is *symmetric*, *i.e.*,

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{xz} = \tau_{zx}, \qquad \tau_{yz} = \tau_{zy}. \tag{1.5}$$

If eq. (1.5) is always assumed to be true, then the law of conservation of angular momentum will be satisfied. When solving a particular problem, we therefore always assume that eq. (1.5) holds. When formulating a problem and setting up the equations, however, it is often convenient to ignore relations (1.5), and use separate notations for τ_{xy} and τ_{yx}, for example. Failure to adhere to this guideline can lead to mistakes, as will be seen when we discuss the stored strain energy.

1.6 Conservation of Linear Momentum

Three more equations are found by appealing to the law of conservation of linear momentum. There are three such equations, because momentum is a vector that has three independent components. In a Cartesian co-ordinate system, these three equations are

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0, \tag{1.6a}$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0, \tag{1.6b}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + f_z = 0, \tag{1.6c}$$

where $\boldsymbol{f} = (f_x, f_y, f_z)$ is the *body-force vector*. This vector represents a force, per unit *volume* of rock, that acts throughout the rock, and not only over its outer surface. In keeping with the "compression = positive" convention, the components of this vector are positive numbers if the force acts in the negative direction. The most common body force is gravity, which has magnitude ρg, and acts in the vertical direction. However, we will see later that temperature changes, as well as changes in the pore fluid pressure in the cracks and voids, also essentially function as *body forces* in rock mechanics. If we were analysing a dynamic problem, rather than a static problem, there would be inertia terms on the right hand sides of eq. (1.6), such as $\rho \ddot{u}$ in eq. (1.6a), where the two dots denote time derivatives. In this course, we will usually deal with quasi-static problems, in which case the inertia terms can be ignored. The three equations (1.6) are often referred to as the "stress-equilibrium equations".

1.7 Conservation of Energy

We now need only six more equations in order to balance out the number of unknown functions. However, if we are trying to solve an isothermal problem, the principle of conservation of energy does not supply us with an additional equation. This principle will appear in *thermoelasticity,* but does not appear explicitly in *elasticity*. The additional six equations are found by invoking a set of *constitutive equations,* which is just another name for the *stress-strain relations*.

1.8 Stress-Strain Relations

The equations presented thus far would apply to the deformation of any solid material. As mentioned above, we need to supplement these equations with constitutive laws in order to have enough equations to allow us to solve for the displacements and stresses. There are numerous types of constitutive laws used in rock mechanics, such as linear elasticity, non-linear elasticity, plasticity, viscoelasticity, *etc*. Although most rocks behave in a complex, non-linear, and often inelastic manner, most analyses of rock deformation utilise the assumption of isotropic, linear elasticity. This model is used in part because it is by far the easiest model to employ, in both analytical and numerical work, and in part because it often offers a very reasonable approximation to the actual rock behaviour. The stress-strain relations (*i.e*., Hooke's law) for an isotropic, linear elastic material are

$$\tau_{xx} = \lambda (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\, \varepsilon_{xx}, \tag{1.7a}$$

$$\tau_{yy} = \lambda (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\, \varepsilon_{yy}, \tag{1.7b}$$

$$\tau_{zz} = \lambda (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\, \varepsilon_{zz}, \tag{1.7c}$$

$$\tau_{xy} = 2\mu\, \varepsilon_{xy}, \qquad \tau_{xz} = 2\mu\, \varepsilon_{xz}, \qquad \tau_{yz} = 2\mu \varepsilon_{yz}. \tag{1.7d}$$

The two parameters λ and μ are Lame's two elastic moduli. The parameter μ is equivalent to the shear modulus, which is often denoted by G. The parameter λ has no simple physical interpretation, but is mathematically convenient to use. These two parameters are related to the more commonly used parameters E, K and ν through the following equations:

$$K = \lambda + \frac{2}{3}\mu, \tag{1.8a}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)}, \tag{1.8b}$$

$$E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}. \tag{1.8c}$$

• Homework problem: Invert eqs. (1.7) to find the strains as functions of the stresses. Hint: add up eqs. (1.7a,b,c) to eliminate the term $(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})$. Compare the resulting equation for ε_{xx} with the equation we used in the ERM course, and thereby derive eq. (1.8c).

1.9 Elastostatic Boundary-Value Problems

Equations (1.2,3,5,7) comprise twenty-one equations in twenty-one unknowns. Roughly speaking, this is enough information to allow us to solve for the displacements and stresses in a rock mass. In order to do this, we of course need to know the body force distribution, and the elastic moduli. However, we also need to know the *boundary conditions* on the outer

surfaces of the rock. In general, the boundary conditions consist of specifying either the *tractions* or the *displacements* along each portion of the outer boundary. As an example, a very common type of boundary condition is the "traction free" boundary, which occurs at an excavation surface that is in contact with the atmosphere. If all of this information is known, there will be a unique solution to the equations. We will prove this important "uniqueness theorem" in Section 3. This theorem is not just a mathematical conceit. The theorem is needed because it tells us exactly what type of boundary information is needed in order to solve a problem. Also, note that the other major classical continuum theory, that of flow of a viscous fluid, does *not* have a uniqueness theorem. For example, flow in a pipe at a given flowrate may be either laminar *or* turbulent - hence, in fluid mechanics there can be more than one solution for a given problem. Fortunately, this situation does not arise in linear elasticity. In order to prove the uniqueness theorem, along with other important and related theorems, it is convenient to rewrite the governing equations in terms of *indicial notation*.

Section 2 - Cartesian Tensor "Indicial" Notation

If we examine equations such as (1.2) or (1.5), it becomes apparent that there is a certain pattern to them. This suggests that there may be some simpler way to write these equations. A system of notation known as Cartesian tensor "indicial" notation has been invented for this purpose. This system of notation takes a while to learn, but there are many advantages to using this notation. For one, all equations are simpler and shorter when written in indicial notation. Secondly, there are many proofs and manipulations that are easy to carry out using this notation that would be extremely tedious, if not nearly impossible, to do using the (x,y,z) notation. Thirdly, the notation is in many ways similar to the type of array indexing that is used when writing computer programs to solve the governing equations numerically. Finally, use of indicial notation is becoming very widespread in rock mechanics, so some knowledge of this notation is useful if only to facilitate your reading of the rock mechanics literature.

2.1 Basic Definition

Consider a vector such as the position vector, which can be denoted by (x,y,z). Note that we could just as well denote this vector by (x_1, x_2, x_3). The basic idea behind indicial notation is to represent this vector by the single symbol x_i. The best way to interpret this symbol is to think of it as being a shorthand notation for the *entire set* of the three components (x_1, x_2, x_3). It is sometimes convenient, *but often misleading,* to think of x_i as symbolising a particular component of the position vector. Similarly, the displacement vector can be written as (u_1, u_2, u_3) instead of (u,v,w). In indicial notation, therefore, it would be written as u_i. Using indicial notation, eq. (1.1) could be written as $x'_i = x_i - u_i$, for example. The index i is referred to as a *free index,* because it is "free" to take on the values 1, 2 or 3. It is important to note that any index can be used for this purpose, although it is traditional to start with i, and then use j, k, *etc*., if more indices are needed. Hence, x_i means *exactly the same thing* as x_j, for example.

Following up on the idea that (x,y,z) can be replaced by (x_1, x_2, x_3), a strain component such as ε_{xx} can also be written as ε_{11}, *etc*. The entire strain tensor can therefore be

represented by the symbol ε_{ij}, where the two free indices *i* and *j* are shorthand for all nine combinations that can be formed by taking *i*=1,2,3 and *j*=1,2,3. Likewise, the stress tensor can be represented by τ_{ij}. When using indicial notation, the "order" of a tensor is indicated by the number of free indices that it has. For example, τ_{ij} has two free indices, so it is a second-order tensor with $3^2 = 9$ components, where the "3" is related to the fact that we are in three-dimensional space. A vector such as x_i is therefore also referred to as a *first-order tensor,* and has $3^1 = 3$ components.

2.2 Summation Convention

The summation convention is a rule that allows most equations written in indicial notation to be simplified. To motivate this rule, consider the bulk strain, which we showed in the ERM course to be expressible as $\varepsilon_{bulk} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$. The bulk strain can also be written as

$$\varepsilon_{bulk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \sum_{i=1}^{3} \varepsilon_{ii} \,. \tag{2.1}$$

The famous physicist Albert Einstein noticed that whenever a term appears that contains two repeated indices, such as ε_{ii}, it almost always has a summation sign in front of it, as it does in eq. (2.1). Hence, the summation sign is actually redundant. He therefore invented what is now known as the *Einstein summation convention,* which is a rule that states whenever there are two indices repeated in a tensor equation, it is implied that the terms should actually be summed over the values *i*=1,2,3. Using this convention, $\varepsilon_{bulk} = \varepsilon_{ii}$. An index that appears twice in an expression is often called a "dummy index".

The summation convention is useful when expressing the product of two vectors, or the product of a matrix and a vector. For example, note that the *dot product* of two vectors can be expressed as $\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum u_i v_i = u_i v_i$. The product of a 3×3 matrix $\boldsymbol{A}$ and a 3×1 vector $\boldsymbol{b}$ would be written as $A_{ij} b_j$. You should write out the three terms in this expression to convince yourself that it does in fact represent the product of a matrix and a vector. Note that the order in which the terms appear in an indicial expression is immaterial; this product could also be written as $b_j A_{ij}$. The important fact is that the index on the vector $\boldsymbol{b}$ matches up with the second index on $\boldsymbol{A}$. In this regard, note that $A_{ij} b_i$ would be the indicial representation of $\boldsymbol{b}^{\mathrm{T}} \boldsymbol{A}$, where the superscript "T" indicates transpose.

2.3 Kronecker delta

One very important second-order tensor that is used frequently in elasticity is the *Kronecker delta*, which is denoted by δ_{ij}, and defined by

$$\delta_{11} = \delta_{22} = \delta_{33} = 1, \tag{2.2a}$$

$$\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0. \tag{2.2b}$$

The Kronecker delta is often defined by saying that "δ_{ij} equals 1 when $i=j$, and equals 0 when $i \neq j$". It is exactly equivalent to the 3×3 identity matrix that is denoted in matrix notation by $\boldsymbol{I}$. The most important property of the Kronecker delta is the *replacement rule*, which states that whenever you see δ_{ij} in an equation, if there is another term that also has index i, you can replace that index by j, and delete the Kronecker delta. For example, $\delta_{ij}x_j = x_i$, since we can use the Kronecker delta to replace the subscript i with the subscript j. This rule can be proven as follows:

$$\delta_{ij}x_j = \sum_{j=1}^{3} \delta_{ij}x_j = \delta_{i1}x_1 + \delta_{i2}x_2 + \delta_{i3}x_3$$

$$= (\delta_{11}x_1 + \delta_{12}x_2 + \delta_{13}x_3,\ \delta_{21}x_1 + \delta_{22}x_2 + \delta_{23}x_3,\ \delta_{31}x_1 + \delta_{32}x_2 + \delta_{33}x_3)$$

$$= (x_1, x_2, x_3) = x_i \ . \tag{2.3}$$

In direct matrix notation, this equation merely states that $\boldsymbol{Ix = x}$. The replacement rule can be used with tensors of any order, *i.e.*, in expressions that have any number of free indices, as long as there is one index that matches one of the indices of the Kronecker delta.

•Homework problem: Simplify the following expressions as far as possible: (a) δ_{ii}; (b) $\delta_{ij}\delta_{ij}$; (c) $\delta_{ij}x_i u_j$.

Note that an index can either appear *once* in an equation, in which case it is a *free index,* or it can appear *twice,* in which case it is a dummy index. If an index appears more than twice when using indicial notation, this is a sure sign of a mistake! For example, $\tau_{ii}x_i$ has no meaning in indicial notation. Another way to check for possible errors in your equations is to note that the free indices must *always* match up on both sides of an equation. Hence, the equation $\delta_{ij}x_j = x_i$ makes sense, but the equation $\delta_{ij}x_j = x_j$ cannot possibly be correct.

2.4 Partial Derivative Notation

Another type of space-saving shorthand that is part of indicial notation is to use the subscript "$,i$" to denote a partial derivative with respect to the co-ordinate x_i. This rule applies to *all* subscripts that appear after the comma. For example,

$$f_{,i} = \frac{\partial f}{\partial x_i}; \qquad f_{,ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}; \qquad etc. \tag{2.4}$$

Recall from calculus that when you differentiate a function with respect to two different variables, the order in which you take the derivatives is immaterial. In indicial notation, this rule is expressed as $f_{,ij} = f_{,ji}$. This law is frequently of use when manipulating the equations of elasticity. One important specific example of a partial derivative that arises frequently is

$x_{i,j} = \delta_{ij}$. This equation is equivalent to the set of nine equations $\partial x_1 / \partial x_1 = 1$, $\partial x_1 / \partial x_2 = 0$, *etc*.

2.5 Equations of Elasticity written in Indicial Notation

Using all of the rules we have just introduced, the equations of elasticity can be written as follows:

Displacement:	$x_i' = x_i - u_i$	(2.5)
Strain:	$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$	(2.6)
Traction:	$t_i = \tau_{ij} n_j$	(2.7)
Angular momentum:	$\tau_{ij} = \tau_{ji}$	(2.8)
Linear momentum:	$\tau_{ij,j} + f_i = 0$	(2.9)
Hooke's Law:	$\tau_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$	(2.10)

If we combine eqs. (2.6,9,10), we can derive the governing equations of elasticity in terms of the displacements. These equations, which are sometimes referred to as the *Navier equations*, are

$$(\lambda + \mu) u_{j,ji} + \mu u_{i,jj} + f_i = 0 \,. \tag{2.11}$$

It is important to understand that all of the equations presented thus far are appropriate only if you are using a *Cartesian co-ordinate system*. If you are using a cylindrical co-ordinate system, for example, it would *not* be correct to replace (x,y,z) with (r,θ,z). For curvilinear co-ordinate systems such as *cylindrical* or *spherical,* indicial notation cannot be used. The form of the governing equations in cylindrical co-ordinates will be presented in Section 6.

Equation (2.11) applies to an isotropic elastic material, in which "all directions are physically equivalent". Rock masses that contain bedding planes, or aligned fractures, however, are usually *anisotropic,* which means *"not* isotropic*"*. Specific forms of Hooke's law for the most common types of anisotropy found in rock masses, transverse isotropy and orthotropy, are given in the book *Rock Mechanics for Underground Mining*, by Brady and Brown. For our purposes, it is better to use the most general form of Hooke's law for an anisotropic medium, which would be written in indicial notation as

$$\tau_{ij} = c_{ijkl} \varepsilon_{kl} \,, \tag{2.12}$$

where c_{ijkl} is the *elastic moduli tensor*. The elastic moduli tensor is a *fourth-order* tensor, as can be seen by the fact that it has *four* indices. [Note that in matrix notation, eq. (2.12) would be written as $\boldsymbol{\tau = C\varepsilon}$; however, despite its compactness, matrix notation has a disadvantage in

that it gives no simple way to indicate the *order* of a tensor.] Many of the important theorems of elasticity are true regardless of whether or not the material is isotropic. Furthermore, many of the mathematical manipulations are easier if we use the general anisotropic form of Hooke's law, which of course must contain eq. (2.10) as a special case.

•Homework problem: It can be shown (see *Cartesian Tensors* by H. Jeffreys) that any isotropic elastic moduli tensor must have the form $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Using this elastic moduli tensor in the anisotropic version of Hooke's law, derive eq. (2.10).

2.6 Generalised Divergence Theorem

The generalised divergence theorem, which is also known as Gauss's theorem and Green's theorem, is used very frequently in elasticity to transform volume integrals into surface integrals, or *vice versa*. In particular, it is crucial to the development of the family of numerical methods known as *boundary integral methods,* or *boundary element methods*. Special cases of this theorem are proven in most vector calculus textbooks. One advantage of using indicial notation is that a very general version of the divergence theorem can be written down in a simple form. Consider a region of three-dimensional space denoted by *R,* with its outer boundary denoted by ∂R. In this context, the symbol ∂ does not denote a partial derivative; it is a standard mathematical symbol used to denote the *boundary* of a region. The generalised divergence theorem can be expressed as

$$\iiint_R (tensor)_{,i}\, dV = \iint_{\partial R} (tensor) n_i dA \tag{2.13}$$

The expression denoted by "tensor" can be *any* tensor of any order, with any (legitimate) combination of free and dummy indices. The only restriction to remember is that the *entire* integrand inside the volume integral must be differentiated with respect to x_i in order to apply this theorem. For example, it could be applied to an integrand of the form $(x_i f)_{,i}$; but not to an integrand of the form $x_i f_{,i}$. n this latter case, however, a version of integration-by-parts could be used, as will be demonstrated in Section 3.3.

Although we will not prove the divergence theorem, the following example should convince you that it works. Recall that the local volumetric strain is given by ε_{ii}, which is seen from eq. (2.6) to equal $u_{i,i}$. Hence, the total volume change of the rock in region *R* could be found by integrating $u_{i,i}$ over the region *R*, i.e.,

$$\Delta V = \iiint_R \varepsilon_{ii} dV = \iiint_R u_{i,i} dV = \iint_{\partial R} u_i n_i dA = \iint_{\partial R} \mathbf{u} \cdot \mathbf{n} dA. \tag{2.14}$$

The term $\boldsymbol{u} \cdot \boldsymbol{n}$ in the last integral represents the component of the displacement that is perpendicular to the outer boundary. Hence, it seems that the last integral is, in fact, equal to total volume change of the body. The fact that we derived this last equation using the divergence theorem is indirect evidence that the divergence theorem is correct.

Section 3 - Energy Principles and the Uniqueness Theorem

There are various theorems and principles in elasticity that involve the concept of the strain energy that is stored inside a stressed body. These principles have many important implications and uses. On a theoretical level, they provide constraints on the allowable values of the components of the elastic moduli tensor. They are also crucial in the proof of the *uniqueness theorem* of elasticity. Roughly speaking this theorem assures us that if we find a solution to an elasticity problem in rock mechanics, it will in fact be *the* solution, and therefore does represent the physical behaviour of the rock mass. Finally, the energy principles form the justification for many important approximate methods used in rock mechanics, such as *finite element methods*.

3.1 Strain Energy

Imagine a small piece of rock of volume δV that is subjected to a uniaxial stress τ_{11}. We showed in the ERM course that the work done by this stress as it deforms the rock will be $\tau_{11}\varepsilon_{11}\delta V/2$. This is exactly analogous to the work $kx/2$ that is done when a spring with spring constant k is stretched by an amount x. Note that even though, in general, there will be strains induced in the 22 and 33 directions, due to the Poisson effect, these terms do not enter into the expression for the work. This is because the work is defined in terms of the dot product of the *force* and the *displacement,* and these other strains are perpendicular to the applied stress. In the most general case, there will be nine such work terms, one for each of the nine different stresses. Since the work done *on* the rock by the applied loads must equal the change in energy stored *in* the rock, the stored strain energy per unit volume of rock will be given by

$$w = \frac{1}{2}\left(\tau_{11}\varepsilon_{11} + \tau_{12}\varepsilon_{12} + \cdots\right) = \frac{1}{2}\left(\tau_{ij}\varepsilon_{ij}\right). \tag{3.1}$$

Note that there are nine different terms in eq. (3.1), not six. This equation illustrates why it is important to remember that although τ_{12} and τ_{21} are numerically equal, they represent two different stresses.

If the stresses and strains vary from point to point, the total strain energy stored in the rock in a region R would be found by integrating eq. (3.1) over the region R, as follows:

$$W = \iiint_R w dV = \frac{1}{2}\iiint_R \tau_{ij}\varepsilon_{ij} dV. \tag{3.2}$$

An interesting relation can be derived by manipulating eq. (3.2). First, we use eq. (2.6) to express the strains in terms of the displacements:

$$W = \frac{1}{2}\iiint_R \tau_{ij}\varepsilon_{ij} dV = \frac{1}{4}\iiint_R \tau_{ij}(u_{i,j} + u_{j,i}) dV = \frac{1}{2}\iiint_R \tau_{ij}u_{i,j} dV. \tag{3.3}$$

The last step is valid *only* because $\tau_{ij} = \tau_{ji}$ (verify this step yourself). We now use integration-by-parts to simplify the integral. First, note that, from the product rule of

differentiation, $(\tau_{ij}u_i)_{,j} = \tau_{ij,j}u_i + \tau_{ij}u_{i,j}$. But $\tau_{ij,j} = -f_i$ from eq. (2.9), so the strain energy can be written as

$$W = \frac{1}{2}\iiint_R [(\tau_{ij}u_i)_{,j} + f_i u_i]dV = \frac{1}{2}\iiint_R (\tau_{ij}u_i)_{,j}dV + \frac{1}{2}\iiint_R f_i u_i dV\,. \tag{3.4}$$

The first integral on the right is now in a form that allows the divergence theorem to be used:

$$W = \frac{1}{2}\iint_{\partial R} \tau_{ij}u_i n_j dA + \frac{1}{2}\iiint_R f_i u_i dV. \tag{3.5}$$

Noting that as the order of terms is irrelevant in indicial notation (unlike matrix notation, for example), we can say that $\tau_{ij}u_i n_j = \tau_{ij}n_j u_i = t_i u_i$, and so

$$W = \frac{1}{2}\iint_{\partial R} t_i u_i dA + \frac{1}{2}\iiint_R f_i u_i dV. \tag{3.6}$$

Recalling that the work done is defined in terms of a *force* acting through a *displacement,* we see that the first integral is the total force done on the rock by the tractions that are applied at its outer boundary, and the second integral is the total work done on the rock by the body forces. Eq. (3.6) is therefore a rigorous statement of the fact that the energy stored in the rock is equal to the total work done on the rock by external forces.

3.2 Positive-Definiteness of the Elastic Moduli Tensor

If you deform a rock, it seems reasonable to expect that you would have to do work *on* the rock, in which case the strain energy stored in the rock would *increase*. This should be true whether you compress or expand the rock. If there were a way to deform a rock so that its internal strain energy *decreased*, then you could use this deformation to extract energy from the rock, and do work on the surroundings. It would be nice if this were possible, but it that doesn't seem to be the way rocks (or any other materials) actually behave. This idea that any deformation will cause energy to be stored in the rock is formalised by saying that the strain energy function w must be *positive definite*. This means that it is always ≥ 0, and that it is only = 0 *if* the strain is exactly equal to zero. If we combine eqs. (2.12) and (3.1), we find that

$$w = \frac{1}{2}\tau_{ij}\varepsilon_{ij} = \frac{1}{2}c_{ijkl}\varepsilon_{kl}\varepsilon_{ij} \tag{3.7}$$

The expression for w in eq. (3.7) is known as a "quadratic form", since, if you expand it out explicitly, it consists of a series of *quadratic* terms such as $(1/2)c_{1111}{\varepsilon_{11}}^2$, *etc*. In order for w to be positive-definite, all of the eigenvalues of the elastic moduli tensor must be positive numbers; this is proven in any course on linear algebra. Note that this is *not* the same as saying that each component of c_{ijkl} must be positive.

The best way to examine the implications of positive-definiteness for an isotropic material is to decompose the stresses and strains into *isotropic* and *deviatoric* parts. Recall from the ERM course that the isotropic and deviatoric parts of the stress tensor are

$$\tau_{ij}^{\ I} = \frac{1}{3}\tau_{kk}\delta_{ij}, \tag{3.8a}$$

$$\tau_{ij}^{\ D} = \tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij}. \tag{3.8b}$$

The isotropic and deviatoric parts of the strain are defined similarly:

$$\varepsilon_{ij}^{\ I} = \frac{1}{3}\varepsilon_{kk}\delta_{ij}, \tag{3.9a}$$

$$\varepsilon_{ij}^{\ D} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}. \tag{3.9b}$$

By combining eqs. (2.10) and (3.8,9), it can be shown that Hooke's law can be written as

$$\tau_{ij}^{\ I} = 3K\varepsilon_{ij}^{\ I}; \tag{3.10a}$$

$$\tau_{ij}^{\ D} = 2\mu\varepsilon_{ij}^{\ D}, \tag{3.10b}$$

where $3K = 3\lambda + 2\mu$. Equation (3.10) shows that if Hooke's law is written in terms of deviatoric and isotropic components, the two components are completely *decoupled* from each other, in the sense that the deviatoric stress depends only on the deviatoric strain, and the isotropic stress depends only on the isotropic strain. If you compare eqs. (2.12) and (3.10), it should become apparent that any isotropic strain is an eigenvector of the elastic moduli tensor, and its eigenvalue is $3K$; similarly, any deviatoric strain is also an eigenvector, and its eigenvalue is 2μ. From the comment above that positive-definiteness implies that the eigenvalues are positive, it follows that the two elastic moduli, K and μ, must be positive. Eq. (1.8c) then shows that the Young's modulus, E, must be positive. Strangely enough, λ does not have to be positive; eq. (1.8a) shows that it must only satisfy the condition $\lambda > -2\mu/3$.

•Homework problem: Fill in the missing steps needed to derive eq. (3.10).

3.2 Uniqueness Theorem for Boundary-Value Problems

Imagine a rock mass that occupies a region in space denoted by R. The rock has a positive-definite elastic moduli tensor, c_{ijkl}. The body force, f_i, that acts on the rock is assumed to be known. Now imagine that the displacements along the outer boundary of the rock mass are known; we can denote these *known* boundary displacements by $u_i^{\,o}$. The complete elastostatic boundary-value problem can be formulated in precise mathematical terms as

$$\tau_{ij,j} + f_i = 0 \qquad \text{for} \quad x_i \in R\,, \tag{3.11a}$$

$$u_i = u_i^o \qquad \text{for} \quad x_i \in \partial R\,, \tag{3.11b}$$

where $x_i \in R$ is just a fancy way of saying "for all points inside the region R".

The uniqueness theorem was proven in 1859 by the physicist Gustav Kirchhoff, using the following argument. Imagine that there are actually *two* displacement fields that solve this problem, u_i^1 and u_i^2. Consider the *difference* between these two displacements, $\hat{u}_i = u_i^1 - u_i^2$. Because the stress-strain and strain-displacement equations are *linear,* the stresses associated with the displacement $\hat{u}_i$ will be $\hat{\tau}_{ij} = \tau_{ij}^1 - \tau_{ij}^2$. However, note that this displacement is *not* a solution to eq. (3.11), because

$$\hat{\tau}_{ij,j} = \tau_{ij,j}^1 - \tau_{ij,j}^2 = -f_i + f_i = 0\,. \tag{3.12}$$

Hence, the displacement $\hat{u}_i$ is a solution to the stress equilibrium equations for the case where the body force is *zero*. Now consider the displacements on the outer boundary:

$$\text{on} \quad x_i \in \partial R\,, \qquad \hat{u}_i = u_i^1 - u_i^2 = u_i^o - u_i^o = 0. \tag{3.13}$$

There are therefore no body forces associated with $\hat{u}_i$, and no displacements on the outer boundary. Equation (3.6) then shows that the strain energy associated with this displacement is zero. But if the elastic moduli tensor is positive definite, zero strain energy implies zero strains, and consequently zero stresses. Hence, we see that $\hat{\tau}_{ij} = 0$, which implies $\tau_{ij}^1 = \tau_{ij}^2$; and $\hat{\varepsilon}_{ij} = 0$, which implies $\varepsilon_{ij}^1 = \varepsilon_{ij}^2$. But the two strain fields can be equal only if their associated displacements differ by, at most, a rigid-body displacement. But the two fields have the same displacement on the outer boundary, so this supposed rigid-body displacement must be zero. Hence, we have proven that the two different solutions are in fact exactly equal, since they have the same displacements, stresses, and strains. QED!

The above proof of the uniqueness theorem was for the general case of an *anisotropic* material; it also holds in the special case where the rock is isotropic. The only requirement was that the elastic moduli tensor be positive-definite. Note that this proof would have worked just as well (essentially; see the HW problem below) if we specified the *tractions* on the outer boundary, rather than the displacements. In fact, one can have different boundary conditions on different parts of the boundary. We have therefore proven that if either the tractions or displacements are specified along the outer boundary, there will be *at most* one solution to the problem. However, the fact that *one* solution does exist (as opposed to having *no* solution) is much harder to prove, although it has been done, mainly by Eli Sternberg and his students. For our purposes, we can be assured that if we know the body forces, the elastic moduli tensor, and the appropriate boundary conditions, the problem at hand will have one and only one solution. The hard part is, of course, finding the solution!

•Homework problem: Complete the proof of the uniqueness theorem for the case where the *tractions* are specified along the outer boundary of the rock mass. Is the solution actually unique in this case, or "almost" unique?

3.3 Principle of Minimum Potential Energy, and Related Principles

The principle of minimum potential energy, and other related principles, form the theoretical basis of many of the numerical methods that are used to solve rock mechanics problems, such as *finite element methods*. The main idea behind this principle, and its use, can be roughly explained as follows. Consider an elastostatic boundary-value problem such as was formulated in section 3.2, with specified displacements on the outer boundary. Now imagine that we "guess" a displacement field for this problem, chosen so that is satisfies the boundary conditions. From this displacement field, we can calculate strains and stresses. However, unless we happened to be very lucky and guessed the correct answer, these stresses will probably not satisfy the stress-equilibrium equations, (2.9). We can nevertheless calculate the strain energy associated with this displacement, using eq. (3.2). According to the principle of minimum potential energy, the "potential energy" (which we will define precisely below) associated with this assumed displacement will be greater than the actual potential energy associated with the exact solution. Hence, if we somehow can tinker with the assumed displacement field in such a way as to continually lower its potential energy, we know that we are getting closer to the correct solution. We therefore have a way of approximating the correct solution, without every having to actually solve eq. (2.9).

To prove this principle, consider the boundary-value problem (3.11). Let $u_i(\boldsymbol{x})$ denote the solution to this problem. To simplify the notation, we will write the displacement as u_i, but we should remember that the displacement varies from point to point. Now let $\tilde{u}_i$ be *any* displacement field that is continuous, and satisfies the boundary conditions $\tilde{u}_i = u_i^o$ on ∂R. Such a displacement is called a "kinematically admissible displacement field". This displacement can always (symbolically) be expressed as $\tilde{u}_i = u_i + u_i^*$, where u_i is the exact solution, and u_i^* is the "error". We now define the "potential energy" associated with the displacement $\tilde{u}_i$ as follows:

$$\Phi(\tilde{u}_i) = \frac{1}{2}\iiint_R c_{ijkl}\tilde{\varepsilon}_{ij}\tilde{\varepsilon}_{kl}dV - \iiint_R f_i\tilde{u}_i dV. \tag{3.14}$$

The first integral on the right represents the strain energy associated with the displacement $\tilde{u}_i$. The second integral represents the work that *would* be done by the body forces *if* they acted through these displacements. This term represents a loss of energy for the agency that supplies the body force (*i.e.*, the Earth's gravitational field), and therefore represents a loss of total potential energy of the system that is composed of the rock plus its external surroundings. As defined in eq. (3.14), the potential energy is a property that is in some sense shared by the rock and its "surroundings". We now use $\tilde{u}_i = u_i + u_i^*$ and $\tilde{\varepsilon}_{ij} = \varepsilon_{ij} + \varepsilon_{ij}^*$ in eq. (3.14) to arrive at

$$\Phi(\tilde{u}_i) = \frac{1}{2}\iiint_R c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}dV + \iiint_R c_{ijkl}\varepsilon_{kl}\varepsilon^*_{ij}dV + \frac{1}{2}\iiint_R c_{ijkl}\varepsilon^*_{ij}\varepsilon^*_{kl}dV - \iiint_R f_i u_i dA - \iiint_R f_i u^*_i dA\,. \quad (3.15)$$

We now use integration-by-parts to transform the second integral on the right side:

$$\iiint_R c_{ijkl}\varepsilon_{ij}\varepsilon^*_{kl}dV = \iiint_R \tau_{kl}\varepsilon^*_{kl}dV = \iiint_R \tau_{kl}u^*_{k,l}dV = \iiint_R (\tau_{kl}u^*_k)_{,l}dV - \iiint_R \tau_{kl,l}u^*_k dV\,. \quad (3.16)$$

The first step in eq. (3.16) utilised the fact that $c_{ijkl} = c_{klij}$, which can be proven by examining the mixed partial derivatives of w with respect to the strains (see *Mathematical Theory of Elasticity*, by I. S. Sokolnikoff). We now apply the divergence theorem to the first integral on the right; in the second integral, we note from eq. (2.9) that $\tau_{kl,l} = -f_k$. This leads to

$$\iiint_R c_{ijkl}\varepsilon_{ij}\varepsilon^*_{kl}dV = \iint_{\partial R} \tau_{kl}u^*_k n_l dA + \iiint_R f_k u^*_k dV\,. \quad (3.17)$$

On the outer boundary, however, $u^*_i = \tilde{u}_i - u_i = u^o_i - u^o_i = 0$, because the kinematically admissible displacement field has, by definition, no error on the boundary. Hence, the first integral is zero. Combining eqs. (3.17) and (3.15) leads to

$$\Phi(\tilde{u}_i) = \frac{1}{2}\iiint_R c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}dV + \frac{1}{2}\iiint_R c_{ijkl}\varepsilon^*_{ij}\varepsilon^*_{kl}dV - \iiint_R f_i u_i dA$$

$$= W(u_i) + W(u_i^*) - \iiint_R f_i u_i dA,$$

$$i.e., \qquad \Phi(\tilde{u}_i) = \Phi(u_i) + W(u_i^*)\,. \quad (3.19)$$

But the strain energy function W is positive-definite, which implies that $W(u_i^*) \geq 0$. Furthermore, $W(u_i^*) = 0$ only when $u_i^* = 0$, *i.e.*, when the error is zero, in which case $\tilde{u}_i = u_i$. We have therefore proven that, for any kinematically-admissible displacement field $\tilde{u}_i$,

$$\text{PMPE:} \qquad \Phi(\tilde{u}_i) \geq \Phi(u_i)\,. \quad (3.20)$$

In other words, the potential energy of the trial displacement field is always greater than or equal to the actual potential energy. Another way of interpreting eq. (3.20) is that the correct displacement is the one that *minimises* the potential energy.

There are many other minimum principles that are related to the PMPE. The book *Variational Methods in Elasticity and Plasticity* by K. Washizu describes them in detail. The important point about them for our purposes is that they allow us to find approximate solutions to elasticity problems, by choosing trial displacement functions that contain adjustable parameters, and then choosing these parameters so as to minimise the potential energy. These energy principles form part of the theoretical justification for finite element methods. They also form the basis of many other areas of mechanics, such as, for example, the various methods that attempt to estimate the effect that cracks and fractures have on the elastic moduli of a rock mass. The role that the minimum principles play in this problem is described in the book *Micromechanics: Overall Properties of Heterogeneous Materials,* by S. Nemat-Nasser and M. Hori.

Section 4 - Two-Dimensional Elasticity

Elasticity problems are generally very hard to solve, and so it is advantageous to try to simplify them as far as possible before trying to solve them. One way to simplify any mathematical problem is to reduce it from three dimensions to two dimensions. Fortunately, there are many rock mechanics problems for which this type of simplification is acceptable. For example, consider a very long borehole drilled into a rock mass. If the *in situ* stress state that existed before the hole was drilled was nearly uniform, it is reasonable to assume that all deformation will take place *perpendicular* to the axis of the hole. Moreover, the deformation in all planes perpendicular to the borehole axis is probably the same. In such situations, the equations of elasticity can be reduced from 3-D to 2-D. With most physical theories, you can reduce the 3-D equations to 2-D by merely ignoring all terms that depend on the third co-ordinate. The situation is no so simple in elasticity, because of the existence of the Poisson effect, which introduces "coupling" between the stresses in, say the x_1 direction, and the strains in the x_2 and x_3 directions. There are two physically distinct types of problems in which the equations of elasticity can be reduced to two dimensions, as shown below.

4.1 Plane Strain

Consider a situation in which there is no deformation in the x_3 direction, and in which the deformation in the x_1 and x_2 directions depend are independent of x_3. This is the type of deformation that we postulated above would exist around a long borehole. The deformation field can in this situation be expressed as

$$\mathbf{u} = [u_1(x_1, x_2), u_2(x_1, x_2), 0]. \tag{4.1}$$

Equation (1.2) then shows that all five strain components that contain at least one subscript "3" will be zero. Equation (1.7) shows that the following shear stresses will be zero:

$$\tau_{13} = \tau_{31} = \tau_{23} = \tau_{32} = 0. \tag{4.2}$$

However, the normal stress in the x_3 direction will not, in general, be zero. In order to maintain zero strain in this direction, a nonzero stress τ_{33} will be needed to balance out the strain in that direction caused by the other two normal stresses. Equation (1.7) shows that in order for ε_{33} to be zero, τ_{33} must be exactly given by

$$\tau_{33} = \nu(\tau_{11} + \tau_{22}). \tag{4.3}$$

If we now insert eq. (4.3) into the stress equilibrium equations, we find that the third equation is automatically satisfied, whereas the first two reduce to

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + f_1 = 0, \tag{4.4a}$$

$$\frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + f_2 = 0. \tag{4.4b}$$

These equations can also be written in an indicial form, if we use Greek indices instead of Roman indices, along with the conventions that Greek indices such as α and β can take on values of 1 or 2, and that a repeated Greek index implies a summation from 1 to 2. Using this notation, the strain-displacement equations can be written as

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \tag{4.5}$$

which has the same form as in three dimensions. The stress-equilibrium equations (4.4) can be written in indicial notation as

$$\tau_{\alpha\beta,\beta} + f_\alpha = 0, \qquad \text{for } \alpha,\beta = 1,2; \tag{4.6a}$$

$$\tau_{33} = \nu(\tau_{11} + \tau_{22})\,. \tag{4.6b}$$

The governing equations for the displacements can be written as

$$(\lambda + \mu)u_{\beta,\beta\alpha} + \mu u_{\alpha,\beta\beta} + f_\alpha = 0\,, \quad \text{for } \varpi = 1,2; \tag{4.7a}$$

$$u_3 = 0. \tag{4.7b}$$

We therefore see that, when written in terms of the displacements, the equations of two-dimensional "plain strain" elasticity are essentially the same as those of three-dimensional elasticity.

4.2 Plane Stress

There is another type of situation that allows the equations of elasticity to be reduced to two dimensions. This is the so-called "plane stress" situation. Although this situation is not as relevant to rock mechanics as is plane strain, it arises frequently in most books on elasticity. The plane stress approximation applies to thin plates that are subjected to loads *within* the plane of the plate, but not *perpendicular* to that plane. If we take the x_3 direction to be normal to the plane of the plate, then the traction on the plane whose outward unit normal vector is in the x_3 direction will be zero. Hence, eq. (2.7) shows that $\tau_{31} = \tau_{32} = \tau_{33} = 0$. From the symmetry of the stress tensor, we then see that $\tau_{13} = \tau_{23} = 0$. Under these conditions, the governing equations can be reduced to a two-dimensional form. The procedure is described in most elasticity texts; for example, *Theory of Elasticity* by Y. A. Amenzade. The final result of this laborious procedure is a set of equations identical to eqs. (4.5,6a,7a) *except* that

(a) The displacement functions u_1 and u_2 must be interpreted as the mean values of the displacements, averaged over the thickness of the plate;

(b) The parameter λ must be replaced by $\lambda^* = 2\lambda\mu / (\lambda + 2\mu)$.

•Homework problem: The equations of plane stress are derived from those of plane strain by replacing λ with $\lambda^* = 2\lambda\mu / (\lambda + 2\mu)$, but keeping μ as is, *i.e.*, $\mu^* = \mu$. If you start with the

plane strain equations in terms of E and ν, what would the appropriate expressions for E^* and ν^* be for the *plane stress* equations?

Once the equations of elasticity have been simplified from three to two dimensions, many powerful mathematical methods that involve *complex variables* can be used to solve these equations. The methods were developed by Kolosov and Muskhelishvili in the Soviet Union, and by Green and Stevenson in the UK. The most accessible book on these methods is *Complex Variable Methods in Elasticity,* by A. H. England. The book *Fundamentals of Rock Mechanics* by Jaeger et al. utilises these methods to solve many problems of stresses and displacements around tunnels and boreholes. There are also numerical methods that are based on the complex variable approach, such as described in the monograph *The Complex Variable Boundary Element Method*, by T. V. Hromodka.

Section 5 - Displacement Potentials and Fundamental (Singular) Solutions

There are certain types of solutions to the elasticity equations that correspond to forces applied to a body over a very localised region. These solutions are extremely useful, because they can be used as building blocks to solve problems in which loads are applied over extended regions of the rock mass. In particular, these so-called "singular" solutions are the basis of the class of numerical methods known as *boundary element methods*. We will discuss one of the more important of these solutions, the *Kelvin solution* for a point load applied to an infinite rock mass. In order to derive this solution, we will introduce the idea of *displacement potentials,* which are very useful in their own right as a means of generating analytical solutions to the Navier equations.

5.1 Papkovich-Neuber Displacement Potentials

The governing Navier equations of elasticity, eqs. (2.11), are a set of three coupled linear partial differential equations. One of the standard ways to develop analytical solutions to a linear PDE is to use the method of *separation of variables* to first find an infinite family of functions that satisfy the PDE, and then use the boundary conditions to find the appropriate "arbitrary constants" in the solution. This approach is widely used in elasticity, with one difference. Instead of attempting to find displacement functions that satisfy the governing equations (2.11), which, are somewhat different and more difficult than most PDEs that are encountered in applied mathematics, it is traditional instead to work with *displacement potentials;* the displacements are given by certain derivatives of the displacement potentials. The difficulty in solving the elasticity equations directly is due mainly to the fact that they are *coupled,* which is to say that each of the three displacement components appears in each of the three Navier equations; this can be seen by writing out eqs. (2.11) in component form. Whereas the displacements themselves are governed by eq. (2.11), the displacement potentials are governed by Laplace's equation. Laplace's equation arises in many areas of applied mathematics, because it governs many steady-state "equilibrium" processes, such as heat conduction, electrical current flux, fluid flow in porous media, *etc*. Hence, it is relatively easy to find functions that satisfy Laplace's equation, because this has already been done in

other fields of applied mathematics. One disadvantage of this approach is that the expressions for the displacements and stresses are complicated when written in terms of the displacement potentials. However, this added complication is compensated for by the fact that it is much easier to find solutions to Laplace's equation than to the Navier equations of elasticity.

There are many different types of displacement potentials, usually named after the scientist who originated their use. The most commonly-used potentials are those of Love, Galerkin, Westergaard, Boussinesq, and Papkovich-Neuber; most texts on elasticity discuss these potentials to one extent or another. We will discuss only the Papkovich-Neuber potentials, which are perhaps the most widely used. It is difficult to give a concise and self-contained "derivation" of these potentials, because they were developed slowly over a period of many years, and because the motivation behind their use cannot be fully appreciated without knowledge of *velocity potentials* that are used in fluid mechanics. Rather than attempt a derivation, it is best just to jump right in and present the Papkovich-Neuber potential functions, and then verify that they work.

Imagine that we have four functions, three of which are considered to be components of a vector field, and the other considered to be a scalar field. (A "field" is just the mathematical term for a function that depends on the co-ordinates (x,y,z)). These functions can be denoted by ψ_i and φ, and are referred to as *displacement potentials*. Now assume that a displacement field can be constructed from these potentials in the following manner:

$$2\mu u_i = -4(1-\nu)\psi_i + (x_k\psi_k + \varphi)_{,i}\,. \tag{5.1a}$$

$$i.e., \qquad 2\mu\boldsymbol{u} = -4(1-\nu)\,\boldsymbol{\psi} + \nabla(\boldsymbol{x}\cdot\boldsymbol{\psi} + \varphi)\,. \tag{5.1b}$$

The latter form is in direct vector notation, which some people find easier to interpret. However, when carrying out the differentiations that are needed in the following development, indicial notation is easier to use, because its use does not require memorisation of the various rules for manipulating gradient symbols, *etc*.

At this point, of course, we have no reason to think that if we pick *any* functions to play the roles of ψ_i and φ, the displacements generated from them through eq. (5.1) will satisfy the governing eqs. (2.11). It seems reasonable to expect that only certain functions will be acceptable as displacement potentials. To see what conditions the potentials should satisfy, we simply insert eq. (5.1) into eqs. (2.11). In order to simplify the calculations, we will assume that the body forces are zero. First, we expand out and simplify eq. (5.1):

$$\begin{aligned} 2\mu u_i &= -4(1-\nu)\psi_i + \psi_i + x_k\psi_{k,i} + \varphi_{,i} \\ &= -(3-4\nu)\,\psi_i + x_k\psi_{k,i} + \varphi_{,i}\,. \end{aligned} \tag{5.2}$$

The first derivatives of the displacement are given by

$$2\mu u_{i,j} = -(3-4\nu)\psi_{i,j} + \psi_{j,i} + x_k\psi_{k,ij} + \varphi_{,ij}\,. \tag{5.3}$$

The two types of second derivatives required in eq. (2.11) are given by

$$2\mu u_{i,jj} = -(3-4\nu)\psi_{i,jj} + 2\psi_{j,ij} + x_k\psi_{k,ijj} + \varphi_{,ijj}; \tag{5.4}$$

$$2\mu u_{i,ji} = -(2-4\nu)\psi_{i,ij} + \psi_{j,ii} + x_k\psi_{k,iji} + \varphi_{,iij}. \tag{5.5a}$$

In deriving eqs. (5.4,5), we have made use of the fact that the *order* of taking partial derivatives is irrelevant, so we can freely switch the order of any indices that appear *after* a comma. Although the expression in eq. (5.5a) may not look as if it is relevant to eq. (2.11), we can *switch* the indices *i* and *j* in eq. (5.5a) to arrive at

$$2\mu u_{j,ij} = -(2-4\nu)\psi_{j,ji} + \psi_{i,jj} + x_k\psi_{k,jij} + \varphi_{,jji}. \tag{5.5b}$$

This ability to switch indices is another useful trick that can be used when working in indicial notation. It is valid as long as you switch indices in a consistent manner, on *both* sides of the equation.

We now insert eqs. (5.4,5b) into eq. (2.11), with the body-force term set to zero, use the relation $\lambda = 2\mu\nu/(1-2\nu)$, and divide through by $4\mu(1-\nu)$, to arrive at

$$x_k\psi_{k,jji} - (1-4\nu)\psi_{i,jj} + \varphi_{,jji} = 0. \tag{5.6}$$

The situation now looks more complicated than when we began, since eq. (5.6) represents three coupled equations in the four unknown functions. However, note that one way to satisfy eq. (5.6) is if all four potential functions satisfy Laplace's equation, *i.e.*,

$$\nabla^2\psi_i = \psi_{i,jj} = 0, \tag{5.7a}$$

$$\nabla^2\varphi = \varphi_{,jj} = 0. \tag{5.7b}$$

To see more clearly that eqs. (5.7) imply eq. (5.6), we note, for example, that $\varphi_{,jji} = (\varphi_{,jj})_{,i} = \partial\varphi_{,jj}/\partial x_i = \partial 0/\partial x_i = 0$. We therefore see that any function that satisfies Laplace's equation can play the role of either the Papkovich-Neuber scalar displacement potential, ϕ, or the role of one of the components of the P-N vector displacement potential, ψ_i. Functions that satisfy Laplace's equation are known as *harmonic functions*, and the study of these functions is known as *potential theory*. There are numerous applied mathematics books that discuss these functions in great detail; perhaps the most famous is *Methods of Mathematical Physics* by Courant and Hilbert.

There are two important points to be made concerning our replacement of eqs. (2.11) with eqs. (5.7). The first is that we have replaced a system of three *coupled* PDEs by a set of four *uncoupled* PDEs; it is always easier to solve uncoupled equations than to solve coupled equations. The second point is that each of the four displacement functions satisfies Laplace's equation, which is the most well-understood partial differential equation in all of applied mathematics. In particular, the general solution to Laplace's equation is known in a variety of co-ordinate systems, such as Cartesian, cylindrical, spherical, spheroidal, *etc*. Hence, we have a ready-made storehouse of functions that can serve as displacement potentials, and thereby generate solutions to the Navier equations.

An interesting aspect to this procedure is that although there are only *three* components to the displacement vector, we are representing that vector in terms of *four* potential functions. There would therefore seem to be some redundancy in the Papkovich-Neuber representation. For many years it was thought that one could ignore any one of the four P-N potentials, and set it equal to zero. However, it turns out that one cannot *arbitrarily* set one of the four P-N potentials equal to zero; the choice of which three to use depends on the shape of the body, among other things. This very complicated and arcane issue is discussed in *Elasticity* by J. R. Barber, and *The Mathematical Theory of Elasticity* by I. S. Sokolnikoff. The important point, for most practical purposes, is that by finding the most general solution to eqs. (5.7), which has already been done in many co-ordinate systems, the solution to an elastostatic boundary value problem is reduced to merely choosing the constants in the general solution so as to satisfy the boundary conditions.

5.2 Kelvin Solution

The Papkovich-Neuber potentials offer one route to the *Kelvin solution*, which is one of the most important of the *fundamental*, or *singular*, solutions to the elasticity equations. The Kelvin solution represents the displacements, stresses, *etc*., that are caused by a concentrated force that acts at a "point" in an infinite elastic medium. We will follow the development of the Kelvin solution that is given in *Elasticity* by J. R. Barber, although our discussion will be greatly simplified by the use of indicial notation. Another derivation, using the Galerkin displacement potential, is given in *Boundary Element Analysis* by J. H. Kane. This latter book contains a detailed discussion of the relationship between the Kelvin solution and *boundary element methods*.

As a first step to presenting the Kelvin solution, consider the special case of a Papkovich-Neuber potential in which the first two components of $\boldsymbol{\psi}$ are zero, and the scalar potential ϕ is also zero. In order to adhere to Barber's notation, we will denote the third component of the vector potential by ω instead of ψ_3. In order to make use of indicial notation, which Barber does not utilise, we can say that

$$\psi_i = \omega \delta_{3i}, \qquad \phi = 0. \tag{5.8}$$

Insertion of this expression for the P-N potentials into eq. (5.2) yields

$$2\mu u_i = -(3-4\nu)\omega\delta_{3i} + x_3\omega_{,i}. \tag{5.9}$$

We now choose ω to be one of the simplest of all harmonic functions, c/*R*, where *c* is a constant, and *R* is the *magnitude* of the position vector, x_i. From the Pythagorean theorem, we know that $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, which can be written in indicial notation as $R = (x_i x_i)^{1/2}$. In order to discuss the displacements and stresses that are associated with this potential, we need the following derivatives:

$$(1/R)_{,i} = \frac{-x_i}{R^3}; \tag{5.10a}$$

$$(1/R)_{,ij} = \frac{-\delta_{ij}}{R^3} + \frac{3x_i x_j}{R^5}. \tag{5.10b}$$

• Homework problem: Derive eqs. (5.10). Verify that $1/R$ is an harmonic function, *i.e.*, verify that $(1/R)_{,kk} = 0$.

With this choice of ω, the displacements given by eq. (5.9) can be written as

$$2\mu u_i = -(3-4\nu)c\frac{\delta_{3i}}{R} - \frac{cx_3x_i}{R^3}. \tag{5.11}$$

The displacement *gradient*, which is needed in order to find the strains (and hence, the stresses), is found by differentiating eq. (5.11):

$$2\mu u_{i,j} = (3-4\nu)c\frac{\delta_{3i}x_j}{R^3} - c\frac{\delta_{3j}x_i}{R^3} + 3c\frac{x_3x_ix_j}{R^5} - c\frac{x_3\delta_{ij}}{R^3}. \tag{5.12}$$

The stresses are found from Hooke's law, eq. (2.10):

$$\tau_{ij} = \frac{(1-2\nu)c}{R^3}\left[\delta_{3i}x_j + \delta_{3j}x_i - \delta_{ij}x_3\right] + \frac{3cx_3x_ix_j}{R^5}. \tag{5.13}$$

We now have a set of displacements and stresses that satisfy the equations of elasticity in an infinite rock mass, under the conditions of zero body force. However, we do not yet know what problem this displacement field corresponds to! In order to find out, we have to examine the solution presented in eqs. (5.11,13) more closely. First, we note that as $R \to \infty$, both the displacements and the stresses decay to zero. This is seen from eqs. (5.11,13) by noting that the denominators of each term are of higher order than the numerators, for large R. Hence, this displacement field corresponds to zero tractions and zero displacements at infinity. However, now that we are thinking about the relative magnitudes of the numerators and denominators, we see that as $R \to 0$, the displacements and stresses blow up, and become infinite! Therefore, the displacement field (5.9) cannot be a solution to the Navier equations (with zero body force) at the point $R = 0$. We now guess, *ala* Kelvin, that this displacement field corresponds to the one caused by a *concentrated body force* that acts at the *origin* of the co-ordinate system. Actually, this is not a very risky guess, since *any* displacement field will be a solution to the Navier equations, for the proper choice of a body force field. (If you don't believe this, just note that you can always solve eq. (2.11) for f_i, if u_i is given!) In the present problem, we know that $f_i = 0$ when $R \neq 0$, so the only place at which a body force can be acting is as it $R = 0$; it remains only to find the *magnitude* and *direction* of this force.

One rigorous (but somewhat complicated) way to do this would be to (1) transform the stresses into *spherical* co-ordinates; (2) find the tractions that act on an "infinitesimally small" small sphere that is centred at the origin, and then (3) integrate the tractions over this sphere to find the resultant force. This calculation is done in the book *Boundary Element*

Analysis by Kane. This calculation would show that although the body force at the origin is "infinite", it has a finite resultant when integrated over a small region surrounding the origin. In order to find this resultant by a calculation that does not require transformation of the stresses into a curvilinear co-ordinate system, we will give a modified version of the calculation done in *Elasticity* by Barber.

Consider a cylindrical-shaped region surrounding the origin, bounded by the two parallel planes $x_3 = +h$ and $x_3 = -h$, and by the cylindrical surface $\rho = a$, where $\rho = (x_1^2 + x_2^2)^{1/2}$ is the distance from the x_3 axis. The rock within this region must be in equilibrium, so the resultant force acting on it must be zero. This resultant force is equal to the integral of the surface tractions over the outer boundary of this cylinder, *plus* the resultant body force acting at the origin. Due to the symmetry of the stress state, the resultant surface forces in the x_1 and x_2 directions will be zero. If we eventually let the radius of this cylinder be very large, it can be shown that the net force in the x_3 direction acting over the sides of the cylinder will go to zero. The only nonzero contribution to the net surface force is the integral of τ_{33} taken over the top and bottom faces of the cylinder. By letting both indices in eq. (5.13) take on the value "3", we find

$$\tau_{33} = \frac{(1-2\nu)cx_3}{R^3} + \frac{3cx_3^3}{R^5}. \tag{5.14}$$

Now consider the top surface of the cylinder, $x_3 = +h$. On this surface, the outward unit normal vector is (0,0,1), so the traction component in the x_3 direction is given by

$$t_3(x_3 = h) = \frac{(1-2\nu)ch}{R^3} + \frac{3ch^3}{R^5} \tag{5.15}$$

The total force acting on this upper surface of the cylinder, in the x_3 direction, is found by integrating the traction t_3 over the entire surface. Even though we are expressing the stresses in a Cartesian co-ordinate system, it is convenient to make use of the cylindrical (ρ,θ,z) co-ordinates for this integration. Noting that $R^2 = \rho^2 + x_3^2 = \rho^2 + h^2$ on this surface, and that the differential of surface area in polar co-ordinates is $2\pi\rho d\rho$ (note that there is no dependence on θ in this problem), the total force acting on the *top* face is given by

$$F_3(x_3 = h) = \int_{\rho=0}^{\rho=a} \left[\frac{(1-2\nu)ch}{(\rho^2+h^2)^{3/2}} + \frac{3ch^3}{(\rho^2+h^2)^{5/2}} \right] 2\pi\rho d\rho$$

$$= -2\pi c \left[\frac{(1-2\nu)h}{(\rho^2+h^2)^{1/2}} + \frac{h^3}{(\rho^2+h^2)^{3/2}} \right]_{\rho=0}^{\rho=a}. \tag{5.16}$$

As any arbitrary region of the rock mass must be in equilibrium, we are allowed to pick the region however we please, so as to facilitate the evaluation of the integral. We noted above that as $a \to \infty$, the net force due to tractions along the curved sides of the cylinder becomes negligible. Hence, we let $a \to \infty$ in eq. (5.16) and arrive at

$$F_3(x_3 = h) = 4\pi(1-\nu)c\,. \tag{5.17}$$

The resultant force acting on the bottom face, $x_3 = -h$, is exactly the same as on $x_3 = +h$; this is because the "-" sign introduced in τ_{33} is cancelled out by the "-" sign in the outward unit normal vector to the bottom surface. Hence, the total surface force acting on this region of rock is $8\pi(1-\nu)c$. For equilibrium to obtain, this force must be balanced by the concentrated body force that acts at the origin. We therefore see that this body force must act in the x_3 direction, and have magnitude $-8\pi(1-\nu)c$. Alternatively, if we denote the magnitude of the concentrated body force by F_3, we can solve for $c = -F_3/8\pi(1-\nu)$. We can now summarise the Kelvin solution as follows.

The Kelvin solution corresponds to a *concentrated load* that acts at the *origin* of an *infinite elastic medium*. The load acts in the x_3 direction, and has magnitude F_3. The displacement vector caused by this load is

$$u_i = \frac{(3-4\nu)F_3\delta_{3i}}{16\pi\mu(1-\nu)R} + \frac{F_3 x_3 x_i}{16\pi\mu(1-\nu)R^3}\,, \tag{5.18}$$

where R is the distance to the origin. The stresses caused by this force are

$$\tau_{ij} = \frac{-(1-2\nu)F_3}{8\pi(1-\nu)R^3}\left[\delta_{3i}x_j + \delta_{3j}x_i - \delta_{ij}x_3\right] - \frac{3F_3 x_3 x_i x_j}{8\pi(1-\nu)R^5}\,. \tag{5.19}$$

If the force were in the x_1 direction rather than the x_3 direction, we would merely replace the subscript "3" with "1" in eqs. (5.18) and (5.19), wherever it appears; similarly for a force in the x_2 direction.

• Homework: Write out all terms of eqs. (5.18) and (5.19).

Finally, consider the case where the force acts at a point whose co-ordinates are $(x_1 = \xi_1, x_2 = \xi_2, x_3 = \xi_3)$. The displacements at an arbitrary point $\boldsymbol{x}$ can be found from eq. (5.18) by replacing x_i with $x_i - \xi_i$ on the right side of the equation, and similarly for x_3. Note that the resulting expressions give the displacements and stresses at the point $\boldsymbol{x}$, *not* at the point $\boldsymbol{\xi}$, which is the location of the applied load. The displacements at point $\boldsymbol{x}$ due to a load applied at point $\boldsymbol{\xi}$ are often denoted by $u_i(\boldsymbol{x};\boldsymbol{\xi})$.

The Kelvin "point-load" solution is often used as the basic building block for boundary element analysis, which is a powerful numerical method that allows elasticity problems to be solved in regions having complex geometries (see *Boundary Element Analysis* by J. H. Kane). Another important singular solution is the Boussinesq solution, which corresponds to a point load applied at the surface of a "semi-infinite half-space". If this solution is integrated over a finite area, we can develop solutions for foundations of various planforms. This procedure is described in *Engineering Rock Mechanics* by Harrison and Hudson.

Section 6 - Betti's Reciprocal Theorem

One of the key ingredients in setting up the "boundary element method" (BEM) for solving elasticity problems is the reciprocal theorem, which was originally proven by E. Betti in 1872. The role of the reciprocal theorem is demonstrated in the paper "A direct formulation of the boundary element method of stress analysis for complete plane strain" by B. H. G. Brady (*Int. J. Rock Mech.*, 1979, pp. 235-244). In light of the importance of this theorem in setting up the boundary element equations, a statement and derivation of the theorem are given below.

The reciprocal theorem can be stated in words very simply. Recall that a rock mass can be subjected to two types of loads - body forces and surface tractions. Let us denote some combination of these loads as a "set of forces". Each set of forces will create a certain displacement field. Note that one can use eq. (3.6) to calculate the work that *would* be done by a set of forces as they acted through a certain displacement, regardless of whether or not those displacements are the ones caused by that particular set of forces. If the forces and displacements used in the integrals in eq. (3.6) are unrelated, then the "work" calculated would be hypothetical; nevertheless, such hypothetical work terms are useful in various of the mathematical manipulations that are needed to set up the BEM equations. The reciprocal theorem states that "if an elastic body is subjected to two different sets of forces, then the work that would be done by the first set when acting through the displacements caused by the second set is equal to the work that would be done by the second set when acting through the displacements caused by the first set" (Sokolnikoff, *Mathematical Theory of Elasticity*, 1956, p. 391).

To prove this theorem, first imagine two sets of forces, distinguished by superscripts 1 and 2. Denote the first set of forces by $\{f_i^1, t_i^1\}$, and let the displacement field caused by these forces be denoted by u_i^1; similarly for the second set of forces. Now let the work that would be done by $\{f_i^1, t_i^1\}$, acting through displacements u_i^2, be denoted by W^{12}. From eq. (3.6),

$$2W^{12} = \iint_{\partial R} t_i^{\,1} u_i^{\,2} dA + \iiint_R f_i^1 u_i^{\,2} dV. \tag{6.1}$$

We now essentially repeat the steps carried out in section 3.1, but in reverse order. First recall that $t_i^{\,1} = \tau_{ij}^1 n_j$, so that (A1) can be written as

$$2W^{12} = \iint_{\partial R} \tau_{ij}^1 n_j u_i^2 dA + \iiint_R f_i^1 u_i^{\,2} dV$$

$$= \iint_{\partial R} \tau_{ij}^1 u_i^{\,2} n_j dA + \iiint_R f_i^1 u_i^{\,2} dV\,. \tag{6.2}$$

We now use the divergence theorem "in reverse" on the first integral in eq. (6.2), to arrive at

$$2W^{12} = \iiint_R (\tau_{ij}^1 u_i^{\,2})_{,j} dV + \iiint_R f_i^1 u_i^{\,2} dV\,. \tag{6.3}$$

Now expand out the derivative in the first integral, using the product rule for derivatives:

$$2W^{12} = \iiint_R (\tau^1_{ij,j} u_i{}^2 + \tau^1_{ij} u^2_{i,j}) dV + \iiint_R f_i{}^1 u_i{}^2 dV, \tag{6.4}$$

which can be rearranged as

$$2W^{12} = \iiint_R (\tau^1_{ij,j} + f^1_i) u_i{}^2 dV + \iiint_R \tau^1_{ij} u^2_{i,j} dV. \tag{6.5}$$

The term enclosed in parentheses in the first integral is zero, because the stress state with superscript 1 must satisfy the stress-equilibrium equations, (2.9). Hence, the first integral vanishes, *regardless* of whether or not the displacement field u^2_i is in any way related to the stresses and displacements associated with the first set of forces!

For the remaining integral in eq. (A5), we use the following "trick". First recall that it is permissible to replace the dummy indices in any indicial expression with a different dummy index, *i.e.*, $\varepsilon_{ii} = \varepsilon_{kk}$, *etc*. Hence, we see that $\tau_{ij} u_{i,j} = \tau_{ji} u_{j,i}$, which can be verified by expanding out both sides of this equation. We can therefore write eq. (6.5) as

$$2W^{12} = \frac{1}{2} \iiint_R (\tau^1_{ij} u^2_{i,j} + \tau^1_{ji} u^2_{j,i}) dV. \tag{6.6}$$

Now we use the fact that the stresses are symmetric, i.e., $\tau_{ij} = \tau_{ji}$, to rewrite eq. (6.6) as

$$\begin{aligned} 2W^{12} &= \frac{1}{2} \iiint_R (\tau^1_{ij} u^2_{i,j} + \tau^1_{ij} u^2_{j,i}) dV \\ &= \frac{1}{2} \iiint_R \tau^1_{ij} (u^2_{i,j} + u^2_{j,i}) dV \\ &= \iiint_R \tau^1_{ij} \varepsilon^2_{ij} dV. \end{aligned} \tag{6.7}$$

We now recall from eq. (2.12) that $\tau_{ij} = c_{ijkl} \varepsilon_{kl}$, in which case eq. (6.7) can be written as

$$W^{12} = \frac{1}{2} \iiint_R c_{ijkl} \varepsilon^1_{kl} \varepsilon^2_{ij} dV. \tag{6.8}$$

If we switch the roles of the two superscripts, we can also say that

$$W^{21} = \frac{1}{2} \iiint_R c_{ijkl} \varepsilon^2_{kl} \varepsilon^1_{ij} dV. \tag{6.9}$$

Although the two systems of forces, 1 and 2, are not necessarily related to each other in any way, they both act on the same rock mass, as so the same elastic moduli tensor c_{ijkl} appears in eqs. (6.8) and (6.9).

Now let us focus our attention on eq. (6.9). We again make use of the ability to switch dummy indices: we switch i and k, and switch j and l, and thereby rewrite eq. (6.9) as

$$W^{21} = \frac{1}{2} \iiint_R c_{klij} \varepsilon^2_{ij} \varepsilon^1_{kl} dV. \tag{6.10}$$

Finally, we recall that the elastic moduli tensor always has the following symmetry: $c_{ijkl} = c_{klij}$. For example, $c_{1231} = c_{3112}$, *etc*. This is true regardless of the degree of anisotropy of rock mass; it is merely a consequence of the equality of mixed partial derivatives of the strain energy density w with respect to the strains. Using this symmetry, we can rewrite eq. (6.10) as

$$W^{21} = \frac{1}{2}\iiint_R c_{ijkl}\varepsilon^1_{kl}\varepsilon^2_{ij}dV\,. \tag{6.11}$$

Comparison of eqs. (6.8) and (6.11) show that $W^{12} = W^{21}$, which proves the reciprocal theorem! Two useful forms of the theorem for computational purposes are (see eqs. 6.1 and 6.8)

$$\iint_{\partial R} t_i{}^1 u_i{}^2 dA + \iiint_R f_i^1 u_i{}^2 dV = \iint_{\partial R} t_i{}^2 u_i{}^1 dA + \iiint_R f_i^2 u_i{}^1 dV\,; \tag{6.12}$$

$$\frac{1}{2}\iiint_R c_{ijkl}\varepsilon^2_{ij}\varepsilon^1_{kl}dV = \frac{1}{2}\iiint_R c_{ijkl}\varepsilon^1_{ij}\varepsilon^2_{kl}dV\,. \tag{6.13}$$

Equation (6.12) is the form used by Brady, with the body forces f_i^1 and f_i^2 taken to be zero.

● Homework problem: Starting from eq. (6.7), use the isotropic version of Hooke's law, as given by eq. (2.10), and directly verify that $W^{12} = W^{21}$.