Imperial College MSc in
GeoEnergy with Machine Learning and Data Science

GEMS 3: Geomechanics and Pressure Transient Analysis

Lecture 1:
Darcy's Law and the Derivation of the Pressure Diffusion Equation

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Darcy's Law and the Definition of Permeability

The basic law governing the flow of fluids through porous media is *Darcy's law*, which was formulated by the French civil engineer Henry Darcy in 1856 on the basis of his experiments on vertical water filtration through sand beds. Darcy found that his data could be described by the following equation:

$$Q = \frac{CA\Delta(P - \rho g z)}{L}$$

where: Q = volumetric flowrate [m^3/s]

P = pressure [Pa]

ρ = density [kg/m^3]

g = gravitational acceleration [m/s^2]

z = vertical coordinate (measured downwards) [m]

L = length of sample [m]

A = cross-sectional area of sample [m^2]

C = constant of proportionality [m^2/Pa s]

Darcy's Experiments and Data

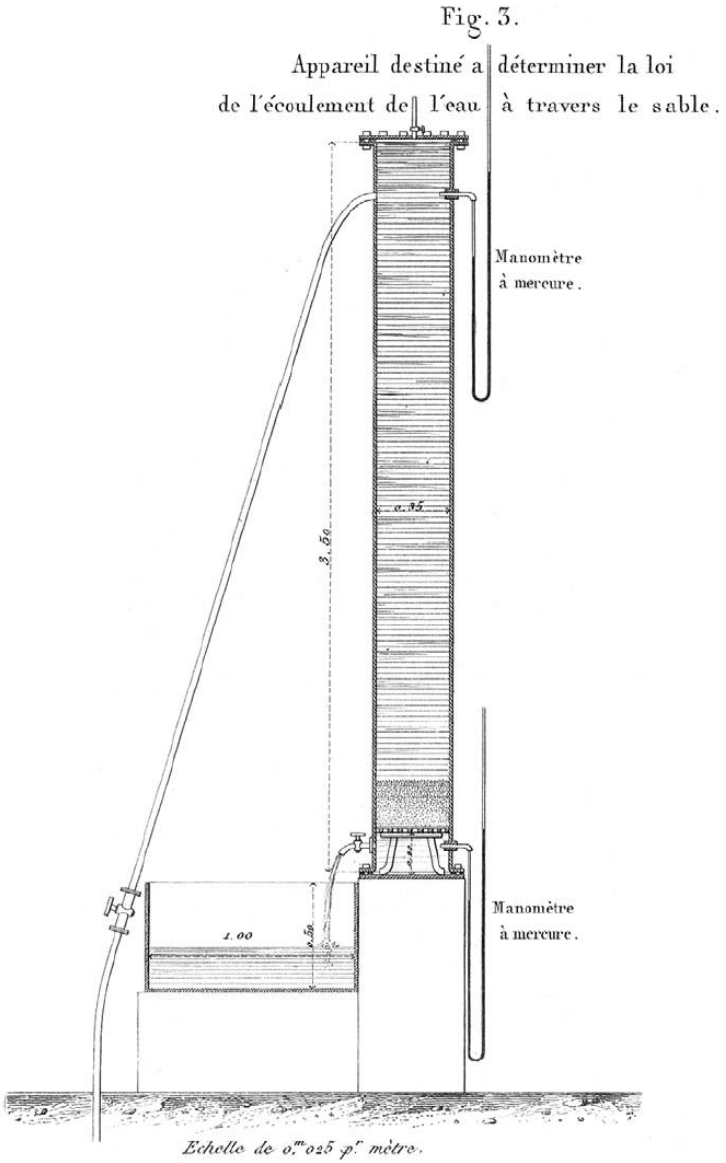
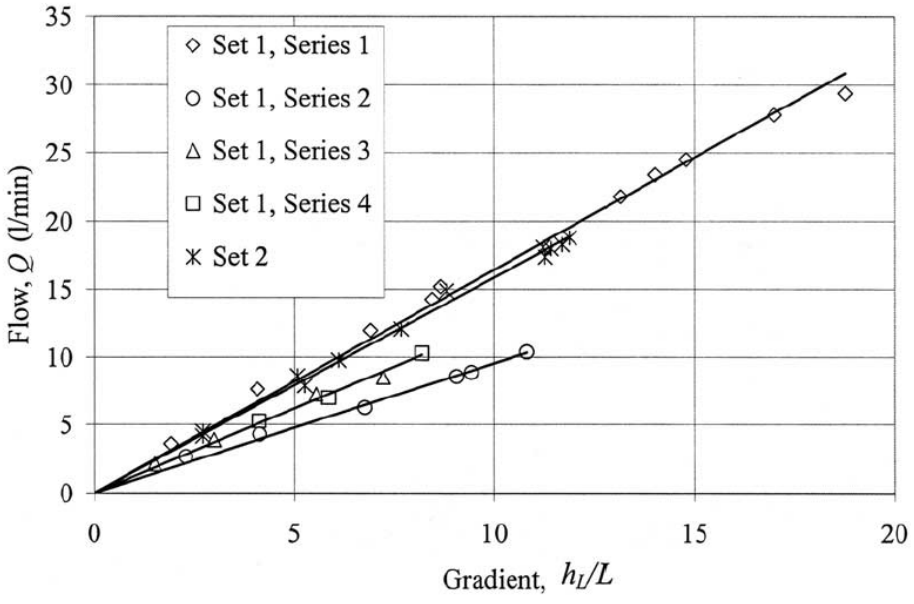


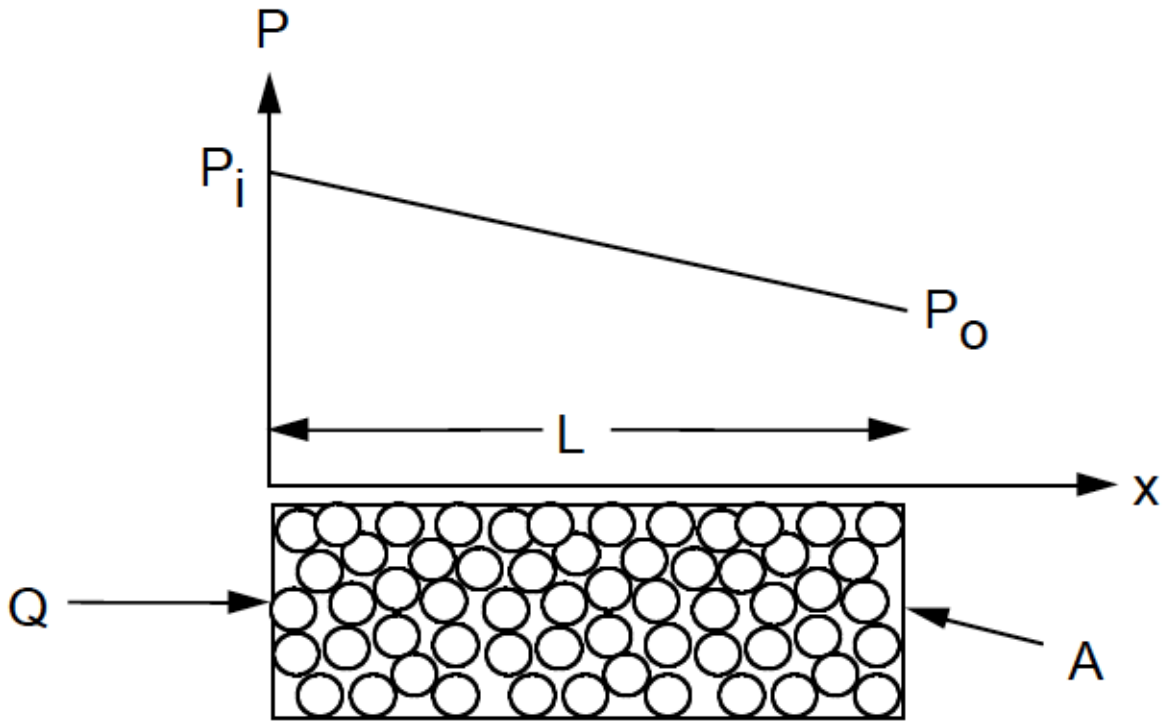
Figure 5. Column apparatus [*Darcy*, 1856, Figure 3, Plate 24].

Figure 1. Portrait of Henry Darcy by F. Perrodin, from the Collection of the Bibliothéque Municipale de Dijon.



Darcy's Law and the Definition of Permeability

In the lab, permeability is usually measured using a horizontal version of Darcy's experimental configuration:



Note that the fluid flows from regions of *high* pressure to regions of *low* pressure

Darcy's Law and the Definition of Permeability

Why does the term "$P - \rho gz$" govern the flowrate?

Recall from elementary fluid mechanics that Bernoulli's equation, which essentially embodies the principle of "conservation of energy", contains the following terms

$$\frac{P}{\rho} - gz + \frac{v^2}{2} = \frac{1}{\rho}\left(P - \rho gz + \frac{\rho v^2}{2}\right)$$

where $P/\rho = Pv$ is related to the enthalpy per unit mass, $h = u + Pv$
gz is the gravitational energy per unit mass
$v^2/2$ is the kinetic energy per unit mass

Fluid velocities in a reservoir are usually very small, and so the third term is usually negligible, in which case we see that the combination $P - \rho gz$ represents an energy-type term.

It seems reasonable that fluid would flow from regions of *higher* energy to *lower* energy, and, therefore, the driving force for flow should be the *gradient* (*i.e.,* the rate of spatial change) of $P - \rho gz$.

Darcy's Law and the Definition of Permeability

Subsequent to Darcy's initial discovery, it has been found that, all other factors being equal, Q is inversely proportional to the fluid viscosity, μ [Pa s].

It is therefore convenient to factor out μ, and put $C = k/\mu$, where k is known as the *permeability*, with dimensions [m^2].

It is also usually more convenient to work with the volumetric flow per unit area, $q = Q/A$, rather than the total flowrate, Q:

$$q = \frac{Q}{A} = \frac{k}{\mu}\frac{\Delta(P - \rho g z)}{L}$$

For transient processes in which the flux varies from point-to-point, we need a *differential* form of Darcy's law. For flow in the "*s*"-direction, we can say

$$q_s = \frac{-k}{\mu}\frac{d(P - \rho g z)}{ds}$$

The minus sign shows that the fluid flows from *higher* to *lower* values of $P - \rho g z$.

For horizontal flow, z is constant, and Darcy's law takes the form

$$q_x = \frac{-k}{\mu}\frac{dP}{dx}$$

Darcy's Law and the Definition of Permeability

The permeability is a function of rock type, and also varies with stress, temperature, *etc*., but does *not* depend on the fluid; the effect of the fluid on the flowrate is accounted for by the viscosity term in Darcy's law.

Permeability has units of m^2, but in the UK and US petroleum industries, it is conventional to use "Darcy" units, defined by

$$1\,\text{Darcy} = 0.987 \times 10^{-12}\ \text{m}^2 \approx 10^{-12}\ \text{m}^2$$

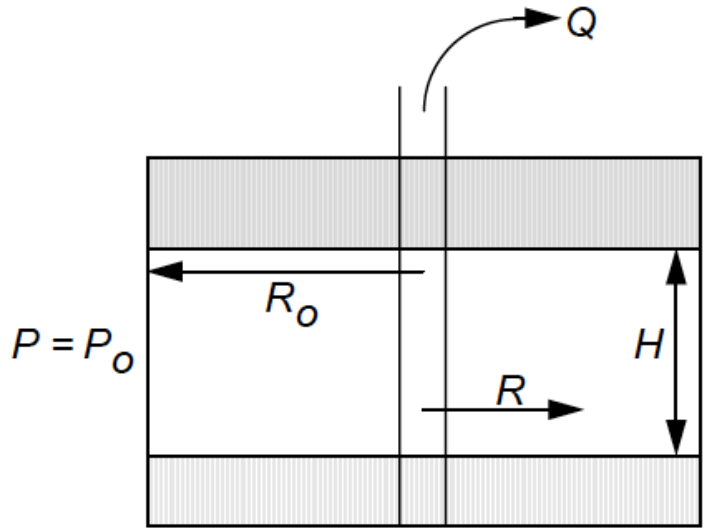
The numerical value of k for a given rock depends on the diameter of the pores in the rock, d, as well as on the degree of interconnectivity of the void space. Very roughly speaking, $k = d^2/1000$; see GEMS 1 for a derivation.

For example, a rock with a typical pore size of 10×10^{-6} m, *i.e.,* 10 microns, would be expected to have a permeability of about 10^{-13} m^2, or 0.1 D.

The permeabilities of petroleum reservoir rocks tend to be in the range of 0.001−1.0 Darcies. It is therefore convenient to quantify the permeability of reservoir rocks in units of "milli-Darcies" (mD), *i.e*., 0.001 D.

Radial, Steady-State Flow to a Well

Before we derive the general transient equation that governs fluid flow through porous media, we will examine a simple, but illustrative, problem that can be solved using only Darcy's law: a circular reservoir that has a constant pressure at its outer boundary, and a constant flowrate into the wellbore.



Consider a reservoir of thickness *H* and horizontal permeability *k,* fully penetrated by a vertical well of radius R_w. Assume that at some radius R_o, the pressure remains at its undisturbed value, P_o.

If we pump fluid out of this well at a rate *Q*, what will be the steady-state pressure distribution in the reservoir?

Radial, Steady-State Flow to a Well

Darcy's law for radial flow takes the form:

$$Q = \frac{-kA}{\mu}\frac{dP}{dR}$$

The cross-sectional area normal to the flow, at a radial distance R from the centre of the well, is $2\pi RH$ (*i.e.,* a cylindrical surface of height H, and perimeter $2\pi R$), so

$$Q = \frac{-2\pi kH}{\mu} R\frac{dP}{dR}$$

Separate the variables, and integrate from the outer boundary, R_o, to some generic location R:

$$\int_{R_o}^{R} \frac{dR}{R} = -\int_{P_o}^{P} \frac{2\pi kH}{\mu Q} dP$$

$$\ln\frac{R}{R_o} = \frac{-2\pi kH}{\mu Q}(P - P_o)$$

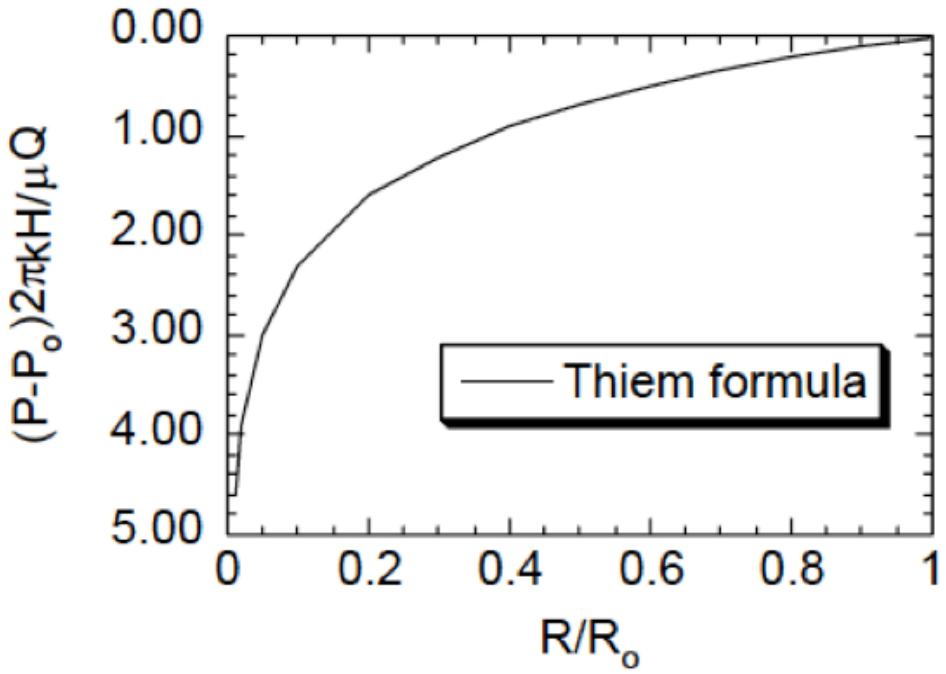
$$P(R) = P_o - \frac{\mu Q}{2\pi kH}\ln\left(\frac{R}{R_o}\right)$$

This is the Dupuit-Thiem equation, derived in 1857 by the French hydrologist Jules Dupuit, and popularised by the German hydrologist Adolf Thiem.

Radial, Steady-State Flow to a Well

Since the pressure varies logarithmically with distance from the wellbore, most of the drawdown occurs near the well, whereas far from the well, the pressure varies slowly.

This same type of logarithmic variation of pressure with distance occurs in transient processes, as we will see in Lecture 2.



Radial, Steady-State Flow to a Well

We can make the following comments about the Thiem equation:

1. If fluid is pumped *from* the well, then (mathematically) Q is *negative*, because the fluid is flowing in the direction opposite to the direction of the radial co-ordinate, R. Hence, $P(R)$ will be less than P_o for any $R < R_o$.

2. The amount by which $P(R)$ is less than P_o is called the *pressure drawdown*.

3. The pressure drawdown *at the well* is found by setting $R = R_w$:

$$P_w = P_o - \frac{\mu Q}{2\pi kH} \ln\left(\frac{R_w}{R_o}\right)$$

4. Since we are often interested in situations in which the fluid is flowing *towards* the well (*i.e.,* "production"), it is common to *re-define* Q to be positive for production, in which case we write the Thiem equation as

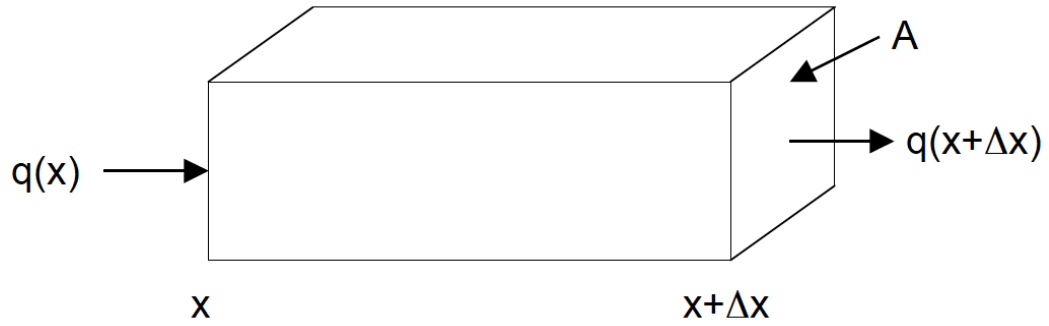
$$P_w = P_o + \frac{\mu Q}{2\pi kH} \ln\left(\frac{R_w}{R_o}\right)$$

Regardless of the sign convention, it is imperative to understand, and to see from the equation, that if we are producing fluid from the reservoir, then the pressure *at the well* is less than the pressure *in the reservoir*!

Conservation of Mass Equation, 1

Darcy's law in itself does not contain sufficient information to allow us to solve transient (*i.e.,* time-dependent) problems involving subsurface flow. To develop a complete governing equation that applies to transient problems, we must first derive a mathematical expression of the principle of *conservation of mass*.

Consider flow through a one-dimensional tube of cross-sectional area A; in particular, let's focus on the region between two locations x and $x+\Delta x$:



The main idea behind the application of the principle of conservation of mass can be expressed in words as follows:

Flux in – Flux out = Increase in amount stored

Note that the quantity that is conserved is fluid *mass*, not *volume*.

Conservation of Mass Equation, 2

Consider the period of time between time t and time $t + \Delta t$. During this time increment, the mass flux *into* this region of rock between will be

$$\text{Mass flux in} = A(x)\rho(x)q(x)\Delta t$$

The mass flux *out of* this region of rock will be

$$\text{Mass flux out} = A(x+\Delta x)\rho(x+\Delta x)q(x+\Delta x)\Delta t$$

The amount of fluid mass stored in the region is denoted by m, so the conservation of mass equation takes the form

$$A(x)\rho(x)q(x)\Delta t - A(x+\Delta x)\rho(x+\Delta x)q(x+\Delta x)\Delta t = m(t+\Delta t) - m(t)$$

For 1D flow, such as through a cylindrical core, $A(x) = A$ = constant.

So, we can factor out A, divide both sides by Δt, and let Δt go to zero:

$$-A[\rho q(x+\Delta x) - \rho q(x)] = [m(t+\Delta t) - m(t)]/\Delta t = dm/dt$$

Conservation of Mass Equation, 3

But $m = \rho V_p$, where V_p is the *pore volume* of the rock contained in the slab between x and $x+\Delta x$. Therefore,

$$m = \rho V_p = \rho\phi V = \rho\phi A\Delta x$$

Combining these last two equations gives

$$-A[\rho q(x+\Delta x) - \rho q(x)] = dm/dt = d(\rho\phi A\Delta x)/dt = A\Delta x d(\rho\phi)/dt$$

Divide through by $A\Delta x$, and let Δx go to 0:

$$d(\rho\phi)/dt = -[\rho q(x+\Delta x) - \rho q(x)]/\Delta x = -d(\rho q)/dx$$

$$i.e., \quad d(\rho\phi)/dt = -d(\rho q)/dx$$

This is the basic equation of conservation of mass for 1-*D* flow in a porous medium. It relates the *spatial* rate of change of stored mass, to the *temporal* rate of change of stored mass.

This equation is *exact*, and applies to gases, liquids, high or low flowrates, *etc*.

Pressure Diffusion Equation in Cartesian Coordinates

Steady-state flow rarely occurs in a reservoir. The typical flow scenario is *transient*, in which pressure, density, flowrate, *etc*., vary in space and time.

To derive the transient *pressure diffusion equation*, we combine Darcy's law, the conservation of mass equation, and an equation that relates the fluid pressure to the amount of fluid that is stored inside the porous rock ("storativity").

Let's examine the derivative term $d(\rho\phi)/dt$ more closely, using the product rule:

$$\frac{d(\rho\phi)}{dt} = \rho\frac{d\phi}{dt} + \phi\frac{d\rho}{dt}$$

$$= \rho\frac{d\phi}{dP}\frac{dP}{dt} + \phi\frac{d\rho}{dP}\frac{dP}{dt}$$

$$= \rho\phi\left[\left(\frac{1}{\phi}\frac{d\phi}{dP}\right) + \left(\frac{1}{\rho}\frac{d\rho}{dP}\right)\right]\frac{dP}{dt}$$

$$= \rho\phi(c_\phi + c_f)\frac{dP}{dt}$$

where c_f is the compressibility of the fluid, and c_ϕ is the "pore compressibility" of the rock formation, also sometimes called the "formation compressibility".

Pressure Diffusion Equation in Cartesian Coordinates

Now let's examine the space derivative term $d(\rho q)/dx$ in the conservation of mass equation, recalling that q is given by Darcy's law:

$$-\frac{d(\rho q)}{dx} = -\frac{d}{dx}\left[\frac{-\rho k}{\mu}\frac{dP}{dx}\right] = \frac{k}{\mu}\left[\rho\frac{d^2P}{dx^2} + \frac{d\rho}{dx}\frac{dP}{dx}\right]$$

$$= \frac{k}{\mu}\left[\rho\frac{d^2P}{dx^2} + \frac{d\rho}{dP}\frac{dP}{dx}\frac{dP}{dx}\right]$$

$$= \frac{\rho k}{\mu}\left[\frac{d^2P}{dx^2} + \left(\frac{1}{\rho}\frac{d\rho}{dP}\right)\left(\frac{dP}{dx}\right)^2\right]$$

$$= \frac{\rho k}{\mu}\left[\frac{d^2P}{dx^2} + c_f\left(\frac{dP}{dx}\right)^2\right]$$

Now equate the expressions for $-d(\rho q)/dx$ and $d(\rho\phi)/dt$:

$$\frac{d^2P}{dx^2} + c_f\left(\frac{dP}{dx}\right)^2 = \frac{\phi\mu(c_f + c_\phi)}{k}\frac{dP}{dt}$$

Pressure Diffusion Equation in Cartesian Coordinates

For liquids, the second (nonlinear!) term on the left is negligible (see FFiPM textbook for a numerical example), in which case the one-dimensional, linearised form of the pressure diffusion equation is:

$$\frac{dP}{dt} = \frac{k}{\phi \mu c_t} \frac{d^2P}{dx^2}$$

in which the total compressibility, c_t, is given by

$$c_t = c_{formation} + c_{fluid} = c_\phi + c_f$$

Typical values of the compressibility of various rock types and reservoir fluids are as follows:

Rock (or Fluid) Type	c (1/Pa)	c (1/psi)
Sand	$10^{-6} – 10^{-8}$	$10^{-2} – 10^{-4}$
Sandstones	$10^{-7} – 10^{-9}$	$10^{-3} – 10^{-5}$
Carbonates	$10^{-9} – 10^{-11}$	$10^{-9} – 10^{-11}$
Shales	$10^{-10} – 10^{-12}$	$10^{-10} – 10^{-12}$
Water	5×10^{-10}	3.5×10^{-6}
Oil	1×10^{-9}	7.0×10^{-6}

Pressure Diffusion Equation in Radial Coordinates

In most areas of energy-related engineering in the subsurface, we are very often interested in fluid flowing towards, or away from, a *vertical well*.

In these situations, it is more convenient to use cylindrical ("radial") coordinates, rather than Cartesian coordinates.

In radial coordinates, the pressure diffusion takes the form (see Section 1.7 of the FFiPM textbook for the detailed derivation):

$$\frac{dP}{dt} = \frac{k}{\phi \mu c_t} \frac{1}{R} \frac{d}{dR}\left(R \frac{dP}{dR} \right)$$

This is the governing equation for transient, radial flow of a liquid through porous rock, to a vertical borehole.

In the next five lectures of the GEMS 3 module, we will develop several methods to solve this equation, and derive solutions for a few important problems governed by this equation.

Deviations from Darcy's Law

Darcy's law holds in most situations of subsurface flow, and it is the most important equation in areas such as oil and gas engineering, carbon sequestration, underground gas storage, and geothermal engineering.

However, there are some situations in which Darcy's law does not hold, and must be replaced by a different law. These situations are discussed in detail in Chapter 9 of the FFiPM textbook.

One such situation occurs when the flowrate is not sufficiently low, in a sense that we will soon quantify. In these situations, we need to replace Darcy's law with an extension thereof, known as the **Forchheimer** equation.

Another situation when Darcy's law does not hold occurs if the pore sizes in a rock (again, in a sense that we will quantify soon) are too small. In this case, we need to replace Darcy's law with an equation called the **Klinkenberg** equation.

Non-Darcy (Forchheimer) Flow, 1

Recall that we earlier "justified" Darcy's law by appealing to the Bernoulli equation, and noting that if the flow velocity was low, we would expect the flowrate to be controlled by the pressure gradient.

Consequently, Darcy's law actually holds only at low flowrates, which can be defined, roughly, as flows for which the Reynolds number is less than one.

The Reynolds number is a dimensionless measure of the relative strengths of inertial forces and viscous forces. Using the definition of Reynolds number, this condition can be written as

$$Re = \frac{\rho v d}{\mu} < 1$$

where ρ is the density of the fluid, μ is the viscosity, d is a mean pore diameter, and v is the mean (microscopic, *i.e.*, pore-scale) velocity.

If this criterion is violated, Darcy's law must be replaced with a nonlinear law, such as Forchheimer's equation:

$$\frac{dP}{dR} = \frac{\mu q}{k} + \beta \rho q^2$$

in which the coefficient β accounts for "non-Darcy" (inertial) effects.

Non-Darcy (Forchheimer) Flow, 2

Dimensional analysis of the Forchheimer equation shows that the factor β has dimensions of L^{-1}.

Since k has dimensions of L^2, it is (very) roughly the case that $\beta = 1/\sqrt{k}$.

It follows that the magnitude of the *nonlinear* part of the pressure gradient in the Forchheimer equation, relative to the magnitude of the *linear* part (the Darcy pressure gradient), is

$$\frac{\text{non}-\text{Darcy term}}{\text{Darcy term}} = \frac{\beta\rho q^2}{(\mu q/k)} = \frac{\beta\rho q k}{\mu} \approx \frac{\rho q\sqrt{k}}{\mu}$$

You will have learned in the GEMS 1 module that k is proportional to d^2. So, aside from some additional dimensionless terms that we will ignore here, we see that the ratio of the *non-Darcy* pressure drop to the *Darcy* pressure drop is roughly proportional to the Reynolds number.

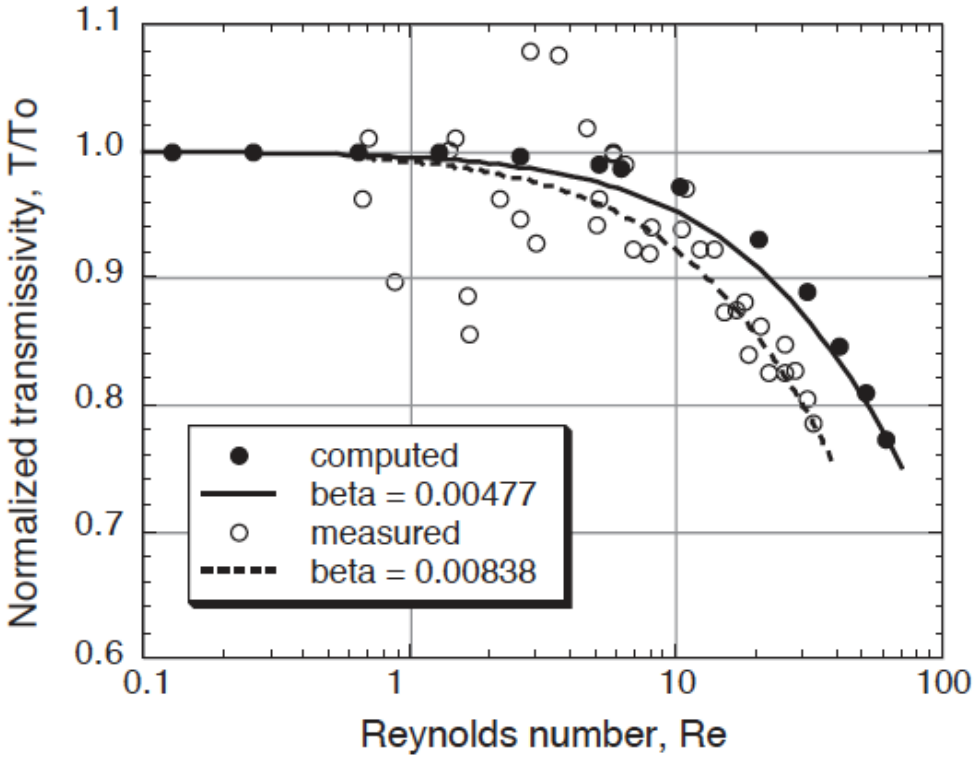
Hence, if the Reynolds number is much less than one, the nonlinear terms in the Forchheimer equation are indeed negligible, and we recover Darcy's law.

Non-Darcy inertial effects are more important for gas flow than for liquid flow (see textbook for details), and more important *near* a well than *far* from a well.

Non-Darcy (Forchheimer) Flow, 3

The plot shown below, taken from an IC PhD thesis by Azzan Al-Yaarubi in 2004, shows the permeability of a fracture as a function of Reynolds number, as measured in the lab, and as computed using the Navier-Stokes equations, which are the fundamental equations of fluid flow. For both data sets, experimental and numerical, the values were fit by a Forchheimer-type equation.

(For fractures, we often talk of the “transmissivity” instead of the “permeability”, but that distinction is not important for our purposes here.)



Klinkenberg Equation, 1

When a liquid flows through a porous medium, the correct “boundary condition” for the flow is the so-called “no-slip” boundary condition, that specifies that the tangential velocity at the wall of the pore is *zero*.

However, this no-slip boundary condition does *not* hold for a gas at very low densities, which is to say at very low pressures, and/or for a gas flowing through a rock that has very small pores.

In order for the gas to behave like a continuum, and obey the no-slip boundary condition, a given gas molecule must collide much more frequently with other gas molecules, than with the pore walls. At low densities, however, each gas molecule will collide with a pore wall much more frequently than it collides with another gas molecule.

In this situation, the gas travels through the pores not as a continuum fluid, but more like a bunch of billiard balls bouncing off the pore walls.

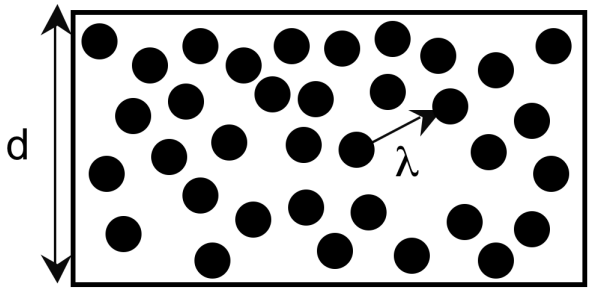
To quantify whether or not this will be the case, we must consider the concept of the “mean free path”, which is the mean distance travelled by a molecule before it collides with another molecule.

Klinkenberg Equation, 2

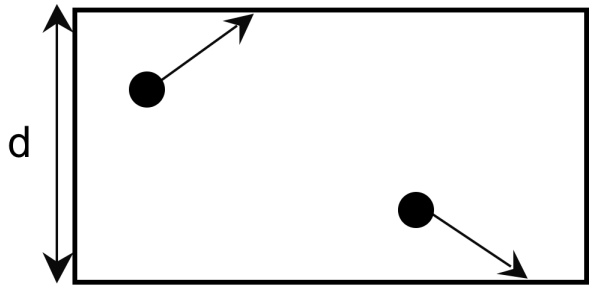
According to the kinetic theory of gases, the mean free path λ is given by

$$\lambda = \frac{k_B T}{\sqrt{2}\pi\sigma^2 P}$$

where k_B is the Boltzmann constant (*i.e.,* the gas constant per molecule), and σ is the effective molecular diameter. Roughly, λ can be thought of as the average distance between a molecule and its nearest neighbour.



Continuum flow



Slip flow

If the pore size *d* is smaller than the mean free path, collisions with the pore walls will be much more frequent than collisions with other molecules, and the gas will flow through the pores as a set of individual molecules, rather than as a fluid continuum. This type of flow is known as “Knudsen flow”, or “slip flow”.

Klinkenberg Equation, 3

In 1941, Klinkenberg assumed that gas flow through a porous medium could be modelled as Knudsen flow through a capillary tube, and showed that the "apparent" permeability measured during gas flow will be related to the "true" (*i.e.*, Darcy's law) absolute permeability *k by*

$$k_{gas} = k\left[1 + \frac{8c\lambda}{d}\right]$$

where λ is the mean free path, d is the pore diameter, and c is a dimensionless coefficient whose value is close to 1.

If we combine this expression for k_{gas} with our equation for λ, we find

$$k_{gas} = k\left[1 + \frac{8c}{\sqrt{2}\pi}\frac{k_B T}{d\sigma^2}\frac{1}{P}\right]$$

We now us the Kozeny-Carman equation (which you will have learned in GEMS 1), which states that (approximately) $k = \phi d^2/96 \approx \phi d^2/100$, to arrive at

$$k_{gas} = k\left[1 + \frac{4c\sqrt{\phi}}{5\sqrt{2}\pi}\frac{k_B T}{\sqrt{k}\sigma^2 P}\right]$$

Klinkenberg Equation, 4

$$i.e., \quad k_{gas} = k\left[1+\frac{4c\sqrt{\phi}}{5\sqrt{2\pi}}\frac{k_B T}{\sqrt{k}\sigma^2 P}\right] \equiv k\left[1+\frac{P^*}{P}\right]$$

where P^* is a characteristic pressure, usually written as b, that is given by

$$P^* = \frac{4c\sqrt{\phi}\,k_B T}{5\pi\sqrt{2}\sigma^2\sqrt{k}}$$

We see from the equation at the top that, roughly, the Klinkenberg effect will be *important* if $P < 10P^*$, and will be *negligible* if $P > 10P^*$.

The temperature, the molecular diameter, and $\sqrt{\phi}$ will not vary by very much, so P^* is mainly controlled by the permeability of the rock.

Consider a rock having a porosity of $\phi = 0.10$, at 300°K. Boltzmann's constant is 1.38×10^{-23} J/°K, and typical molecular diameters of a gas are about 4Å, so:

k (Darcy units)	P^* (MPa)	P^* (psi)
1 Darcy	0.0015	0.22
1 milli-Darcy	0.047	6.8
1 micro-Darcy	1.5	220
1 nano-Darcy	47.0	6,800

.

Klinkenberg Equation, 5

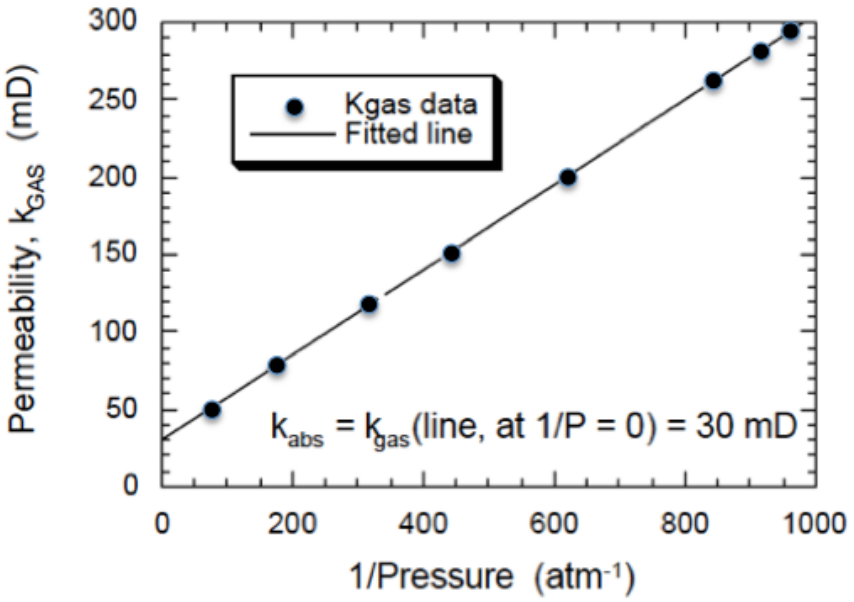
There are two situations in which the Klinkenberg effect is important:

1. In very low-permeability rocks such as shales, reservoir pressures will not be much larger than P^*, so this effect must be accounted for when modelling flow through the rock. In this case, an accurate estimate of P^* will be needed.

2. For rocks with permeabilities in the usual range of mDarcies, we often measure the permeability in the lab using a gas at low pressures. In this case our measurements give us "k_{gas}", but we want to know the actual permeability, k. To find k, we plot k_{gas} as a function of $1/P$, and extrapolate back to $1/P = 0$.

$$k_{gas} = k\left[1 + \frac{P^*}{P}\right]$$

If we fit a straight line through the k vs. $1/P$ data, and extrapolate back to $1/P = 0$, we find k. Note that $1/P = 0$ corresponds to high pressures, at which the gas behaves like a continuum. Note that this procedure does not require knowledge of the Klinkenberg parameter, P^*!



Problems for Lecture 1

Problem 1.1. A well located in a 100 ft. thick reservoir having a permeability of 100 mD produces 100 barrels/day of oil from a 10 in. diameter wellbore. The viscosity of the oil is 0.4 cP. The pressure at a distance of 1000 feet from the wellbore is 3000 psi. What is the pressure at the wellbore?

Conversion factors are as follows:

1 barrel = 0.1589 m^3
1 Poise = 0.1 Newton-seconds/m^2
1 foot = 0.3048 m
1 psi = 6895 N/m^2 = 6895 Pa

Problem 1.2. Carry out a derivation of the diffusion equation for *spherically-symmetric* flow, in analogy to the derivation given in the textbook for radial flow. (The *spherical flow* equation can be used to model flow to a well in situations when only a small length of the well has been perforated, in which case the flow field will, at early times, be roughly spherical.)

The result of your derivation should be an equation similar to the one for 2D radial flow, but with a slightly different term on the right-hand side.

Imperial College MSc in
GeoEnergy with Machine Learning and Data Science

GEMS 3: Geomechanics and Pressure Transient Analysis

Lecture 2:
Line Source Solution for a Vertical Well in an Infinite Reservoir

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Line Source Solution for a Vertical Well in an Infinite Reservoir

A basic problem in subsurface energy engineering is to calculate the pressures in the reservoir, and at the well, when fluid is being produced from, or injected into, a vertical well at a constant rate, from an homogeneous, laterally infinite reservoir. This problem can be formulated as follows:

Geometry: A *vertical* well that *fully penetrates* a reservoir which is of uniform thickness, H, and which extends *infinitely* far in all horizontal directions.

Reservoir Properties: The reservoir is assumed to be *isotropic* and *homogeneous*, with uniform properties (*i.e.,* permeability, porosity, *etc*.) that *do not vary with pressure*.

Initial and Boundary Conditions: The reservoir is initially at a *uniform pressure*. Starting at $t = 0$, fluid is extracted from the wellbore at a *constant rate*, Q.

Wellbore diameter: The diameter of the wellbore is assumed to be *infinitely small*; this leads to a much simpler problem than the more realistic finite-diameter problem, but with very little loss of practical applicability.

Problem: To determine the *pressure* at all points in the reservoir, including at the wellbore, as a function of the *elapsed time* since the start of production.

Derivation of the Line-Source Solution

The governing equation for this problem is the linearised pressure diffusion equation, in radial coordinates, that we derived in Lecture 1:

$$\frac{dP}{dt} = \frac{k}{\phi \mu c_t} \frac{1}{R} \frac{d}{dR}\left(R \frac{dP}{dR}\right)$$

This partial differential equation is second-order in R, and first-order in t. So, in order to solve it, we need *two boundary conditions*, and *one initial condition.* These conditions can be stated in words, as follows:

Initial Condition: At the start of production, the pressure in the reservoir is assumed to be at some uniform value, P_i.

Boundary condition at infinity: Infinitely far from the well, the pressure will always remain at its initial value, P_i.

Boundary condition at the wellbore: At the wellbore, which is assumed to be infinitely small, the flux must be equal to Q at all times $t > 0$, defined here so that $Q > 0$ for the case in which we produce fluid *from* the reservoir.

We can therefore formulate the problem in precise *mathematical* terms as follows, where for notational simplicity we replace c_t with c:

Derivation of the Line-Source Solution, 1

Governing PDE:

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{dP}{dR}\right)=\frac{\phi\mu c}{k}\frac{dP}{dt} \tag{1}$$

Initial condition:

$$P(R,t=0)=P_i \tag{2}$$

BC at wellbore:

$$\lim_{R\to 0}\left(\frac{2\pi kH}{\mu}R\frac{dP}{dR}\right)=Q \tag{3}$$

BC at infinity:

$$\lim_{R\to \infty} P(R,t)=P_i \tag{4}$$

Strictly speaking, we can't impose a boundary condition *at* $R = 0$, since when $R = 0$, the term R inside the parenthesis in eq. (3) goes to zero, and the term dP/dR goes to infinity. So, we need to first multiply these two terms together, and *then* take the "limit" as R goes to 0.

Derivation of the Line-Source Solution, 2

There are many ways to solve this equation, but we will solve it using a method that does not require advanced techniques such as Laplace transforms or Green's functions.

First, we define a new variable η that combines, in clever way, the spatial variable *R* and the time variable *t*. This "trick" to simplifying a diffusion equation was discovered by the German physicist Ludwig Boltzmann in 1894, and is now referred to as the *Boltzmann transformation*:

$$\eta = \frac{\phi\mu c R^2}{kt}$$

Next, we rewrite eq. (1) in terms of this new variable, η. The right-hand side transforms as follows:

$$\frac{dP}{dt} = \frac{dP}{d\eta}\frac{d\eta}{dt} = -\frac{\phi\mu c R^2}{kt^2}\frac{dP}{d\eta} = \frac{-\eta}{t}\frac{dP}{d\eta}$$

The derivatives with respect to *R* transform as follows:

$$\frac{dP}{dR} = \frac{dP}{d\eta}\frac{d\eta}{dR} = \frac{2\phi\mu c R}{kt}\frac{dP}{d\eta} = \frac{\phi\mu c R^2}{kt}\frac{2}{R}\frac{dP}{d\eta} = \frac{2\eta}{R}\frac{dP}{d\eta}$$

Derivation of the Line-Source Solution, 3

Proceeding in this way, we eventually find that, when written in terms of the Boltzmann variable η, the governing PDE gets transformed into the following ODE:

$$\frac{d}{d\eta}\left(\eta\frac{dP}{d\eta}\right) = -\frac{\eta}{4}\frac{dP}{d\eta}$$

We must now also transform the boundary/initial conditions, so that they apply to the function $P(\eta)$. First, note that both limits, $R \to \infty$ and $t \to 0$, correspond to the *same* limit in the Boltzmann domain, $\eta \to \infty$.

So, conditions (2) and (4) coalesce to the following condition,

$$\lim_{\eta\to\infty} P(\eta) = P_i$$

We saw on the previous slide that $R(dP/dR)$ transforms to $2\eta(dP/d\eta)$, and so the BC at the wellbore takes the following form:

$$\lim_{\eta\to 0}\left(\eta\frac{dP}{d\eta}\right) = \frac{\mu Q}{4\pi kH}$$

The three equations shown above constitute a “two-point ODE boundary-value problem”, which we must now solve.

Derivation of the Line-Source Solution, 4

The next step in solving this equation is to note that although it is a 2nd-order equation for *P*, it is actually a 1st-order equation for $\eta(dP/d\eta)$. If we define $y = \eta(dP/d\eta)$ as our new variable, the ODE takes the form:

$$\frac{dy}{d\eta} = -\frac{y}{4}, \quad \text{where} \quad y = \eta\frac{dP}{d\eta}$$

Now separate the variables, and integrate from $\eta = 0$ out to some arbitrary value of η:

$$\frac{dy}{y} = -\frac{d\eta}{4} \quad \rightarrow \quad \int_{y(0)}^{y(\eta)} \frac{dy}{y} = -\int_{0}^{\eta} \frac{d\eta}{4}$$

$$\rightarrow \quad \ln\left[\frac{y(\eta)}{y(0)}\right] = -\frac{\eta}{4} \quad \rightarrow \quad y(\eta) = y(0)e^{-\eta/4}$$

At the well, the boundary condition on the function *y* is seen from the the previous slide to be given by $y(0) = \mu Q/4\pi KH$. So:

$$y(\eta) = \frac{\mu Q}{4\pi kH} e^{-\eta/4}$$

Recalling that $y = \eta(dP/d\eta)$, we can say that

$$\frac{dP(\eta)}{d\eta} = \frac{\mu Q}{4\pi kH}\frac{e^{-\eta/4}}{\eta}$$

Derivation of the Line-Source Solution, 5

This equation can now be integrated to find $P(\eta)$, which will give us the pressure in the reservoir as a function of η, and therefore as a function of R and t.

We can't start the integral at $\eta = 0$, because we don't know the pressure at the the point $R = 0$. We do, however, know that the pressure at $\eta = \infty$ must be equal to the initial reservoir pressure, P_i. Therefore, we start the integral at $\eta = \infty$:

$$\int_{P_i}^{P(\eta)} dP = \int_{\infty}^{\eta} \frac{\mu Q}{4\pi kH} \frac{e^{-\eta/4}}{\eta} d\eta \quad \rightarrow \quad P(\eta) = P_i - \frac{\mu Q}{4\pi kH} \int_{\eta}^{\infty} \frac{e^{-\eta/4}}{\eta} d\eta$$

Recalling that $\eta = \phi\mu cR^2/kt$, we replace η with $\phi\mu cR^2/kt$ on the left-hand side, and also at the lower limit of integration on the right, but not *inside* the integral, because inside the integral, η is merely a *dummy integration variable*:

$$P(R,t) = P_i - \frac{\mu Q}{4\pi kH} \int_{\phi\mu cR^2/kt}^{\infty} \frac{e^{-\eta/4}}{\eta} d\eta$$

Lastly, to further simplify the integral, we make another change of variables, $u = \eta/4$, which leads us to:

$$P(R,t) = P_i - \frac{\mu Q}{4\pi kH} \int_{\phi\mu cR^2/4kt}^{\infty} \frac{e^{-u}}{u} du$$

Derivation of the Line-Source Solution, 6

The integral that appears in this equation is called the “exponential integral function”, which is defined as:

$$-Ei(-x) = \int_x^\infty \frac{e^{-u}}{u} du$$

This function was defined by mathematicians long before it was used to solve the problem of a well in an infinite reservoir, and so it contains two awkward minus signs on the left, which are now traditional. However, note that Python removes the two – signs, and calls the result the E_1(x) function. We will use the *Ei* notation.

The above solution was first published by the American hydrologist Charles Theis in 1935, and is referred to as the *Theis solution*. We can summarise it as follows:

$$P(R,t) = P_i + \frac{\mu Q}{4\pi kH} Ei(-x) \qquad \text{(i)}$$

$$\text{where} \quad -Ei(-x) = \int_x^\infty \frac{e^{-u}}{u} du \qquad \text{(ii)}$$

$$\text{and} \quad x = \phi\mu c R^2 / 4kt \qquad \text{(iii)}$$

If we want to know the pressure at distance *R* from the centre of the well, at time *t,* we first use eq. (iii) to compute *x*. We then compute the value of -*Ei*(-*x*) from eq. (ii), or look it up from a table or graph (see next slide). The pressure at distance *R* and time *t* is then given by eq. (i).

Exponential Integral Function

Exponential Integral Function, $-Ei(-x)$

x	1	2	3	4	5	6	7	8	9
$\times 1$	.219	.049	.013	.0038	.0011	3.6e-4	1.2e-4	3.8e-5	1.2e-5
$\times 10^{-1}$	1.82	1.22	0.91	0.70	0.56	0.45	0.37	0.31	0.26
$\times 10^{-2}$	4.04	3.35	2.96	2.68	2.47	2.30	2.15	2.03	1.92
$\times 10^{-3}$	6.33	5.64	5.23	4.95	4.73	4.54	4.39	4.26	4.14
$\times 10^{-4}$	8.63	7.94	7.53	7.25	7.02	6.84	6.69	6.55	6.44
$\times 10^{-5}$	10.94	10.24	9.84	9.55	9.33	9.14	8.99	8.86	8.74
$\times 10^{-6}$	13.24	12.55	12.14	11.85	11.63	11.45	11.29	11.16	11.04
$\times 10^{-7}$	15.54	14.85	14.44	14.15	13.93	13.75	13.60	13.46	13.34
$\times 10^{-8}$	17.84	17.15	16.74	16.46	16.23	16.05	15.90	15.76	15.65
$\times 10^{-9}$	20.15	19.45	19.05	18.76	18.54	18.35	18.20	18.07	17.95
$\times 10^{-10}$	22.45	21.76	21.35	21.06	20.84	20.66	20.50	20.37	20.25
$\times 10^{-11}$	24.75	24.06	23.65	23.36	23.14	22.96	22.81	22.67	22.55
$\times 10^{-12}$	27.05	26.36	25.96	25.67	25.44	25.26	25.11	24.97	24.86
$\times 10^{-13}$	29.36	28.66	28.26	27.97	27.75	27.56	27.41	27.28	27.16
$\times 10^{-14}$	31.66	30.97	30.56	30.27	30.05	29.87	29.71	29.58	29.46
$\times 10^{-15}$	33.96	33.27	32.86	32.58	32.35	32.17	32.02	31.88	31.76

For example, if $x = 5\times 10^{-7}$, then $-Ei(-x) = 13.93$.

Dimensionless Form of the Line-Source Solution

Although the pressure seems to depend on many variables and parameters, there are actually only two *independent*, *dimensionless* mathematical variables in the line-source solution.

Traditionally, these variables are defined as the *dimensionless time*,

$$t_D = \frac{kt}{\phi\mu c R^2}$$

and the *dimensionless pressure drawdown*,

$$\Delta P_D = \frac{2\pi kH(P_i - P)}{\mu Q}$$

In terms of these dimensionless parameters, the line-source solution takes the form

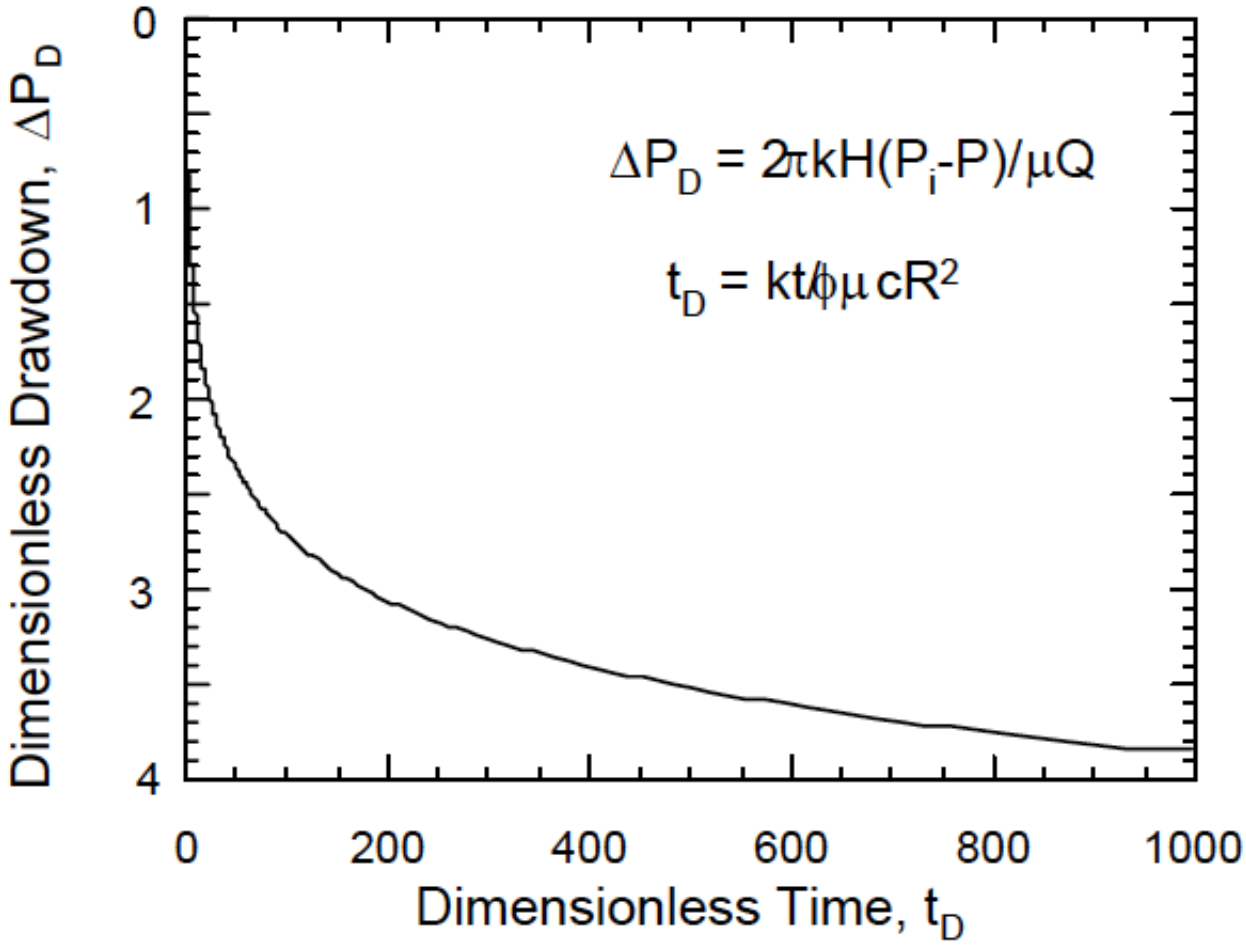
$$\Delta P_D = -\frac{1}{2} Ei(-1/4t_D)$$

By definition, the dimensionless time is different at each location R in the reservoir. Most often, however, we are interested in the pressure at the well, in which case $R = R_w$, and the dimensionless time *at the well* is given by

$$t_{D_w} = \frac{kt}{\phi\mu c R_w^2}$$

Dimensionless Form of the Line-Source Solution

$$\Delta P_D = -\frac{1}{2} Ei(-1/4t_D)$$

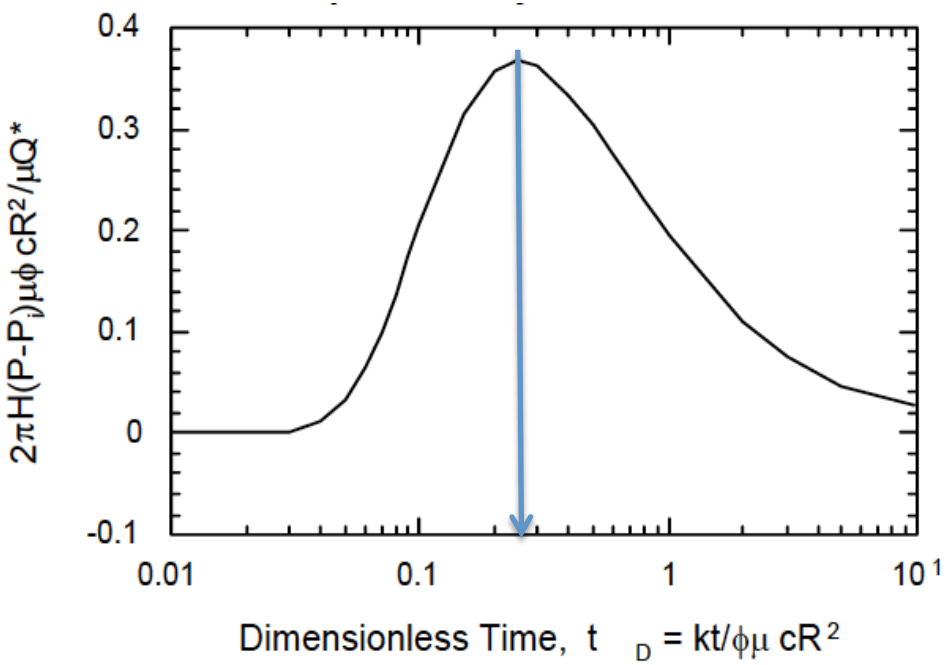


Physical Interpretation of the Dimensionless Time

Recall that the dimensionless time is defined as $t_D = kt / \phi\mu cR^2$, and the line source solution depends on the variable $4t_D$.

Can we give a physical interpretation to this parameter, or is it just a convenient mathematical tool?

In Section 2.5 of the textbook, it is shown that if we inject a small amount of fluid, Q^*, into the wellbore, the pressure pulse that is created will reach the location R when the dimensionless time equals about 0.25 (see right):



It follows that values of $4t_D << 1$ correspond to times at which the pressure pulse emanating from the wellbore has not yet arrived at location R, whereas times such that $4t_D >> 1$ correspond to times at which the pressure pulse has penetrated much farther than a distance R into the reservoir.

Logarithmic Approximation to the Line-Source Solution

It is easy to show that, for large values of x, the *Ei* function becomes exponentially small, *i.e.*, $-Ei(-x) < e^{-x}$. This corresponds to the fact that, very far from the well, the drawdown is always, for practical purposes, zero!

The more interesting and important case is that of small values of *x*, which are relevant for small values of *R*, *i.e.*, at the wellbore, and/or for large values of time (recall that $x = \phi\mu cR^2/4kt$). Fortunately, for small values of *x*, the exponential integral essentially becomes a *logarithm function*, which makes it very easy to use.

To derive this "late-time" approximation, known as Jacob's approximation, we proceed as follows. For large times, *x* will be small, and we can break up the integral into two parts:

$$-Ei(-x) = \int_x^\infty \frac{e^{-u}}{u} du = \int_x^1 \frac{e^{-u}}{u} du + \int_1^\infty \frac{e^{-u}}{u} du$$

Use the Taylor series for exp(-*u*) in the first integrand on the right:

$$\int_x^1 \frac{e^{-u}}{u} du = \int_x^1 \left[\frac{1 - \frac{u}{1!} + \frac{u^2}{2!} - \frac{u^3}{3!} + \cdots}{u} \right] du$$

Logarithmic Approximation to the Line-Source Solution

Break up the integral on the right side into a series of integrals, and evaluate these integrals term-by-term:

$$\int_x^1 \frac{e^{-u}}{u}du = \int_x^1 \frac{1}{u}du - \frac{1}{1!}\int_x^1 du + \frac{1}{2!}\int_x^1 u\,du + \frac{1}{3!}\int_x^1 u^2\,du - \cdots$$

$$= \ln u\Big]_x^1 - u\Big]_x^1 + \frac{1}{2!}\frac{u^2}{2}\Big]_x^1 - \frac{1}{3!}\frac{u^3}{3}\Big]_x^1 + \cdots$$

$$= (\ln 1 - \ln x) - (1-x) + \frac{1}{2!2}(1-x^2) - \frac{1}{3!3}(1-x^3) + \cdots$$

$$= -\ln x + x - \frac{1}{2!2}x^2 + \frac{1}{3!3}x^3 + \cdots - \left\{1 - \frac{1}{2!2} + \frac{1}{3!3} + \cdots\right\}$$

Substituting this result this back into the full equation for *Ei*, allows us to say

$$-Ei(-x) = -\ln x - \ln\gamma + x - \frac{1}{2!2}x^2 + \frac{1}{3!3}x^3 + \cdots$$

where

$$\ln\gamma = \left\{1 - \frac{1}{2!2} + \frac{1}{3!3} + \cdots\right\} - \int_1^\infty \frac{e^{-u}}{u}du = 0.5572$$

Logarithmic Approximation to the Line-Source Solution

It follows that, for $x < 0.01$, the *Ei* function can be approximated very well by the following simple logarithmic function,

$$-Ei(-x) \approx -\ln x - \ln\gamma$$

where $\ln\gamma = \ln(1.781) = 0.5772$, and γ is known as Euler's number.

If we use this approximation in the equation for the pressure, we arrive at:

$$P(R,t) = P_i + \frac{\mu Q}{4\pi kH} Ei(-x) = P_i + \frac{\mu Q}{4\pi kH}(\ln x + \ln\gamma)$$

$$= P_i + \frac{\mu Q}{4\pi kH}\ln\left(\frac{\phi\mu c R^2\gamma}{4kt}\right) = P_i - \frac{\mu Q}{4\pi kH}\ln\left(\frac{4kt}{\phi\mu c R^2\gamma}\right)$$

$$= P_i - \frac{\mu Q}{4\pi kH}\left[\ln\left(\frac{kt}{\phi\mu c R^2}\right) + \ln\left(\frac{4}{\gamma}\right)\right]$$

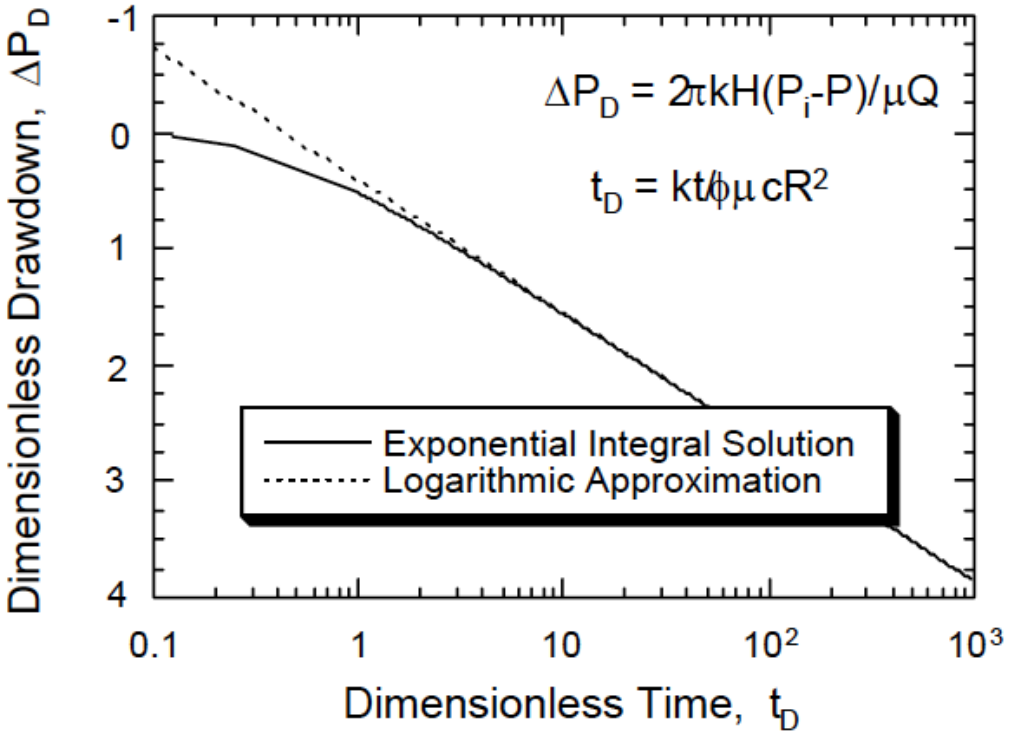
$$= P_i - \frac{\mu Q}{4\pi kH}\left[\ln\left(\frac{kt}{\phi\mu c R^2}\right) + 0.8091\right]$$

Logarithmic Approximation to the Line-Source Solution

In dimensionless form, the logarithmic approximation takes the following form:

$$\Delta P_D = \frac{1}{2}\left[\ln(t_D) + 0.80907)\right]$$

Since x = $\phi\mu cR^2/4kt$ = $1/4t_D$, and the logarithmic approximation is valid for $x < 0.01$, we see that this approximation is highly accurate for $t_D > 25$, as seen in the graph below:



Estimating Permeability from a Drawdown Test

A common use of the line-source solution, and the other solutions that we will derive in this module, is to solve the "inverse" problem: we measure the flowrate and drawdown in the well, and fit the data to our analytical solution, so as to estimate reservoir properties such as permeability, *etc*.

This process is called *well-test analysis*. We will now do one simple example, to get a feeling for how this process works.

Using the line-source solution, the pressure at the well can be written as

$$P(R_w,t) = P_i - \frac{\mu Q}{4\pi kH}\left[\ln\left(\frac{4kt}{\phi\mu c R_w^2 \gamma}\right)\right] = P_i - \frac{\mu Q}{4\pi kH}\left[\ln t + \ln\left(\frac{4k}{\phi\mu c R_w^2 \gamma}\right)\right]$$

The second logarithmic term, although its numerical value is *a priori* unknown, is a *constant*. Hence, if we plot $P_w(t)$ *vs*. ln$t$, the data will eventually, at large enough values of t, fall on a straight line! The late-time slope of this semi-log straight line, which we call m, will have the following value:

$$\left|\frac{dP_w}{d\ln t}\right| = \left|\frac{\Delta P_w}{\Delta \ln t}\right| \equiv m = \frac{\mu Q}{4\pi kH} \quad \rightarrow \quad k = \frac{\mu Q}{4\pi Hm}$$

If we find the slope m graphically, and if we measure the flowrate Q, and the fluid viscosity μ, and if we know the reservoir thickness H, we can estimate the permeability k.

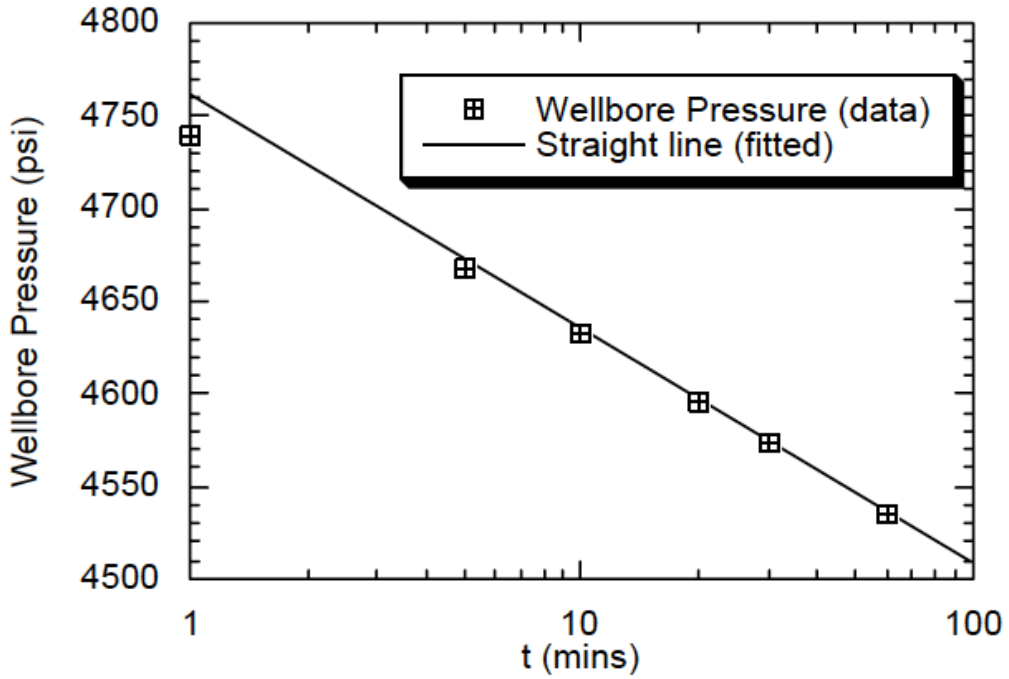
Estimating Permeability from a Drawdown Test

Example: A well with 4 in. radius produces oil with viscosity 0.3 cP, at a constant rate of 200 barrels/day, from a reservoir that is 15 ft. thick. The wellbore pressure as a function of time is measured as follows:

t (mins)	1	5	10	20	30	60
P_w (psi)	4740	4667	4633	4596	4573	4535

Use the "semi-log straight line" method to estimate the permeability, k.

(1) We plot the wellbore pressure against the logarithm of time, and fit the late-time data to a straight line:



Estimating Permeability from a Drawdown Test

(2) Calculate the slope of the late-time straight line:

$$m = \left|\frac{\Delta P}{\Delta \ln t}\right| = \frac{4760\,\text{psi} - 4510\,\text{psi}}{2 \times 2.303} = 54.3\,\text{psi} \times \frac{6895\ \text{Pa}}{\text{psi}} = 374{,}400\,\text{Pa}$$

(Note that $\Delta \ln t$ will have the same value, *regardless* of the units we use for t!)

(3) Calculate k from $k = \mu Q/4\pi Hm$, after converting all data to SI units:

$$\mu = 0.3\ \text{cP} \times \frac{0.001\ \text{Pa}\cdot\text{s}}{\text{cP}} = 0.0003\ \text{Pa}\cdot\text{s}$$

$$Q = 200\,\frac{\text{bbl}}{\text{day}} \times \frac{0.1589\ \text{m}^3}{\text{bbl}} \times \frac{\text{day}}{24\,\text{hr}} \times \frac{\text{hr}}{3600\,\text{s}} = 3.68 \times 10^{-4}\,\frac{\text{m}^3}{\text{s}}$$

$$H = 15\,\text{ft} \times \frac{0.3048\ \text{m}}{\text{ft}} = 4.572\ \text{m}$$

$$\rightarrow k = \frac{\mu Q}{4\pi mH} = \frac{(0.0003\,\text{Pa s})(3.68 \times 10^{-4}\,\text{m}^3/\text{s})}{4\pi(4.572\,\text{m})(374{,}400\,\text{Pa})}$$

$$= 5.13 \times 10^{-15}\,\text{m}^2 = 5.1\,\text{mD}$$

Problems for Lecture 2

Problem 2.1. A well with 3 in. radius is located in a 40 ft. thick reservoir that has a permeability of 30 mD and a porosity of 0.20. The total compressibility of the oil/rock system is 3×10^{-5}/psi. The initial pressure in the reservoir is 2800 psi. The well produces 448 barrels/day of oil that has a viscosity of 0.4 cP. Conversion factors can be found in Problem 1.1.

(a) What is the pressure at the wellbore after six days of production, according to the line-source solution?

(b) How long will it take in order for Jacob's logarithmic approximation to be valid at the wellbore?

(c) What is the pressure at the wellbore after six days of production, according to the logarithmic approximation?

(d) Answer questions (a)-(c) for a location that is 800 ft. (horizontally) away from the wellbore.

Problems for Lecture 2

Problem 2.2. A well with a radius of 0.3 ft. produces 200 barrels/day of oil, with viscosity 0.6 cP, from a 20 ft. thick reservoir. The wellbore pressures are as follows:

t (mins)	0	5	10	20	60
P_w (psi)	4000	3943	3938	3933	3926

t (mins)	120	480	1440	2880	5760
P_w (psi)	3921	3911	3904	3899	3894

Estimate the permeability, using the semi-log method described earlier in this lecture.

Then, estimate the storativity (ϕc) of the reservoir, using the method discussed in the *FFiPM* textbook.

Imperial College MSc in
GeoEnergy with Machine Learning and Data Science

GEMS 3: Geomechanics and Pressure Transient Analysis

Lecture 3:
Superposition in Time and Space, Convolution, and Image Wells

Prof. Robert Zimmerman

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Linearity and the principle of superposition

A basic property of the pressure diffusion equation that governs flow of a single-phase compressible liquid through a porous medium is its *linearity*. Linearity allows us to use the principle of *superposition* to construct solutions to the equation. Most of the analytical methods that are used to solve differential equations, such as Laplace transforms, Green's functions, separation of variables, *etc.*, can be used *only* on linear differential equations. These analytical methods will be discussed to in later lectures in this module.

In this lecture, we will discuss a simple form of the principle of superposition that will allow us to solve many important reservoir engineering problems, such as pressure buildup tests, or multi-rate flow tests.

During well tests, the well may be flowed for a period of time, and then "shut in", after which the well pressure rebounds towards the initial reservoir pressure. Or, a well may be flowed at a sequence of different rates, to obtain data that can be used to determine various reservoir properties. These sequences result in complex pressure signals. In this lecture, we will develop methods for modelling these complex pressure signals, which will help us to infer the values of important reservoir properties, such as permeability and storativity.

Linearity and the principle of superposition

A differential operator, which we will call "*M*", that operates on a function *F*, is a *linear* operator if it has the following two properties:

$$M(F_1 + F_2) = M(F_1) + M(F_2) \qquad \text{(a)}$$

$$M(aF_1) = aM(F_1) \qquad \text{(b)}$$

for any two functions, F_1 and F_2, and any constant, a.

The process of partial differentiation is a linear operation, since

$$\frac{d}{dt}\left[P_1(R,t) + P_2(R,t)\right] = \frac{dP_1(R,t)}{dt} + \frac{dP_2(R,t)}{dt}$$

$$\frac{d}{dt}\left[aP_1(R,t)\right] = a\frac{dP_1(R,t)}{dt}$$

To test this property in a concrete situation, recall the pressure diffusion equation from Lecture 1:

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{dP}{dR}\right) = \frac{\phi\mu c}{k}\frac{dP}{dt}$$

Linearity and the principle of superposition

Let's check to see if the term on the left is "linear". First, consider property (a):

$$M(P_1 + P_2) = \frac{1}{R}\frac{d}{dR}\left[R\frac{d(P_1 + P_2)}{dR}\right] = \frac{1}{R}\frac{d}{dR}\left(R\frac{dP_1}{dR} + R\frac{dP_2}{dR}\right)$$

$$= \frac{1}{R}\frac{d}{dR}\left(R\frac{dP_1}{dR}\right) + \frac{1}{R}\frac{d}{dR}\left(R\frac{dP_2}{dR}\right) = M(P_1) + M(P_2)$$

Now check property (b):

$$M(aP_1) = \frac{1}{R}\frac{d}{dR}\left[R\frac{d(aP_1)}{dR}\right] = \frac{1}{R}\frac{d}{dR}\left[aR\frac{dP_1}{dR}\right] = \frac{a}{R}\frac{d}{dR}\left(R\frac{dP_1}{dR}\right) = aM(P_1)$$

Similarly, we can easily verify that, if the parameters $\{\phi,\mu,c,k\}$ are each constant, then the right-hand side of the pressure diffusion equation is also linear. So, the "standard" PDE is linear.

I leave it to you to convince yourself that if any of the parameters $\{\phi,\mu,c,k\}$ vary with R in some known way (*i.e.*, a "heterogeneous reservoir"), then the equation is still linear.

Linearity and the principle of superposition

However, if any of the coefficients were functions of *pressure*, the equation would no longer be linear. This situation occurs, for example, with gas flow, for which the compressibility varies strongly with pressure. It is also the case for "stress-sensitive" reservoirs, in which the permeability varies with pressure.

As a simple example, let's imagine that the permeability varies with pore pressure, according to $k(P) = k_o(1+\alpha P)$. In this case, the RHS of the PDE becomes *nonlinear*. The easiest way to see this is to consider property (b):

$$M(aP) = \frac{\phi\mu c_t}{k(aP)}\frac{d(aP)}{dt} = \frac{a\phi\mu c_t}{k_o(1+\alpha aP)}\frac{dP}{dt} \neq \frac{a\phi\mu c_t}{k_o(1+\alpha P)}\frac{dP}{dt} = aM(P)$$

A simple rule-of-thumb is that a differential equation will be nonlinear if it contains any term in which the dependent variable (in our case, P), or any of its derivatives, appear to a power higher than one, or are multiplied by one another.

Linearity and the principle of superposition

The importance of linearity is that it allows us to create new solutions to the diffusion equation by adding together previously known solutions. We must be careful, however, with the initial conditions and boundary conditions.

For example, if P_1 and P_2 are two pressure functions that each satisfy the diffusion equation and the initial condition $P(R,t = 0) = P_i$, then the sum of P_1 and P_2 will also satisfy the diffusion equation, but will *not* satisfy the initial condition, because

$$P_1(R,t = 0) + P_2(R,t = 0) = P_i + P_i = 2P_i$$

This difficulty can be avoided by working with the *drawdown*, defined by $\Delta P(R,t) = P_i - P(R,t)$, instead of working with the pressure itself. Since the drawdown satisfies zero initial condition, by definition, the sum of two drawdown functions will also satisfy the correct initial condition.

Similarly, the drawdown must be zero infinitely far from the well. So, if two drawdown functions satisfy the zero boundary condition at infinity, their sum will also satisfy this far-field boundary condition.

Pressure Buildup Tests in an Infinite Reservoir

In a *pressure buildup test*, a well that has been producing fluid at a constant rate Q for some time t is then "shut in" – *i.e.,* production is stopped**.**

After this occurs, fluid will continue to flow towards the well, due to the pressure gradient in the reservoir, but will not be able to exit at the wellhead. Consequently, the pressure at the well will rise back towards it initial value, P_i. The rate of this pressure recovery at the well can be used to estimate both the transmissivity, kH, and the initial pressure, P_i, of the reservoir.

The analysis of a pressure buildup test is based on the *principle of superposition*, and proceeds as follows.

First, imagine that we produce at a rate Q_0, starting at $t = 0$, in which case the pressure drawdown *due to this production* will be

$$\Delta P_1 = P_i - P_1(R,t) = -\frac{\mu Q_0}{4\pi kH} Ei\left(\frac{-\phi\mu c R^2}{4kt}\right)$$

Now consider the following fictitious problem, in which, at some time t_1, we begin to *inject* fluid into the same well, at rate Q_0. The pressure drawdown due to this injection would be given by the same line source solution, *except* that:

Pressure Buildup Tests in an Infinite Reservoir

(a) The variable that we use in the line source solution to represent the "elapsed time" must be measured from the start of injection, *i.e.,* the variable must be $t - t_1$.

(b) Since we are now "injecting" rather than producing, we must use "$-Q_0$" in this second solution.

Therefore, the pressure drawdown due to this fictitious injection is

$$\Delta P_2 = P_i - P_2(R,t) = \frac{\mu Q_0}{4\pi kH} Ei\left[\frac{-\phi\mu c R^2}{4k(t-t_1)}\right]$$

Note: it is implicitly understood that the value of the *Ei* function is taken to be zero when the term in brackets is positive, which is to say, when $t < t_1$. This corresponds to the fact that production (or injection) starting at time t_1 cannot possibly cause any drawdown for times $t < t_1$!

We now superimpose these two solutions (for the drawdown, not the pressure itself!), putting $\Delta P = \Delta P_1 + \Delta P_2$:

$$\Delta P(R,t) = -\frac{\mu Q_0}{4\pi kH}\left[Ei\left(\frac{-\phi\mu c R^2}{4kt}\right) - Ei\left(\frac{-\phi\mu c R^2}{4k(t-t_1)}\right)\right]$$

Pressure Buildup Tests in an Infinite Reservoir

The key idea behind this use of "superposition" is that if the actual flowrate is decomposed into the superposition of two different problems with two different flowrates, then the pressure drawdown in the well will be the superposition of the drawdowns that correspond to these two different problems.

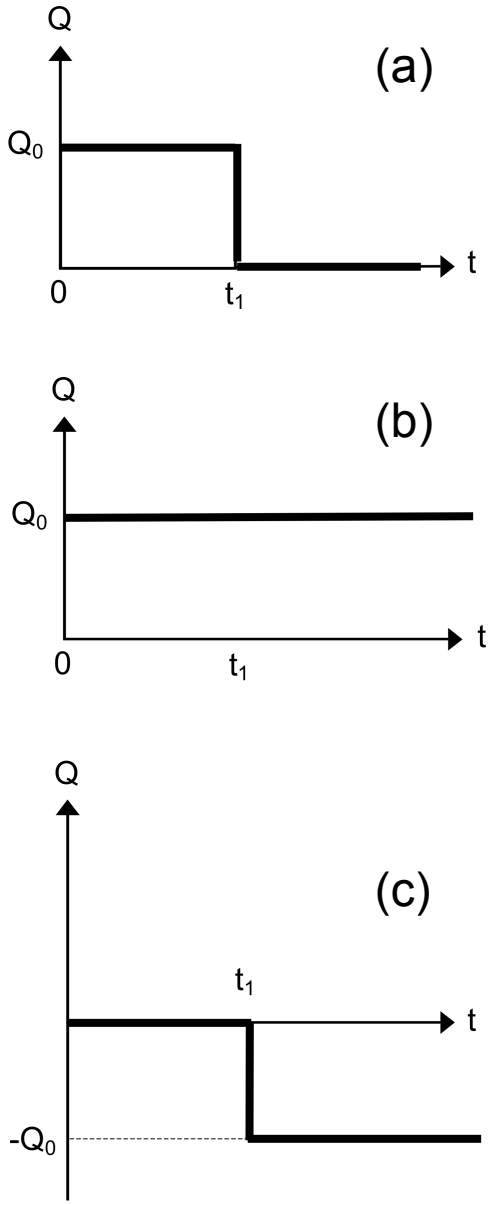
The actual flowrate is shown in the top figure on the right.

We decompose flowrate (a) into the sum of flowrates (b) and (c), *i.e.*, (a) = (b) + (c)

For $0 < t < t_1$, $Q(a) = Q(b) + Q(c) = Q_0 + 0 = Q_0$

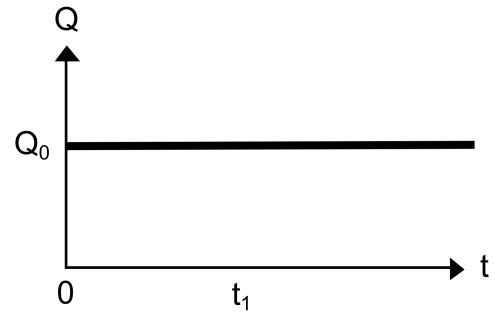
For $t > t_1$, $Q(a) = Q(b) + Q(c) = Q_0 + (-Q_0) = 0$

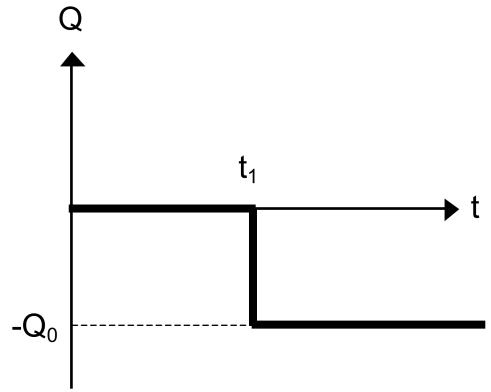
The reason for doing this is that, whereas it would be (very) hard to solve the pressure diffusion equation for case (a), we already know how to solve cases (b) and (c)!

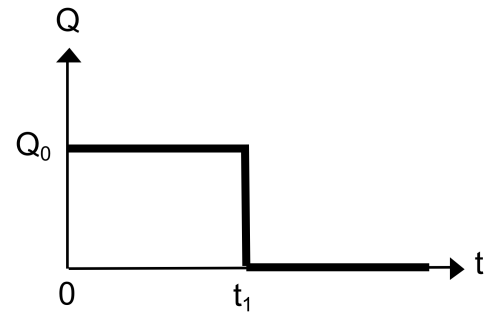


Pressure Buildup Tests in an Infinite Reservoir

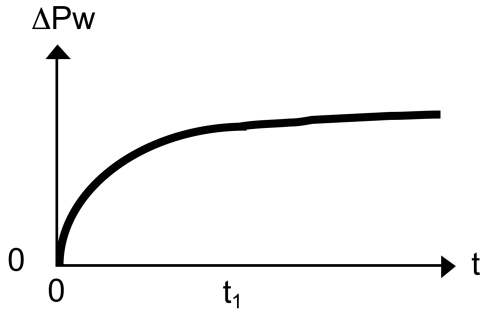
The production rate at the well for the three scenarios:

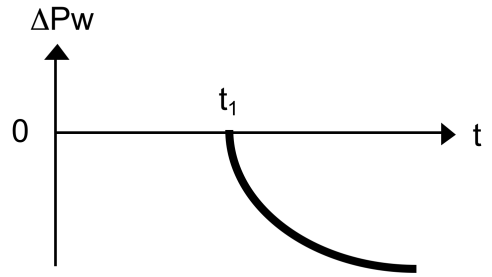


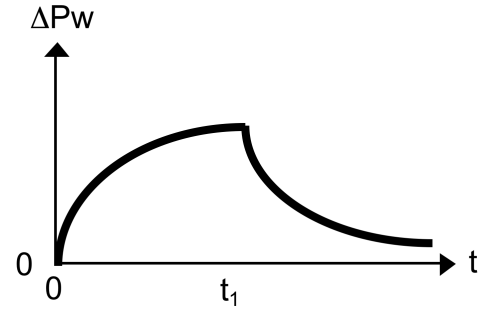




The pressure at the well for the three scenarios:







Multi-Rate Flow Tests

The superposition principle that was used to solve the problem of a buildup test can also be used in the more general situation in which the production rate is changed by discrete amounts at various time intervals.

First, imagine that the production rate is given by

$$Q = Q_0 \quad \text{for} \quad 0 < t < t_1$$

$$Q = Q_1 \quad \text{for} \quad t > t_1$$

To find the drawdown, we superpose the solution for production at rate Q_0 starting at time t = 0, plus a solution starting at t_1 that corresponds to the *increment* in the production rate, Q_1–Q_0:

$$\Delta P(R,t) = -\frac{\mu Q_0}{4\pi kH} Ei\left(\frac{-\phi\mu cR^2}{4kt}\right) - \frac{\mu(Q_1 - Q_0)}{4\pi kH} Ei\left(\frac{-\phi\mu cR^2}{4k(t-t_1)}\right)$$

To verify that it is correct to use the flowrate *increment* in the second *Ei* function, note that for $t > t_1$, the first *Ei* function corresponds to a flowrate of Q_0, and the second corresponds to a rate of Q_1–Q_0, so the total flowrate is

$$Q(t > t_1) = Q_0 + (Q_1 - Q_0) = Q_0 + Q_1 - Q_0 = Q_1$$

Multi-Rate Flow Tests

We can use this same idea of superposition to find the drawdown in the general case, in which flowrate Q_i starts at time t_i, *etc*., by adding up a sequence of additional line source solutions:

$$\Delta P(R,t) = \frac{-\mu Q_0}{4\pi kH} Ei\left(\frac{-\phi\mu cR^2}{4kt}\right) - \sum_{i=1} \frac{\mu(Q_i - Q_{i-1})}{4\pi kH} Ei\left(\frac{-\phi\mu cR^2}{4k(t-t_i)}\right).$$

We can simplify the notation by defining the pressure drawdown *per unit of flowrate*, with production starting at $t = 0$, as $\Delta P_Q(t)$.

For the line source in an infinite reservoir, this definition implies

$$\Delta P_Q(R,t) \equiv \frac{\Delta P(R,t;Q)}{Q} \equiv \frac{-\mu}{4\pi kH} Ei\left(\frac{-\phi\mu cR^2}{4kt}\right)$$

with the understanding that $\Delta P_Q(t) = 0$ when $t < 0$.

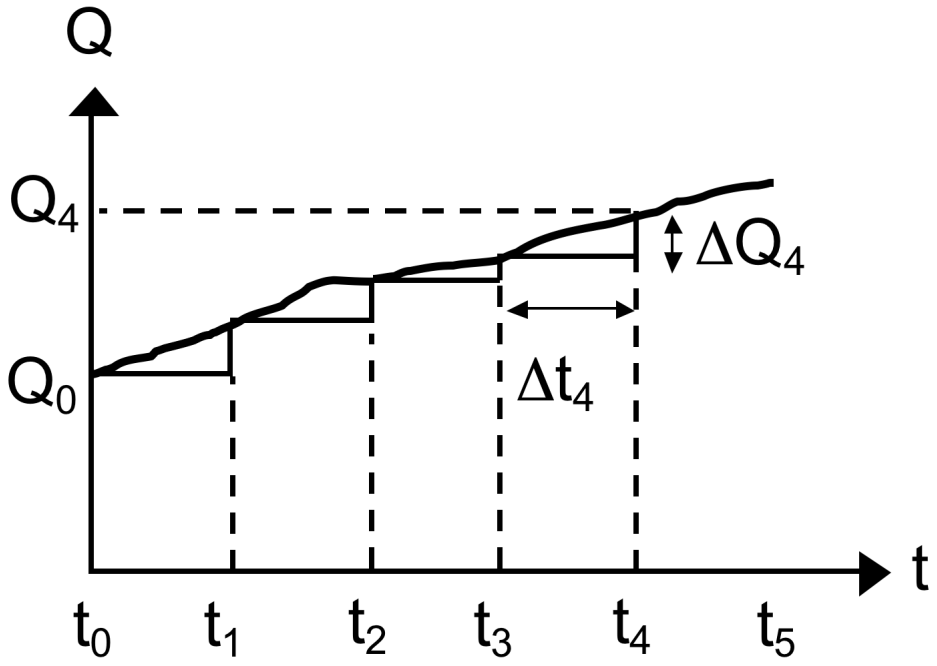
Using this definition, the drawdown in a multi-rate test can be written as

$$\Delta P(R,t) = Q_0 \Delta P_Q(R,t) + \sum_{i=1} (Q_i - Q_{i-1}) \Delta P_Q(R, t-t_i)$$

Convolution Integral for Variable-Rate Flow Tests

The superposition formula can be generalised further, to the case where the flowrate at the well varies according to some arbitrary function of time, $Q(t)$.

First note that an arbitrary production schedule can always be approximated by a discrete number of time periods during which the flowrate is constant, as shown in the figure below. The actual flowrate is given by the thick, irregular curve. The approximate flowrate is given by the "staircase" function.



Convolution Integral for Variable-Rate Flow Tests

Note that the time derivative of the flowrate, at time t_i, can be approximated as follows:

$$\left.\frac{dQ}{dt}\right|_{t_i} \approx \frac{(\Delta Q)_i}{(\Delta t)_i} = \frac{Q_i - Q_{i\text{-}1}}{t_i - t_{i\text{-}1}}$$

We can rearrange this equation to find an expression for $Q_i - Q_{i\text{-}1}$:

$$Q_i - Q_{i-1} \approx \frac{dQ(t_i)}{dt}(t_i - t_{i-1})$$

If we plug this expression into our equation for the drawdown, we find

$$\Delta P(R,t) = Q_0 \Delta P_Q(R,t) + \sum_{i=1} \frac{dQ(t_i)}{dt} \Delta P_Q(R, t - t_i)(t_i - t_{i-1})$$

We now simplify the notation by rewriting this in the following equivalent form:

$$\Delta P(R,t) = Q_0 \Delta P_Q(R,t) + \sum_{i=1} Q'(t_i) \Delta P_Q(R, t - t_i) \Delta t_i$$

As we make each time increment smaller, the above approximation for the flowrate increment $Q_i - Q_{i\text{-}1}$ becomes more accurate, and the "step-function" approximation to $Q(t)$ also becomes more accurate. In the limit as each time increment goes to zero, the errors due to these approximations will vanish.

Convolution Integral for Variable-Rate Flow Tests

The summation in this equation looks complicated, but it actually represents something simple and familiar.

Recall from basic calculus that if we have a function $f(t)$, the integral of f, from 0 to t, is *defined* as the limit, as each Δt goes to zero, of the following sum:

$$\int_0^t f(\tau)d\tau \equiv \lim_{\Delta t \to 0} \sum_{i=1}^{N} f(t_i)\Delta t_i$$

So, we see that as all of the Δt increments go to zero, the summation on the previous slide becomes an integral, and we arrive at

$$\Delta P(R,t) = Q_0 \Delta P_Q(R,t) + \int_{t_i=0}^{t_i=t} Q'(t_i)\,\Delta P_Q(R, t - t_i)dt_i$$

Finally, we note that in the limit, as each of the time increments become infinitely small, the finite number of times that we were denoting by t_i evolve into a continuous variable, which we will denote by τ. The drawdown can then be written as

$$\Delta P(R,t) = Q_0 \Delta P_Q(R,t) + \int_0^t \frac{dQ(\tau)}{d\tau} \Delta P_Q(R, t - \tau)d\tau$$

Convolution Integral for Variable-Rate Flow Tests

This integral is known as a *convolution integral*, and specifically represents "the convolution of the two functions dQ/dt and ΔP_Q".

The importance of the convolution integral is that it allows us to find the drawdown for any production schedule, by merely doing a single integral based on the drawdown for the "constant-flowrate" case!

This implies that, for any reservoir geometry, we only need to solve the problem of constant flowrate; the drawdown for any other situation can be found by convolution. This idea is a cornerstone of well test analysis.

Another form of the convolution integral that is sometimes more convenient to use can be derived by applying integration-by-parts to the integral on the previous slide (see textbook for details). The result is:

$$\Delta P(R,t) = \int_0^t Q(\tau)\frac{d\Delta P_Q(R,t-\tau)}{dt}d\tau.$$

The short-hand notation for convolution, which is used often in Chapter 7 of the textbook, is the "star" symbol, *i.e.*,

$$\Delta P(R,t) = \int_0^t Q(\tau)\frac{d\Delta P_Q(R,t-\tau)}{dt}d\tau \equiv Q(t) * \frac{d\Delta P_Q(t)}{dt}$$

Convolution Integral for Variable-Rate Flow Tests

In Problem 3.2, you are asked to use convolution to solve an important, and realistic (but difficult!) problem. To gain some experience with doing convolution integrals, let's now do a (relatively) easy problem:

What is the convolution of the two functions $f(t) = t$, and $g(t) = \exp(-t)$?

Solution: First, let's write the convolution integral in the following general form, involving two arbitrary functions, f and g:

$$f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau.$$

With $f(t) = t$, and $g(t) = \exp(-t)$, this integral takes the form:

$$f(t) * g(t) = \int_0^t \tau \exp[-(t-\tau)]\, d\tau.$$

First, note that $\exp[-(t-\tau)] = \exp(-t)\exp(\tau)$. Since the integration variable is τ (*not* t), the variable t acts like a *constant* inside the integral, and so $\exp(-t)$ can be taken *outside* of the integral:

$$f(t) * g(t) = \exp(-t) \int_0^t \tau \exp(\tau)\, d\tau.$$

Convolution Integral for Variable-Rate Flow Tests

To go further, we need to use “integration by parts”. (This trick/method is almost always needed when we try to do a convolution integral).

Recall the formula for integration by parts:

$$\int_0^t u(\tau)v'(\tau)d\tau = u(\tau)v(\tau)\Big]_0^t - \int_0^t u'(\tau)v(\tau)d\tau$$

It may look like we are going around in a circle, since we still have an integral on the right side. But the usefulness of integration by parts is that often we can cleverly pick our *u* and *v* functions so that, although the integral on the left is too hard to evaluate, the one on the right is easy.

In our case, the left hand side is the integral of $\tau\exp(\tau)$. So, let’s identify τ with $u(\tau)$, and let’s identify $\exp(\tau)$ with $v'(\tau)$. In this case:

$$u'(\tau) = d\tau/d\tau = 1, \quad \text{and} \quad v(\tau) = \exp(\tau).$$

If we plug this into the above expression, we find:

$$\int_0^t \tau\exp(\tau)d\tau = \tau\exp(\tau)\Big]_0^t - \int_0^t \exp(\tau)d\tau$$

Convolution Integral for Variable-Rate Flow Tests

Notice that the integral on the right is easy! So, we thereby find that

$$\int_0^t \tau \exp(\tau) d\tau = \tau \exp(\tau)\Big]_0^t - \int_0^t \exp(\tau) d\tau$$

$$= t\exp(t) - 0\exp(0) - \exp(\tau)\Big]_0^t$$

$$= t\exp(t) - \exp(t) + 1$$

Finally, we see that

$$f(t) * g(t) = \exp(-t)\int_0^t \tau \exp(\tau)\, d\tau = e^{-t}(te^{t} - e^{t} + 1) = t - 1 + e^{-t}$$

Note 1: We could also have tried to identify τ with $v'(\tau)$, and $\exp(\tau)$ with $u(\tau)$ But if we tried this, we would have arrived at an integral on the right that is even harder to do than the one that we started with on the left. If we had tried that, there was no harm done; you would then just try the other choice.

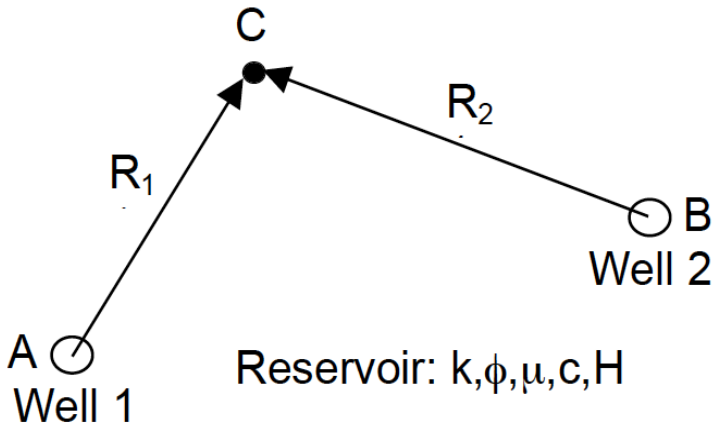
Note 2: Integration by parts doesn't always work, because sometimes the integral on the right is still too hard to evaluate. But it works in a surprising number of cases, and you should always try to use it when confronted with a difficult integral.

Superposition of Sources/Sinks in Space, 1

We can also use superposition *in space* to solve many interesting problems in reservoir engineering.

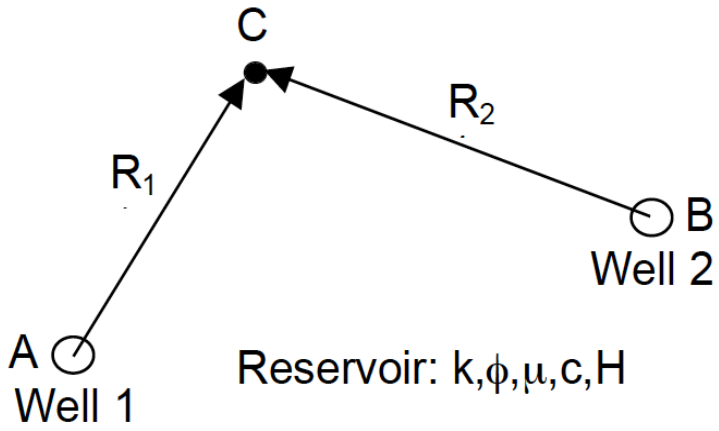
This method is discussed in detail in Chapter 4 of the *FFiPM* textbook. In this lecture, we will introduce the method, and use it to solve the problem of a vertical well in a reservoir that contains an infinite vertical fault.

Consider first two wells located in an infinite reservoir, as shown below (top view). Well 1 is located at point A, and Well 2 is located at point B. It may be convenient to imagine an observation well located at point C, fitted with a pressure gauge. Well 1 starts producing at rate Q_1 at time t_1, and well 2 starts producing at rate Q_2, starting at time t_2.



Superposition of Sources/Sinks in Space, 2

As proven in detail in the textbook, the drawdown at point C in the reservoir, at some time *t*, is given by the sum of the two relevant line source solutions:



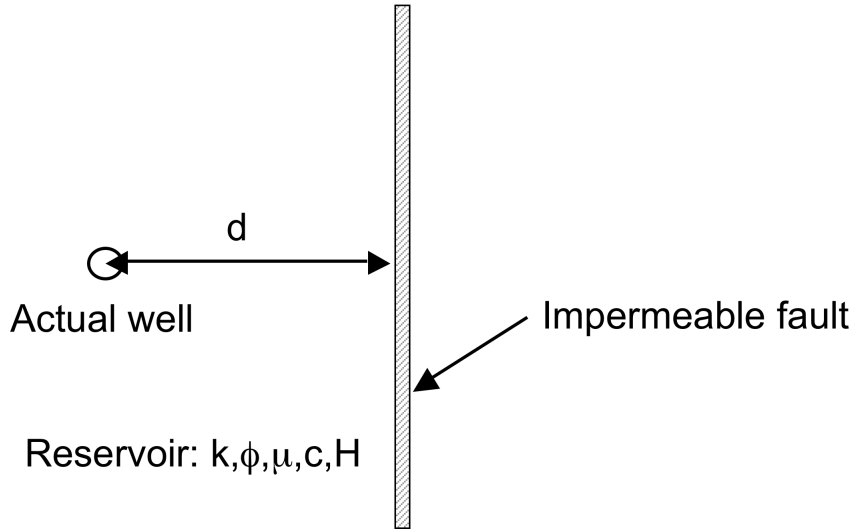
$$P(R,t) = P_i + \frac{\mu Q_1}{4\pi kH} Ei\left(\frac{-\phi\mu c R_1^2}{4k(t-t_1)}\right) + \frac{\mu Q_2}{4\pi kH} Ei\left(\frac{-\phi\mu c R_2^2}{4k(t-t_2)}\right)$$

Recall that in the basic line source solution, the variable *R* always refers to the distance *from* the well *to* the point where we want to measure or know the pressure. The variable *t* always refers to the time that has elapsed *from* the start of production *to* the time at which we want to measure or know the pressure. If you remember this, it should be easy to write down the above equation.

Effect of an Impermeable Vertical Fault, 1

Reservoirs are often transected by faults, many of which are nearly vertical. Due to mineral deposition on the fault surfaces, accumulation of fault gouge, and other geological processes, faults are often *impermeable* to flow. To properly interpret the results of well tests, it is important to understand the effect that an impermeable boundary will have on a drawdown test.

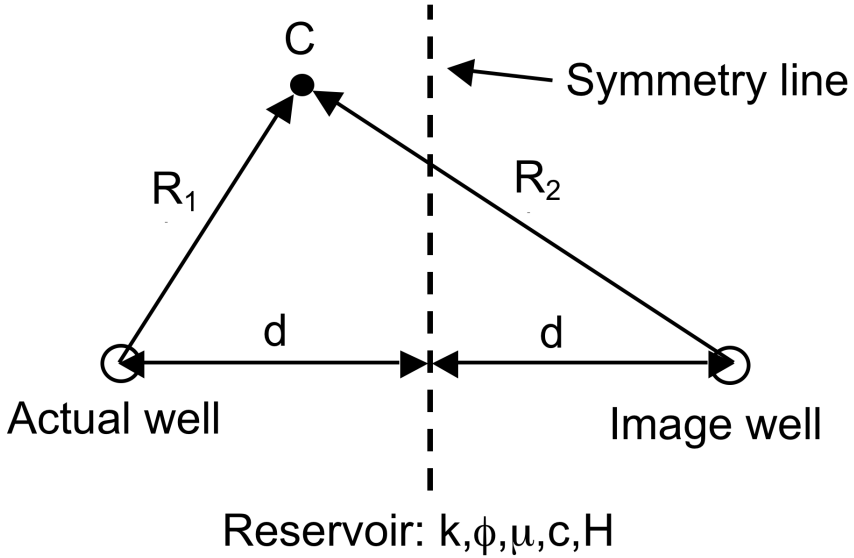
Consider a well located at a perpendicular distance (*i.e.,* nearest distance) *d* from an impermeable fault, which appears in planview as a straight line that extends infinitely far in both directions, as below. This well produces fluid at a constant rate *Q*, starting at $t = 0$.



Effect of an Impermeable Vertical Fault, 2

Next, let's *ignore* the impermeable boundary, and assume that the reservoir extends infinitely far in all horizontal directions. But we will replace the fault by a dotted line, so as to remind us of where the fault is actually located.

Now imagine a fictitious "image well" that is situated as the "mirror image" of the first well (*i.e.,* located at a distance *d* on the other side of this dotted line), which also produces fluid from the reservoir, at rate *Q*, starting at $t = 0$.

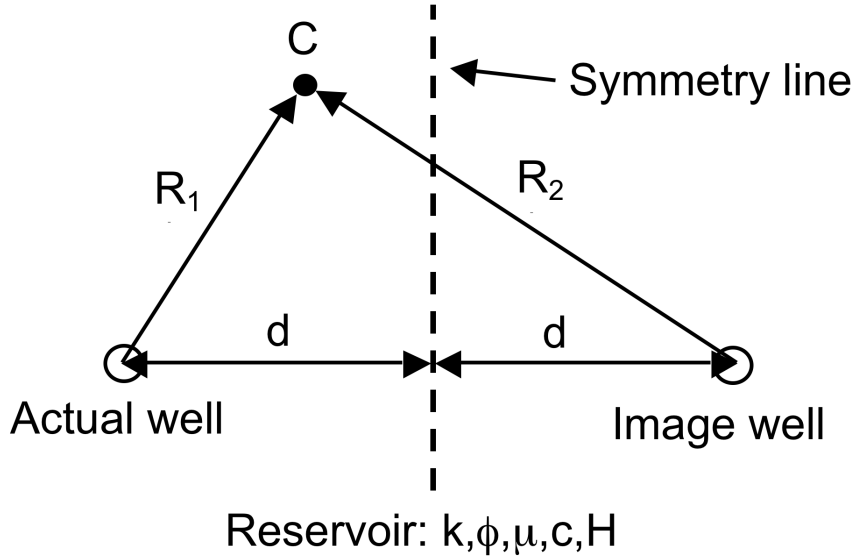


Any fluid to the left of this dotted line will flow towards the actual well, whereas any fluid to the right of this line will flow towards the image well. Hence, no fluid will cross this dotted line, and it will effectively act as a *no-flow boundary.* Hence, this scenario gives us the solution for a well near an impermeable fault!

Effect of an Impermeable Vertical Fault, 3

The pressure drawdown in this situation is the superposition of the drawdown due to the *actual* well, plus the drawdown due to the fictitious *image* well:

$$P(R,t) = P_i + \frac{\mu Q}{4\pi kH} Ei\left(\frac{-\phi\mu c R_1^2}{4kt}\right) + \frac{\mu Q}{4\pi kH} Ei\left(\frac{-\phi\mu c R_2^2}{4kt}\right)$$



To find the pressure at the wellbore, we put (see figure above, and imagine that point C is located at the right edge of the actual borehole) $R_1 = R_w$, and $R_2 = 2d - R_w \approx 2d$:

$$P_w(t) = P_i + \frac{\mu Q}{4\pi kH} Ei\left(\frac{-\phi\mu c R_w^2}{4kt}\right) + \frac{\mu Q}{4\pi kH} Ei\left(\frac{-\phi\mu c (2d)^2}{4kt}\right)$$

Effect of an Impermeable Vertical Fault, 4

We now consider the various time regimes that occur in this problem:

(i) There is an early time regime, during which the logarithmic approximation is not yet valid for either the actual well solution, or for the image-well solution. From Lecture 2, we know that this regime is defined by

$$t < \frac{25\phi\mu c R_w^2}{k}, \quad \text{or} \quad t_{Dw} < 25$$

However, we saw in Lecture 2 that the duration of this regime is usually very short, and so there is no need to study it further.

(ii) There is an intermediate time regime, in which the logarithmic approximation can be used for the well solution, but the drawdown in the wellbore *due to the image solution* is still negligible. This regime starts when $t_{Dw} = 25$.

During this regime, the slope of the wellbore pressure curve *vs.* time, on a semi-log plot, will be

$$\frac{dP_w}{d\ln t} = \frac{\Delta P_w}{\Delta \ln t} = \frac{-\mu Q}{4\pi kH}$$

i.e., the same as if there were no fault!

Effect of an Impermeable Vertical Fault, 5

(iii) There is a late-time regime during which the logarithmic approximation is valid for *both* the actual well solution, *and* the image well solution.

In this regime, the pressure at the actual well is given by

$$P_w(t) = P_i - \frac{\mu Q}{4\pi kH}\ln\left(\frac{2.246kt}{\phi\mu cR_w^2}\right) - \frac{\mu Q}{4\pi kH}\ln\left(\frac{2.246kt}{\phi\mu c(2d)^2}\right)$$

$$= P_i - \frac{\mu Q}{4\pi kH}\left[\ln t + \ln\left(\frac{2.246k}{\phi\mu cR_w^2}\right) + \ln t + \ln\left(\frac{2.246k}{\phi\mu c4d^2}\right)\right]$$

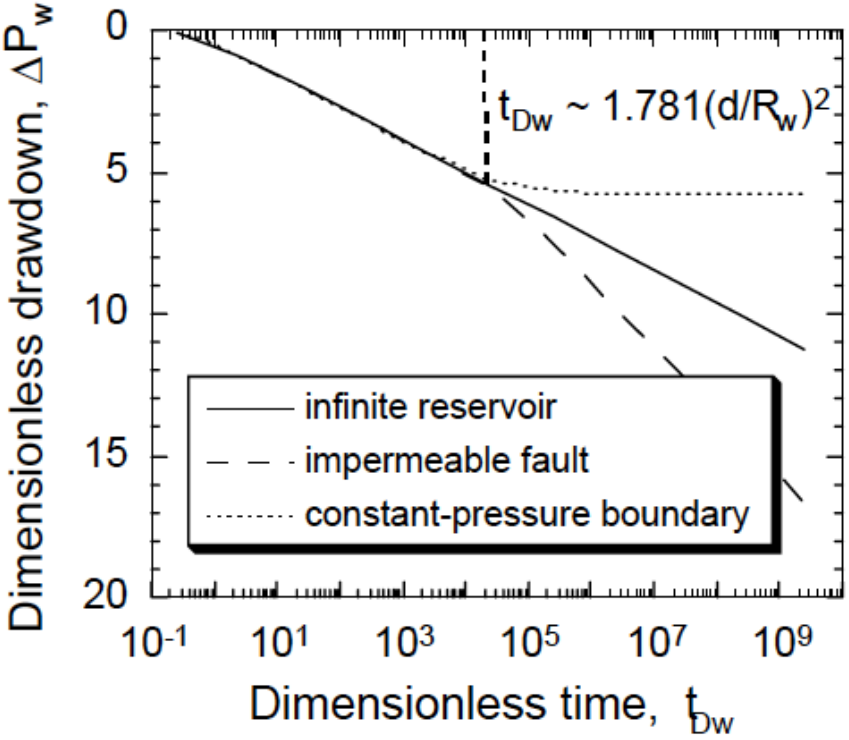
$$= P_i - \frac{\mu Q}{4\pi kH}\left[2\ln t + \ln\left(\frac{2.246k}{\phi\mu cR_w^2}\right) + \ln\left(\frac{2.246k}{\phi\mu c4d^2}\right)\right]$$

This equation will also yield a straight line on a plot of P_w vs. ln$t$, but with a slope that is *twice* that of the earlier slope, *i.e.*,

$$\frac{dP_w}{d\ln t} = \frac{\Delta P_w}{\Delta \ln t} = \frac{-2\mu Q}{4\pi kH} = \frac{-\mu Q}{2\pi kH}$$

Effect of an Impermeable Vertical Fault, 6

So, if we see from the measured data that the slope of the graph of P_w vs. ln$t$ *doubles*, this is an indication of a nearby impermeable vertical boundary.



Note: the distance from the production well to the fault can be estimated from the intersection time of the two semi-log straight lines (see Problem 4.1 in *FFiPM* textbook).

Physical explanation of the slope-doubling: Because of the impermeable boundary, oil is being produced from a “semi-infinite” reservoir, rather than an infinite reservoir. So, at large times, only half as much oil will be produced, for a given wellbore pressure. Therefore, in order to maintain a constant flowrate, the drawdown must be doubled!

Problems for Lecture 3

Problem 3.1. Which, if any, of the following differential equations are linear, and why (or why not)?

(a) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = 0$ (b) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ (c) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$

Problem 3.2. Find an expression for the wellbore pressure in a vertical well in a laterally infinite reservoir, if the production rate increases linearly as a function of time, according to $Q(t) = Q^*t/t^*$, where Q^* and t^* are constants. Use convolution, in either of the following two forms derived earlier:

$$\Delta P(R,t) = Q_0 \Delta P_Q(R,t) + \int_0^t \frac{dQ(\tau)}{d\tau} \Delta P_Q(R, t-\tau) d\tau$$

$$\Delta P(R,t) = \int_0^t Q(\tau) \frac{d\Delta P_Q(R, t-\tau)}{dt} d\tau.$$

and recall that $\Delta P_Q(R,t)$ for a well in an infinite reservoir is given by

$$\Delta P_Q(R,t) \equiv \frac{\Delta P(R,t;Q)}{Q} \equiv \frac{-\mu}{4\pi kH} Ei\left(\frac{-\phi\mu c R^2}{4kt}\right)$$

Imperial College MSc in
GeoEnergy with Machine Learning and Data Science

GEMS 3: Geomechanics and Pressure Transient Analysis

Lecture 4:
Production from, or Injection into, Bounded Reservoirs

Prof. Robert Zimmerman

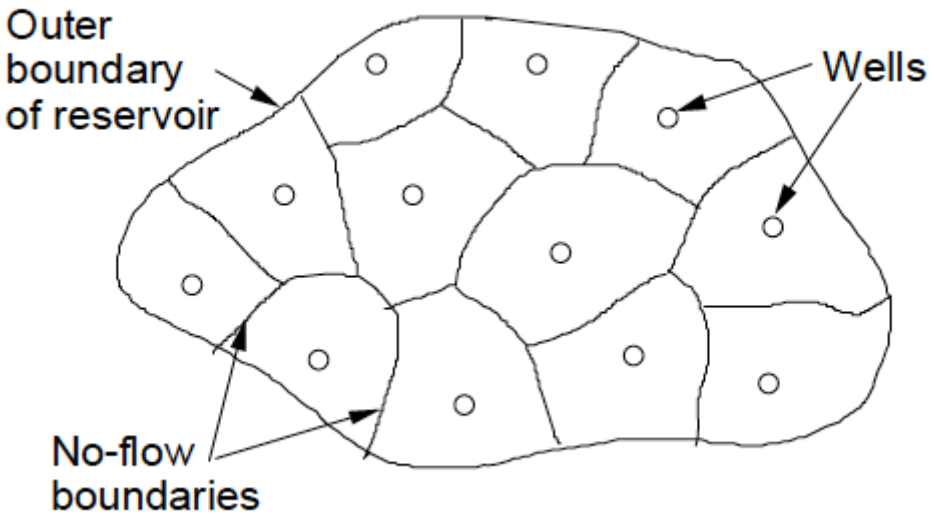
March 2024

Production from Bounded Reservoirs and/or Finite Drainage Regions

Thus far, we have assumed that our wells are located in an infinite reservoir. In reality, the reservoir is always of finite size.

This outer boundary may be an “aquifer”, *i.e.,* a vast expanse of water-filled rock that surrounds the hydrocarbon-filled reservoir. In this case, the appropriate outer boundary condition may be one of constant pressure.

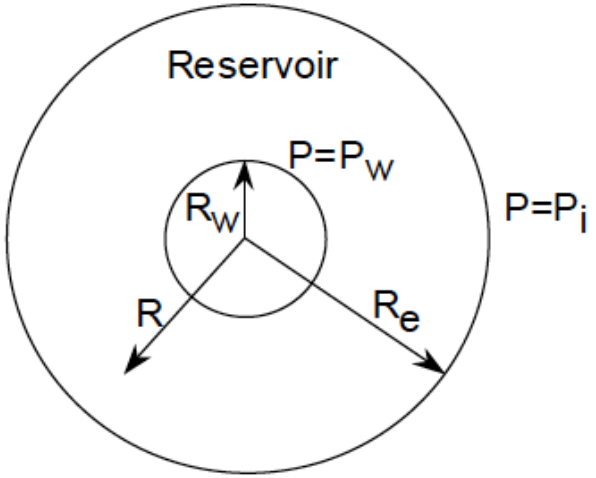
On the other hand, if a number of wells are producing from the same reservoir, each will drain fluid from only a finite region, and so each well will effectively behave as if it were surrounded by a no-flow boundary, as illustrated below.



Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure

Imagine that we have a well located at the centre of a circular reservoir, which is initially at some initial pressure P_i. At time t = 0, the pressure in the wellbore is immediately lowered to some value P_w, and is maintained at that value.

In contrast to previous problems, in which the flowrate was controlled, and we calculated the drawdown, in this problem the *drawdown* is specified, and we must calculate the *flowrate.*



This problem can be formulated as follows:

Governing PDE: $$\frac{1}{R}\frac{d}{dR}\left(R\frac{dP}{dR}\right)=\frac{\phi\mu c}{k}\frac{dP}{dt}$$

BC at wellbore: $$P(R=R_w,t)=P_w$$

BC at outer boundary: $$P(R=R_e,t)=P_i$$

Initial condition: $$P(R,t=0)=P_i$$

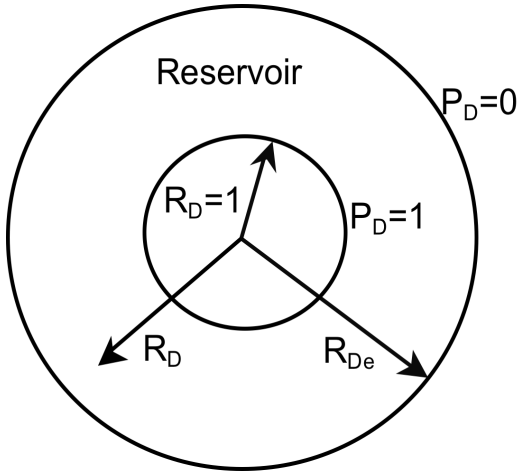
Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure

The first step in solving this problem is to simplify its appearance by introducing dimensionless variables. Define

Dimensionless radius: $R_D = R / R_w$

Dimensionless time: $t_D = \dfrac{kt}{\phi \mu c R_w^2}$

Dimensionless pressure: $P_D = \dfrac{P_i - P}{P_i - P_w}$



The dimensionless radius is defined so that it equals 1 at the well, and increases as we move into the reservoir.

The dimensionless time is defined in the usual manner.

The dimensionless pressure is defined so that it equals 1 at the well, and equals zero at the outer boundary of the reservoir.

Note that the production rate Q is *a priori* unknown, and will not be constant in time, so we can't define P_D in terms of Q, as we did before.

Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure, 1

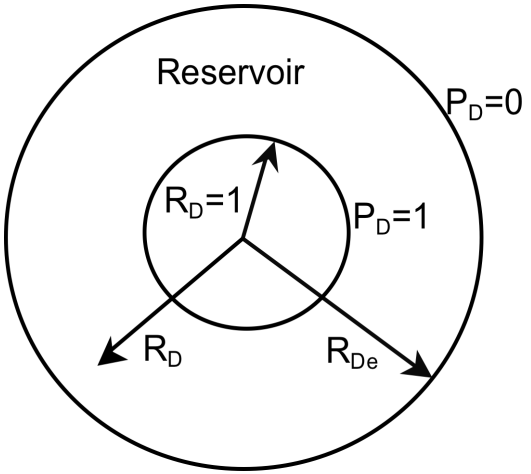
In terms of the dimensionless variables, the governing equations and subsidiary conditions take the form

Governing PDE: $$\frac{1}{R_D}\frac{d}{dR_D}\left(R_D\frac{dP_D}{dR_D}\right)=\frac{dP_D}{dt_D}$$

BC at wellbore: $$P_D(R_D=1,t_D)=1$$

BC at outer boundary: $$P_D(R_D=R_e/R_w\equiv R_{De},t_D)=0$$

Initial condition: $$P_D(R_D,t_D=0)=0$$



To solve this problem, we again use superposition, in a somewhat different form, to break up the pressure into a *steady-state* part, $P_D^s(R_D)$, and a *transient* part, $p_D(R_D,t_D)$:

$$P_D(R_D,t_D)=P_D^s(R_D)+p_D(R_D,t_D)$$

The steady-state solution, by definition, must satisfy the PDE, but with the time derivative set to zero. It must also satisfy both boundary conditions. But it will NOT satisfy the initial condition – that's why we need to add on the transient part of the solution.

Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure, 2

As time is no longer relevant for the steady-state pressure, $P_D{}^S(R_D)$ is governed by the following ordinary differential equation:

Governing ODE: $$\frac{1}{R_D}\frac{d}{dR_D}\left(R_D\frac{dP_D^s}{dR_D}\right)=0$$

BC at wellbore: $$P_D^s(R_D=1)=1$$

BC at outer boundary: $$P_D^s(R_D=R_{De})=0$$

This problem can easily be solved by integrating the ODE using indefinite integration, and imposing the two boundary conditions. The result is:

$$\rightarrow\ P_D^s(R_D)=1-\frac{\ln R_D}{\ln R_{De}}=\frac{-\ln(R_D/R_{De})}{\ln R_{De}}$$

Note 1: The steady-state pressure satisfies the PDE on the previous slide, and the two boundary conditions, but it *doesn't* satisfy the initial condition; this is why we also need the *transient* component of the pressure.

Note 2: The above equation is just the Thiem equation from Lecture 1, in dimensionless form!

Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure, 3

We now substitute $P_D(R_D,t_D) = P_D^s(R_D) + p_D(R_D,t_D)$ into the PDE on slide 80, and find that the transient pressure function must satisfy the following equations:

Governing PDE: $$\frac{1}{R_D}\frac{d}{dR_D}\left(R_D\frac{dp_D}{dR_D}\right) = \frac{dp_D}{dt_D}$$

BC at wellbore: $$p_D(R_D = 1, t_D) = 0$$

BC at outer boundary: $$p_D(R_D = R_{De}, t_D) = 0$$

Initial condition: $$p_D(R_D, t_D = 0) = \frac{\ln(R_D/R_{De})}{\ln R_{De}}$$

To solve the above problem, we again make use of superposition, and first search for as many functions as we can find that *each* satisfy the PDE and the boundary conditions; later, we will superpose these functions to satisfy the initial condition.

First, we assume that these functions can be written in the form

$$p_D(R_D,t_D) = F(R_D)G(t_D)$$

Inserting this into the PDE leads to

$$G(t_D)\left[\frac{1}{R_D}\frac{d}{dR_D}\left(R_D\frac{dF}{dR_D}\right)\right] = F(R_D)\frac{dG}{dt_D}$$

Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure, 4

We now divide through both sides by $F(R_D)G(t_D)$ to arrive at

$$\frac{1}{R_D F(R_D)}\frac{d}{dR_D}\left(R_D\frac{dF}{dR_D}\right) = \frac{1}{G(t_D)}\frac{dG}{dt_D}$$

The left-hand side of this equation doesn't depend on t_D, and the right-hand side doesn't depend on R_D. Since they are equal to each other, they cannot depend on t_D or on R_D. Hence, each side must equal a *constant*, which we call $-\lambda^2$:

$$\frac{1}{R_D F(R_D)}\frac{d}{dR_D}\left(R_D\frac{dF}{dR_D}\right) = \frac{1}{G(t_D)}\frac{dG}{dt_D} = -\lambda^2$$

Let's first focus on the equation for the function F, which can be written as

$$F''(R_D) + \frac{1}{R_D}F'(R_D) + \lambda^2 F(R_D) = 0$$

If we make a change of variables, $x = \lambda R_D$, this equation takes the form

$$F''(x) + \frac{1}{x}F'(x) + F(x) = 0$$

Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure, 5

This equation is known as a *Bessel equation of order zero*. It is a second-order ordinary differential equation, so it must have two independent solutions. One solution is easily found by assuming a power-series solution:

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

Insert this series into the ODE, and we find

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

In order to have x appear to the nth power in each term, so that we can easily add them together, we replace n with n+2 in the first two series (which is valid because n is just a "dummy index"), which leads to

$$0a_0 x^{-2} + a_1 x^{-1} + \sum_{n=0}^{\infty} [(n+2)^2 a_{n+2} + a_n] x^n = 0$$

The coefficient of *each* power of x in this equation must be zero. The x^{-2} term already has a zero in front of it, so a_0 can have any value. The simplest choice is to pick $a_0 = 1$.

In order for the x^{-1} term to vanish, we *must* pick $a_1 = 0$.

Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure, 6

For all higher-order terms to vanish, we must have $(n+2)^2 a_{n+2} + a_n = 0$, which leads to the following *recursion relationship*:

$$a_{n+2} = \frac{-a_n}{(n+2)^2}$$

We already know that $a_0 = 1$ and $a_1 = 0$. This recursion relation allows us to compute a_2, a_3, *etc*. For example, with $n = 0$, the recursion generates $a_2 = -a_0/4 = -1/4$, and with $n = 1$, it generates $a_3 = -a_1/9 = 0$. Continuing in this way, we find:

$$a_3 = a_5 = a_7 = \ldots = 0$$

$$a_2 = \frac{-1}{4}, \quad a_4 = \frac{1}{64}, \quad a_6 = \frac{-1}{2304} \ldots$$

The function $F(x)$ that we have just found is called the Bessel function of the first kind, of order zero, and is denoted by $J_0(x)$:

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \ldots$$

or:
$$J_0(x) = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \ldots$$

where $n! \equiv 1 \times 2 \times 3 \times \ldots \times n$ is the *factorial* function.

Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure, 6

The derivation of the second independent solution to the ODE on the bottom of slide 85 requires a lengthier procedure, which will not be given here. This solution, called the Bessel function of the *second* kind, of order zero, is defined by

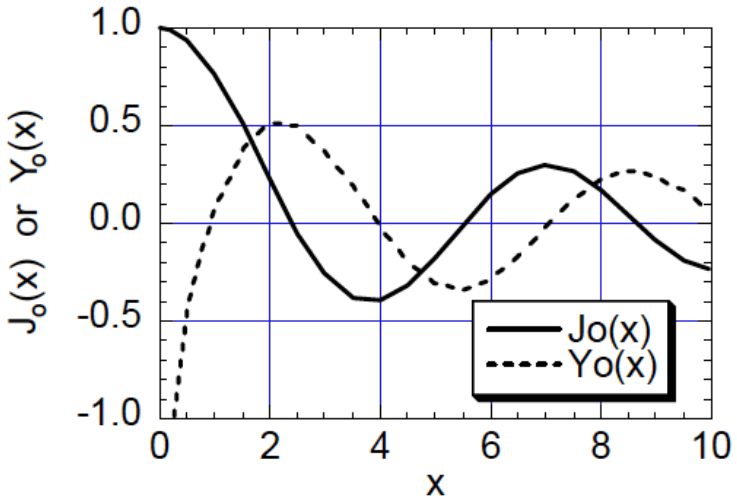
$$Y_0(x) = \frac{2\ln(\gamma x/2)}{\pi} J_0(x) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n h_n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$

where γ = 1.781, and

$$h_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$$

Graphs of these two Bessel functions are shown below:

Although Bessel functions may be unfamiliar to you, and may seem complicated, they are actually not much more complicated than sines and cosines, and in polar coordinates they play roles similar to the roles that sines and cosines play in Cartesian coordinates.



Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure, 7

The general solution to the ODE on the bottom of slide 85 can be written as a linear combination of these two kinds of Bessel functions:

$$F(x) = AJ_0(x) + BY_0(x)$$

where A and B are at this stage two unknown constants.

Recalling that $x = \lambda R_D$, we can say that the following function will be a solution to our ODE, for *any* value of λ:

$$F(R_D) = AJ_0(\lambda R_D) + BY_0(\lambda R_D)$$

However, the above expression will only satisfy the boundary conditions shown on slide 84 for certain special values of λ, which we will now find.

Inserting the above expression for $F(R_D)$ into the BC on slide 84, we find

$$AJ_0(\lambda) + BY_0(\lambda) = 0$$

$$AJ_0(\lambda R_{De}) + BY_0(\lambda R_{De}) = 0$$

which can be written in matrix form as

$$\begin{bmatrix} J_0(\lambda) & Y_0(\lambda) \\ J_0(\lambda R_{De}) & Y_0(\lambda R_{De}) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure, 8

In order for this matrix equation to have non-zero solutions for A and B, the determinant of the matrix must be zero, *i.e.*,

$$J_0(\lambda)Y_0(\lambda R_{De}) - Y_0(\lambda)J_0(\lambda R_{De}) = 0$$

The values of λ that satisfy this equation are called the eigenvalues of this problem. They will depend on the dimensionless size of the reservoir, R_{De}. There will always be an infinite number of eigenvalues, and they can be arranged in order as

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \rightarrow \infty$$

Each eigenvalue generates its own eigenfunction, *i.e.*,

$$F_n(R_D) = A_n J_0(\lambda_n R_D) + B_n Y_0(\lambda_n R_D)$$

It also follows from the equations on the bottom of slide 89 that

$$B_n = \frac{-A_n J_0(\lambda_n)}{Y_0(\lambda_n)}$$

Combining the two previous equations, and simplifying the notation, yields

$$F_n(R_D) = C_n\left[Y_0(\lambda_n)J_0(\lambda_n R_D) - J_0(\lambda_n)Y_0(\lambda_n R_D)\right] = C_n U_n(R_D)$$

Note that at this stage, we don't yet know the values of the constants C_n.

Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure, 9

We now return to the time-dependent part of the solution, which, according to slide 85, must satisfy the following ODE:

$$\frac{dG_n}{dt_D} = -\lambda_n^2 G_n(t_D)$$

The solution to this equation is

$$G_n(t_D) = e^{-\lambda_n^2 t_D}$$

Now that we have found the eigenvalues and the eigenfunctions, we can say that the the general solution for the transient part of the pressure function, that satisfies the PDE and the BCs, is given by

$$p_D(R_D, t_D) = \sum_{n=1}^{\infty} C_n \left[Y_0(\lambda_n) J_0(\lambda_n R_D) - J_0(\lambda_n) Y_0(\lambda_n R_D) \right] e^{-\lambda_n^2 t_D}$$

We now need to pick the constants C_n so that this expression satisfies the initial condition shown on slide 84. This can be done using the "orthogonality" properties of eigenfunctions, as shown in Chapter 10 of *The Flow of Homogeneous Fluids through Porous Media* (Muskat, 1937). The result, skipping the details, is

$$C_n = \frac{\pi J_0(\lambda_n) J_0(\lambda_n R_{De})}{J_0^2(\lambda_n) - J_0^2(\lambda_n R_{De})}$$

Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant wellbore pressure, 10

Lastly, we add together the transient and the steady-state parts of the solution. The resulting expression for the pressure throughout the reservoir is:

$$P_D(R_D,t_D)=\frac{-\ln(R_D/R_{De})}{\ln R_{De}}+\sum_{n=1}^{\infty}\frac{\pi J_0(\lambda_n)J_0(\lambda_n R_{De})}{J_0^2(\lambda_n)-J_0^2(\lambda_n R_{De})}U_n(\lambda_n R_D)e^{-\lambda_n^2 t_D}$$

If we revert to dimensional variables, the solution takes the form

$$P(R,t)=P_i-(P_w-P_i)\frac{\ln(R/R_e)}{\ln(R_e/R_w)}$$

$$+(P_w-P_i)\sum_{n=1}^{\infty}\frac{\pi J_0(\lambda_n)J_0(\lambda_n R_e/R_w)}{J_0^2(\lambda_n)-J_0^2(\lambda_n R_e/R_w)}U_n(\lambda_n R/R_w)e^{-\lambda_n^2 kt/\phi\mu c R_w^2}$$

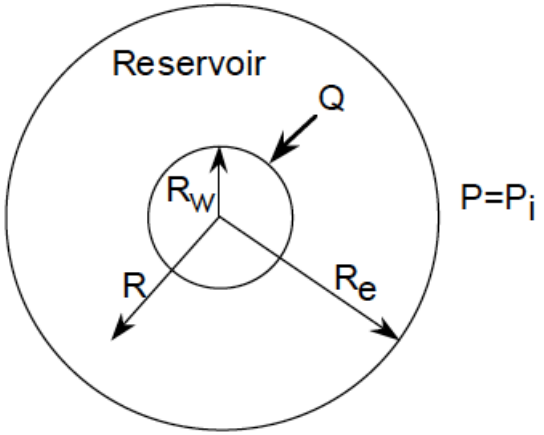
The flowrate into the well, as a function of time, can be found by differentiating the pressure with respect to R, and applying Darcy's law at the wellbore.

As time goes on, all of the exponential terms will eventually die off, and the solution will approach the steady-state solution, which is in fact equivalent to the Thiem equation that was derived in Lecture 1:

$$P(R,t)=P_i-(P_w-P_i)\frac{\ln(R/R_e)}{\ln(R_e/R_w)}$$

Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant flowrate into the wellbore, 1

This problem can also be solved using the method of eigenfunction expansions. The solution is:



$$\Delta P_D(R_D,t_D) = -\ln\frac{R_D}{R_{De}} - \sum_{n=1}^{\infty}\frac{\pi J_0^2(\lambda_n R_{De})U_n(\lambda_n R_D)}{\lambda_n[J_0^2(\lambda_n R_{De}) - J_1^2(\lambda_n)]}e^{-\lambda_n^2 t_D}$$

where the eigenfunctions U_n are defined by

$$U_n(\lambda_n R_D) = Y_1(\lambda_n)J_0(\lambda_n R_D) - J_1(\lambda_n)Y_0(\lambda_n R_D)$$

the eigenvalues λ_n are the roots of the following equation:

$$U_n(\lambda_n R_{De}) = Y_1(\lambda_n)J_0(\lambda_n R_{De}) - J_1(\lambda_n)Y_0(\lambda_n R_{De}) = 0$$

the functions J_1 and Y_1 are *Bessel functions of order one*, defined by

$$J_1(x) \equiv -\frac{dJ_0(x)}{dx}, \quad Y_1(x) \equiv -\frac{dY_0(x)}{dx}$$

and the dimensionless variables are defined by

$$R_D = \frac{R}{R_w} \qquad t_D = \frac{kt}{\phi\mu c R_w^2} \qquad \Delta P_D = \frac{2\pi kH(P_i - P)}{\mu Q}$$

Well at the centre of a circular reservoir with constant pressure on its outer boundary and constant flowrate into the wellbore, 2

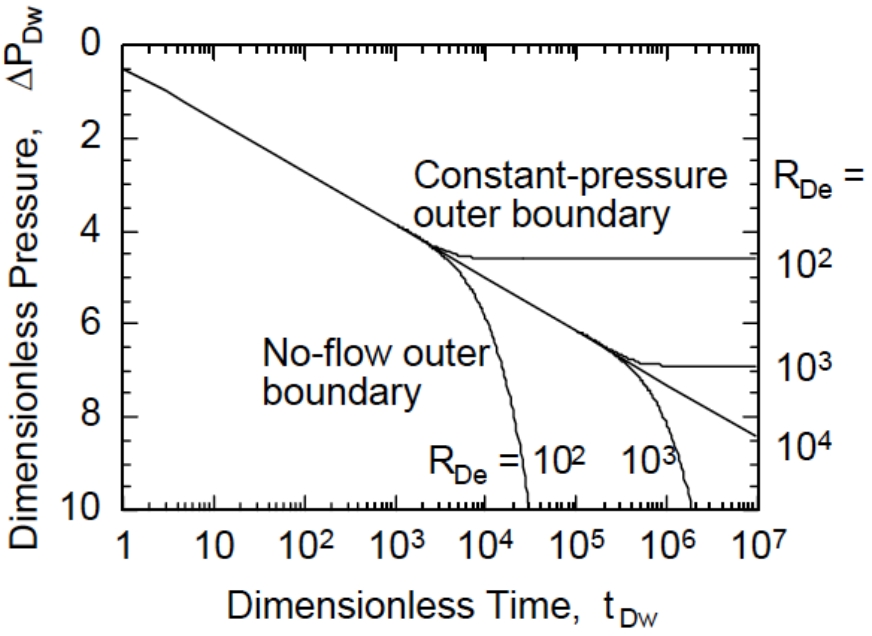
The pressure in the wellbore, ΔP_{Dw}, is found by setting $R_D = 1$, and using the following Bessel function identity: $Y_1(x)J_0(x) - J_1(x)Y_0(x) = -2/\pi x$.

The result is: $$\Delta P_{Dw}(t) = \ln R_{De} + \sum_{n=1}^{\infty} \frac{2J_0^2(\lambda_n R_{De})e^{-\lambda_n^2 t_D}}{\lambda_n^2[J_0^2(\lambda_n R_{De}) - J_1^2(\lambda_n)]}$$

A detailed analysis of this solution would show that at early times, the pressure agrees with that given by the Theis line source solution.

At early times the pressure pulse will not have reached the outer boundary of the reservoir, and so the finite-reservoir solution should coincide with the infinite-reservoir solution.

Eventually, the wellbore reaches a steady-state pressure (see figure) that is equivalent to that which occurs in the Thiem problem of Lecture 1.



Well at the centre of a circular reservoir with a no-flow outer boundary and constant flowrate into the wellbore, 1

This problem is identical to the previous problem, except that the boundary conditions are now:

wellbore: $\left(\frac{2\pi kH}{\mu} R \frac{dP}{dR}\right)_{R=R_w} = Q$ outer boundary: $\left(\frac{2\pi kH}{\mu} R \frac{dP}{dR}\right)_{R=R_e} = 0$

The solution to this problem was found by Muskat of Gulf Oil Company in 1937 using the eigenfunction method, and was re-derived by van Everdingen and Hurst at Shell in 1949 using Laplace transforms. The solution is

$$\Delta P_D(R_D, t_D) = \frac{1}{R_{De}^2 - 1}\left[\frac{R_D^2}{2} + 2t_D - R_{De}^2 \ln R_D\right] - \left[\frac{3R_{De}^4 - 4R_{De}^4 \ln R_{De} - 2R_{De}^2 - 1}{4(R_{De}^2 - 1)^2}\right] + \sum_{n=1}^{\infty} \frac{\pi J_1^2(\lambda_n R_{De}) U_n(\lambda_n R_D)}{\lambda_n [J_1^2(\lambda_n R_{De}) - J_1^2(\lambda_n)]} e^{-\lambda_n^2 t_D}$$

where the eigenfunctions U_n are given by

$$U_n(\lambda_n R_D) = J_1(\lambda_n) Y_0(\lambda_n R_D) - Y_1(\lambda_n) J_0(\lambda_n R_D)$$

and the eigenvalues λ_n are defined implicitly by

$$J_1(\lambda_n) Y_1(\lambda_n R_{De}) - Y_1(\lambda_n) J_1(\lambda_n R_{De}) = 0$$

Well at the centre of a circular reservoir with a no-flow outer boundary and constant flowrate into the wellbore, 2

The pressure in the wellbore is found by setting R_D = 1, and again using the relation $Y_1(x)J_0(x) - J_1(x)Y_0(x) = -2/\pi x$:

$$\Delta P_{Dw} = \frac{1}{R_{De}^2 - 1}\left[\frac{1}{2} + 2t_D\right] - \left[\frac{3R_{De}^4 - 4R_{De}^4 \ln R_{De} - 2R_{De}^2 - 1}{4(R_{De}^2 - 1)^2}\right] + \sum_{n=1}^{\infty} \frac{2J_1^2(\lambda_n R_{De})e^{-\lambda_n^2 t_D}}{\lambda_n^2[J_1^2(\lambda_n R_{De}) - J_1^2(\lambda_n)]}$$

In cases of practical interest, $R_D >> 1$ and $t_{Dw} >> 1$, and so we can simplify this to

$$\Delta P_{Dw}(t_D) = \frac{2t_D}{R_{De}^2} + \ln R_{De} - \frac{3}{4} + \sum_{n=1}^{\infty} \frac{2J_1^2(\lambda_n R_{De})e^{-\lambda_n^2 t_D}}{\lambda_n^2[J_1^2(\lambda_n R_{De}) - J_1^2(\lambda_n)]}$$

There are three important time regimes in this problem:

(i) A regime in which the leading edge of the pressure pulse has not yet reached the outer boundary of the reservoir. This regime is defined by:

$$t_{Dw} < 0.1R_{De}^2$$

As we would expect, the drawdown given by the above equation reduces to the line-source solution during this time regime, although this is not obvious or easy to prove from the equation shown above.

Well at the centre of a circular reservoir with a no-flow outer boundary and constant flowrate into the wellbore, 3

(ii) A second regime that is “late” enough that the presence of the closed outer boundary is felt at the well, but still early enough that the exponential terms in the drawdown function have not yet died out. The duration of this regime is delineated by (see Problem 6.1)

$$0.1R_{De}^2 < t_{Dw} < 0.3R_{De}^2$$

In this regime we must use the entire series to calculate the wellbore pressure, and the drawdown curve has no simple description.

(iii) A third regime, in which the time is large enough that the exponential terms have effectively died out. This regime is defined by

$$t_{Dw} > 0.3R_{De}^2$$

In this regime the wellbore drawdown is given by

$$\Delta P_{Dw} = \frac{2t_{Dw}}{R_{De}^2} + \ln R_{De} - \frac{3}{4}$$

These regimes have been given many conflicting and confusing names in the literature. We will refer to the first regime as the “infinite reservoir” regime, the second as the “transition regime”, and the last as the “finite reservoir” regime.

Well at the centre of a circular reservoir with a no-flow outer boundary and constant flowrate into the wellbore, 4

An important feature of the finite reservoir regime is that the pressure in the well *declines linearly with time*. The rate of pressure decline can be used to find the drainage area of the well, as follows.

First, rewrite the late-time drawdown in terms of the actual variables, rather than the dimensionless variables, to find the pressure at the well in the form

$$P_w(t) = P_i - \frac{Q\mu}{2\pi kH}\left\{\frac{2kt}{\phi\mu c R_e^2} + \ln\left(\frac{R_e}{R_w}\right) - \frac{3}{4}\right\}$$

Next, take the derivative of P_w with respect to t:

$$\frac{dP_w}{dt} = -\frac{Q\mu}{2\pi kH}\cdot\frac{2k}{\phi\mu c R_e^2} = \frac{-Q}{\pi R_e^2 H\phi c} = \frac{-Q}{AH\phi c}$$

The rate of change of the late-time well pressure can therefore be used to find the drainage area, A:

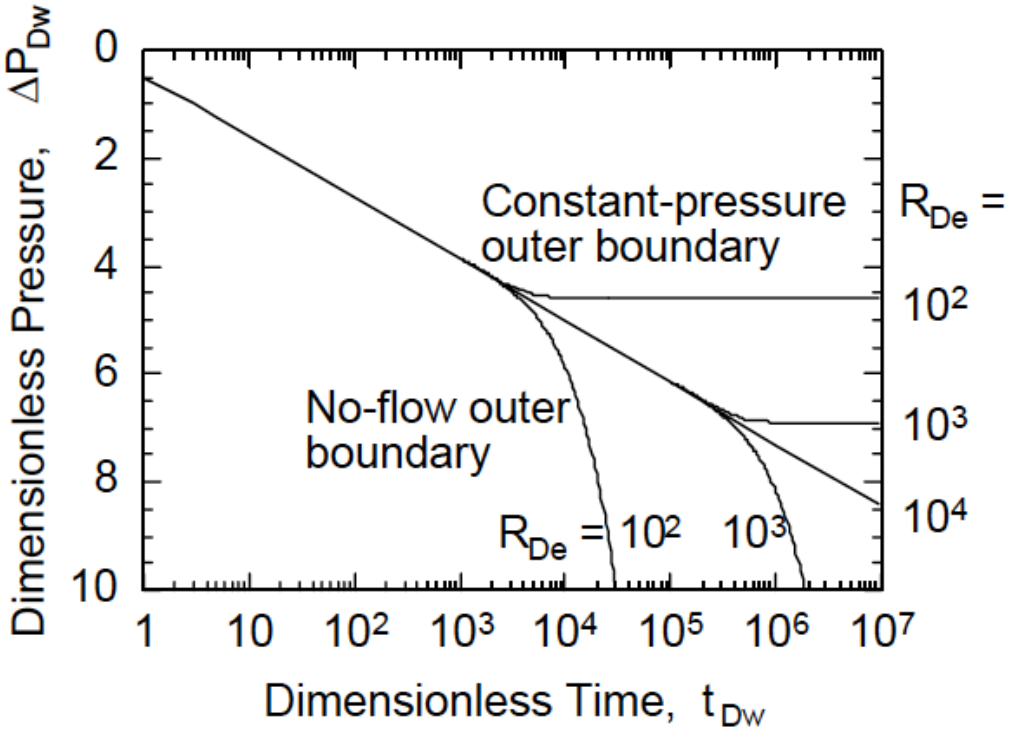
$$A = \frac{-Q}{\phi cH(dP_w/dt)}$$

The textbook gives a simpler and more general derivation of this result, which shows that it holds for drainage areas that have *any* shape.

Well at the centre of a circular reservoir with a no-flow outer boundary and constant flowrate into the wellbore, 5

The dimensionless wellbore pressure for a well located at the centre of a circular reservoir is plotted below, for various values of the dimensionless reservoir size, R_{De}, for the two cases of constant pressure at the outer boundary, and no-flow at the outer boundary.

As claimed above, the "infinite reservoir" regime ends when $t_{Dw} = 0.1R_{De}^2$. For example, when R_{De} = 1000, the curves begins to deviate from the semi-log straight line at about $t_{Dw} = 0.1R_{De}^2 = 1\times10^5$.



Problems for Lecture 4

FFiPM Problems 6.1 and 6.2. Some of you may enjoy working through these difficult problems, which are described in the textbook.

FFiPM Problem 6.3. Starting with the equation for the pressure distribution in the reservoir, as given on slide 93, calculate the average pressure in a circular reservoir that has a constant-pressure outer boundary, and constant flowrate into the wellbore, during the finite-reservoir time regime in which all of the exponential terms have died out.

Use this result to find an equation for the well productivity, which relates the production rate to the difference between the average reservoir pressure and the pressure at the well.

FFiPM Problem 6.4. Imagine a vertical well of radius 4 in, at the centre of a closed circular reservoir whose outer radius is 1000 ft. The well is producing oil at a rate of 100 barrels per day. The oil has a viscosity of 1 cP. The thickness of the reservoir is 50 ft, the permeability of the reservoir is 100 mD, the porosity is 0.2, and the total compressibility of the rock-fluid system is 10×10^{-6}/psi.

What will be the drawdown at the well, after three days of production?

Imperial College MSc in
GeoEnergy with Machine Learning and Data Science

GEMS 3: Geomechanics and Pressure Transient Analysis

Lecture 5:
Laplace Transform Methods in Reservoir Engineering

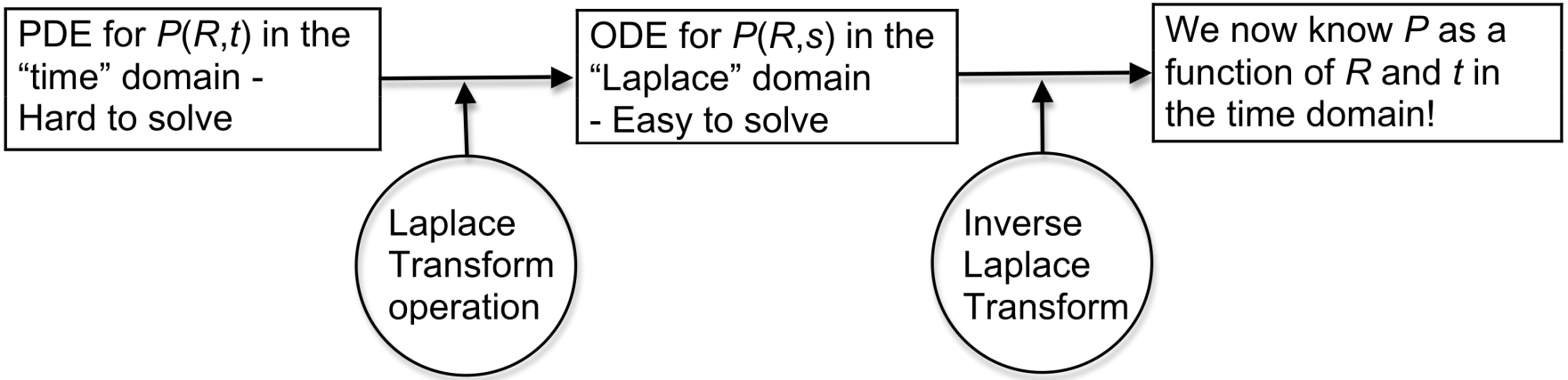
Prof. Robert Zimmerman

March 2024

Introduction to Laplace Transforms

The most widely used method for solving the pressure diffusion equation, for different reservoir geometries and different boundary conditions, is the method of Laplace transforms.

In this approach, the *partial* differential diffusion equation in the "time domain" is transformed into an *ordinary* differential equation in the "Laplace domain", which is generally much easier to solve. The solution in the Laplace domain is then transformed back into the "time domain", using well-known analytical or numerical algorithms.



In this lecture we will introduce the Laplace transform method, present many of its useful properties, and then use the method to solve an important reservoir engineering problem – flow to well containing a vertical hydraulic fracture.

Introduction to Laplace Transforms

Definition: If we have a function $f(t)$, its Laplace transform is defined by

$$L\{f(t)\} \equiv \hat{f}(s) \equiv \int_0^\infty f(t)e^{-st}dt$$

Both notations, $L\{f(t)\}$ and $\hat{f}(s)$, are useful, depending on the context. In the general theory of Laplace transforms, the parameter s must be regarded as a complex variable, but for our purposes we can think of it as a real variable.

If we have a function of two variables, such as $P(R,t)$, we can define its Laplace transform in the same manner:

$$\hat{P}(R,s) \equiv \int_0^\infty P(R,t)e^{-st}dt$$

Note that the Laplace transform is taken with respect to the time variable, not the spatial variable. To simplify the notation, we will often hide the R variable when discussing the general theory.

A key and very useful fact about Laplace transforms is that L is a linear operator, *i.e.,*

$$L\{cf(t)\} = cL\{f(t)\}$$

$$L\{f(t)+g(t)\} = L\{f(t)\}+L\{g(t)\}$$

Introduction to Laplace Transforms

Let's compute a simple Laplace transform, to see how this definition works in practice.

The simplest function $f(t)$ is the constant function, $f(t) = 1$.

According to our definition, the Laplace transform of the function 1 is given by

$$L\{f(t)\} = L\{1\} = \int_0^\infty 1e^{-st}dt = \frac{-1}{s}e^{-st}\Big]_0^\infty = \frac{1}{s}$$

In other words, the Laplace transform of 1 is $1/s$. This is the most important Laplace transform, and you must remember it.

Because of the linearity of Laplace transforms, we immediately see that the Laplace transform of a constant, c, is equal to c/s.

As a more pertinent example, since the function we are interested in is usually the pressure, P, we can see, for example, that the Laplace transform of the initial pressure, P_i, is P_i/s.

Properties of the Laplace Transform

The most important property of the Laplace transform is that differentiation in the time domain essentially corresponds to multiplication by s in the Laplace domain.

To prove this, consider the Laplace transform of the time derivative of $f(t)$. By definition, it is given by the following integral:

$$L\{f'(t)\} \equiv \int_0^\infty f'(t)e^{-st}dt$$

Recall the general formula for integration by parts, for two functions $f'(t)$ and $g(t)$:

$$\int_0^\infty f'(t)g(t)dt = f(t)g(t)\Big]_0^\infty - \int_0^\infty f(t)g'(t)dt$$

If we apply this formula, with e^{-st} playing the role of $g(t)$, we find

$$L\{f'(t)\} = f(t)e^{-st}\Big]_0^\infty + s\int_0^\infty f(t)e^{-st}dt = f(\infty)e^{-s\cdot\infty} - f(0)e^{-s\cdot 0} + sL\{f(t)\} = sL\{f(t)\} - f(0)$$

$$i.e., \quad L\{f'(t)\} = sL\{f(t)\} - f(0)$$

In other words, the Laplace transform of the derivative of $f(t)$ is equal to the Laplace transform of $f(t)$ multiplied by s, minus the initial value of $f(t)$.

Properties of the Laplace Transform

This important result is repeated here:

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

This equation illustrates two important properties of Laplace transforms:

(i) Differentiation in the time domain essentially corresponds to multiplication by s in the Laplace domain.

(ii) The initial conditions in the time domain become incorporated directly into the governing differential equation in the Laplace domain. This is unlike the situation in "time domain" methods, where initial conditions must be considered separately after we have found the general solution to the differential equation.

Note: Since our function "f" is usually the pressure drawdown, which is defined to be zero when $t = 0$, the $f(0)$ term will often drop out of our calculations.

In this case, we simply can say that

$$L\{dP/dt\} = sL\{P(t)\} = s\bar{P}(s)$$

Properties of the Laplace Transform

Many other important and useful properties of Laplace transforms are proven in the textbook. Most of the proofs are easy, and you should be able to follow them. The most important of these properties are given below.

1. Integration with respect to time corresponds to *division* by *s* in the Laplace domain, *i.e.*,

$$L\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\hat{f}(s)$$

An interesting application of this theorem is to use it to calculate the Laplace transform of the function *t*.

First, note that *t* is the integral of the function $f(t) = 1$.

We already know that the Laplace transform of 1 is $1/s$.

And, we see from the equation above that the Laplace transform of the integral of a function is simply the Laplace transform of the function, divided by *s*.

So, the Laplace transform of *t* is $1/s^2$.

Using the theorem/property shown at the top of this slide, and the other properties shown on the following slides, we can often calculate the Laplace transform of a function *without having to actually compute the integral*.

Properties of the Laplace Transform

2. If we take a function $f(t)$, and "damp it out" by multiplying it by e^{-at}, this is equivalent to replacing s with $s+a$ in the Laplace domain, *i.e.,*

$$L\left\{e^{-at}f(t)\right\} = \hat{f}(s+a)$$

As an interesting example of the use of this theorem, let $f(t) = 1$.

We already know that if $f(t) = 1$, then

$$\hat{f}(s) = L\left\{f(t)\right\} = L\{1\} = \frac{1}{s}$$

So, using this theorem with $f(t) = 1$, we have

$$L\left\{e^{-at}f(t)\right\} = L\left\{e^{-at}\cdot 1\right\} = L\left\{e^{-at}\right\} = \hat{f}(s+a) = \frac{1}{s+a}$$

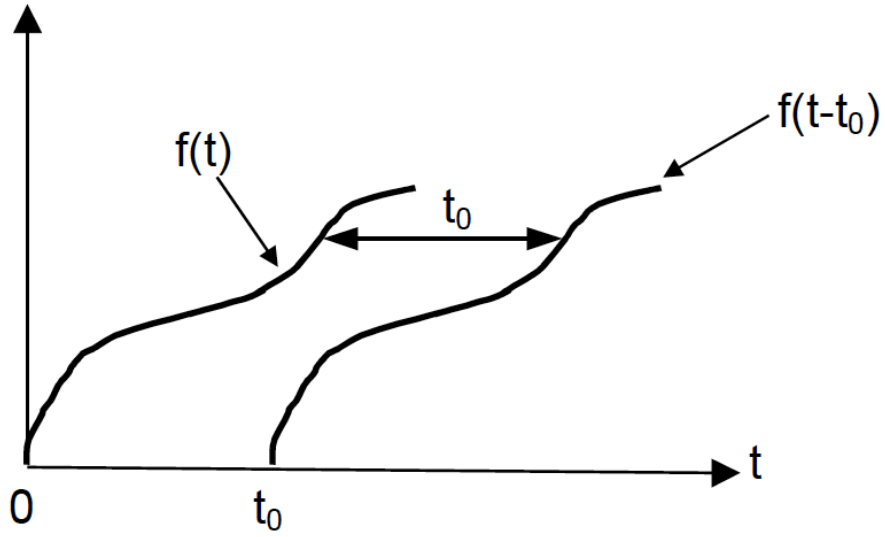
In other words, the Laplace transform of e^{-at} is $1/(s+a)$.

Using this result, and the making use of the linearity property of Laplace transforms, we can find the Laplace transforms of some more complicated functions. For example, since $\sinh(at) = (e^{at} - e^{-at})/2$, we can find that

$$L\left\{\sinh(at)\right\} = \frac{1}{2}L\left\{e^{at}\right\} - \frac{1}{2}L\left\{e^{-at}\right\} = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{(s+a)-(s-a)}{(s+a)(s-a)}\right] = \frac{a}{s^2-a^2}$$

Properties of the Laplace Transform

3. In reservoir engineering, we often consider functions that "start" at some time t_0, rather than at time 0. Consider $f(t)$, and the "time-shifted" function $f(t - t_0)$, defined by the graph below:



Note that $f(t - t_0)$ is assumed to be 0 for $t < t_0$, which is consistent with the fact that, when using Laplace transforms, we always consider *all* functions to be zero when $t < 0$. This assumption reflects the physical fact that the pressure drawdown will be zero before production begins!

As proven in the textbook, the Laplace transform of $f(t - t_0)$ is found by taking the Laplace transform of $f(t)$, and multiplying it by e^{-st_0}, *i.e.*,

$$L\left\{f(t-t_0)\right\} = e^{-st_0}\bar{f}(s)$$

Laplace Transform of Power Functions

Two types of specific functions that often arise when solving the pressure diffusion equation are *t* taken to an integer power, and *t* taken to a half-integer power. The Laplace transforms of these functions are derived in the textbook.

1. The Laplace transform of *t* taken to an integer power is

$$L\left\{t^n\right\} = \frac{n!}{s^{n+1}}$$

in which *n*! is the factorial function, defined as *n*! = 1×2×3×…×*n*. (When *n* = 0, the factorial function is defined as 0! = 1).

For example, if *n* = 1, this theorem tells us that $L\{t\} = 1/s^2$, which agrees with what we already knew. It also tells us, for example, that $L\{t^2\} = 2/s^3$, *etc.*

2. An important function is $t^{-1/2}$, whose Laplace transform is

$$L\{t^{-1/2}\} = \sqrt{\pi / s}$$

3. The general case of *t* taken to a "half-integer" power is, for *n* = 1,2,…:

$$L\{t^{n-(1/2)}\} = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)\sqrt{\pi}}{2^n s^{n+(1/2)}}$$

For example, *n* = 1 shows us that $L\{t^{1/2}\} = \sqrt{\pi} / 2s^{3/2}$.

Laplace Transform of a Spatial Derivative

There is one last, but very important, rule about Laplace transforms that we must learn before we can (finally) start solving reservoir engineering problems.

The pressure diffusion equation involves derivatives with respect to time, but also involves derivatives with respect to a space variable, such as *R* or *x*.

The rule for taking the Laplace transform of the derivative of *P* with respect to *R* is simple (see textbook for the proof):

 The Laplace transform of the partial derivative of *P* with respect to *R* is equal to the partial derivative, with respect to *R*, of the Laplace transform of *P*, *i.e.,*

$$L\left\{\frac{dP}{dR}\right\} = \frac{d}{dR}\left[L\left\{P(R,t)\right\}\right] = \frac{d\hat{P}(R,s)}{dR}$$

Mathematicians describe fact this by saying that “the operation of taking the Laplace transform with respect to *t*, and the operation of taking the derivative with respect to *R*, are commutative with respect to each other”.

Do NOT confuse this rule with the rule for the Laplace transform of *dP*/*dt*, which is

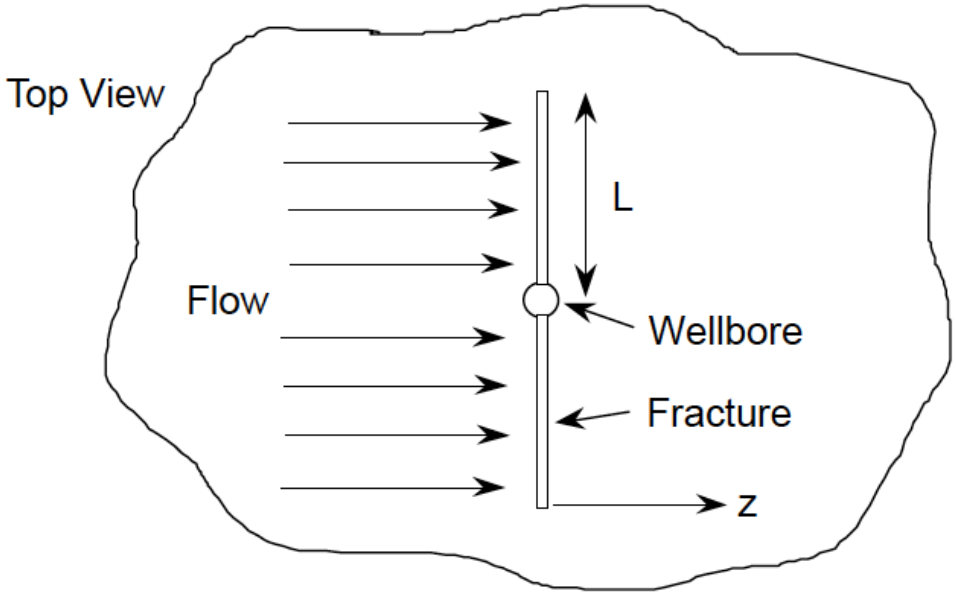
$$L\left\{\frac{dP}{dt}\right\} = s\hat{P}(R,s)$$

Flow to an Hydraulically Fractured Well, 1

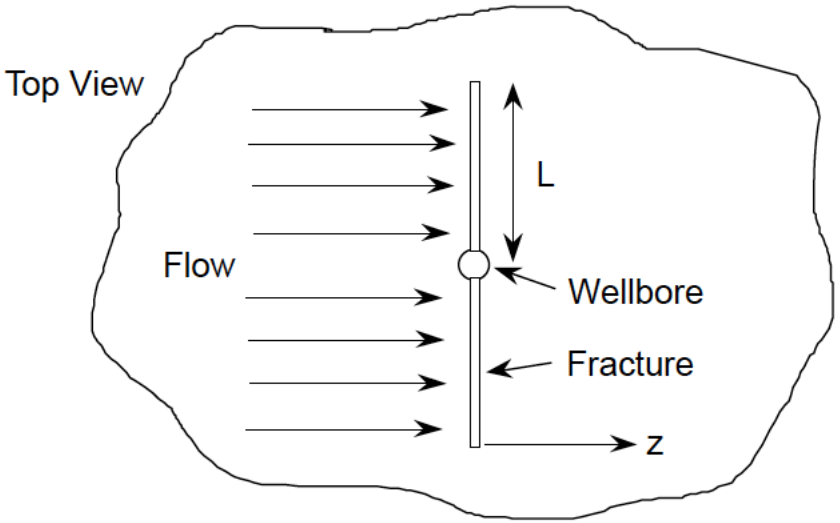
We have now learned enough about Laplace transforms to be able to use this method to solve an important reservoir engineering problem.

In low permeability reservoirs, we often introduce fractures into the rock by pumping fluid into the borehole at high pressure. These “hydraulic fractures” provide very conductive paths for the oil to reach the wellbore. Oil first flows to the fracture, and then through the fracture to the well.

Imagine that the thickness of the reservoir is *H*, and that the hydraulic fracture extends out from the borehole to a distance *L* in each direction:



Flow to an Hydraulically Fractured Well, 2



Initially, fluid flows straight to the nearest part of the fracture, after which it travels through the hydraulic fracture to the wellbore. If the length L is large, or equivalently, at early times, we can model this process as uniform, one-dimensional, horizontal flow.

The governing equation for this problem is the 1-D pressure diffusion equation in Cartesian co-ordinates, derived in Lecture 1 (consider only $z > 0$; the solution for $z < 0$ will be the mirror image):

$$\text{PDE:} \quad \frac{dP}{dt} = D\frac{d^2P}{dz^2} \qquad D = k/\phi\mu c_t$$

If the total flowrate into the well is Q, distributed uniformly over an area $4LH$ (two fractures with two faces, each of area LH), then the subsidiary conditions are

$$\text{IC:} \quad P(z, t=0) = P_i$$

$$\text{far-field BC:} \quad P(z \to \infty, t) = P_i$$

$$\text{fracture BC:} \quad \frac{dP}{dz}(z=0, t) = \frac{\mu Q}{4kLH}$$

Flow to an Hydraulically Fractured Well, 3

To solve this problem, we first define the Laplace transform of $P(z,t)$:

$$\hat{P}(z,s) \equiv \int_0^\infty P(z,t)e^{-st}dt$$

Next, we take the Laplace transform of *both sides* of our governing PDE. For the left side,

$$L\left\{\frac{dP}{dt}\right\} = sL\{P(z,t)\} - P(z,t=0) = s\hat{P}(z,s) - P_i$$

For the right side, we use (twice!) our rule for the Laplace transform of a spatial derivative:

$$L\left\{D\frac{d^2P}{dz^2}\right\} = D\frac{d^2}{dz^2}\left[L\left\{P(z,t)\right\}\right] = D\frac{d^2\hat{P}(z,s)}{dz^2}$$

So, the transformed version of our PDE is

$$D\frac{d^2\hat{P}(z,s)}{dz^2} - s\hat{P}(z,s) = -P_i$$

Although $\hat{P}(z,s)$ is a function of two variables, z and s, this equation contains no derivatives of $\hat{P}(z,s)$ with respect to s. In other words, s is a parameter, not a variable. Consequently, the above equation is an ODE, rather than a PDE.

Flow to an Hydraulically Fractured Well, 4

The initial condition is already incorporated into this ODE. However, we must take the Laplace transforms of the two boundary conditions.

The far-field boundary condition:

$$L\{P(z=\infty,t)\} = \hat{P}(z=\infty,s) = L\{P_i\} = \frac{P_i}{s}$$

The fracture boundary condition:

$$L\left\{\frac{dP}{dz}(z=0,t)\right\} = \frac{d\hat{P}}{dz}(z=0,s) = L\left\{\frac{\mu Q}{4kLH}\right\} = \frac{\mu Q}{4kLHs}$$

For clarity, let's summarise the situation. We have reduced a 2nd-order PDE to a 2nd-order ODE. Being a second-order ODE, it must satisfy *two* boundary conditions. The problem in the Laplace domain can therefore be expressed as

$$\text{ODE}: \qquad D\frac{d^2\hat{P}(z,s)}{dz^2} - s\hat{P}(z,s) = -P_i$$

$$\text{Far-field BC}: \qquad \hat{P}(z=\infty,s) = \frac{P_i}{s}$$

$$\text{Fracture BC}: \qquad \frac{d\hat{P}}{dz}(z=0,s) = \frac{\mu Q}{4kLHs}$$

Flow to an Hydraulically Fractured Well, 5

The general solution to our ODE is (you should be able to derive this easily):

$$\hat{P}(z,s) = Ae^{z\sqrt{s/D}} + Be^{-z\sqrt{s/D}} + \frac{P_i}{s}$$

where A and B are arbitrary constants.

If we apply the far-field boundary condition to the general solution, we see that A must be zero (or else the pressure would become infinite, far from the well).

Applying the fracture boundary condition to the general solution, implies that

$$\frac{d\hat{P}}{dz}(z=0,s) = -B\sqrt{s/D} = \frac{\mu Q}{4kLHs}$$

$$\Rightarrow \quad B = \frac{-\mu Q}{4kLHs\sqrt{s/D}}$$

So, we have now found A and B, and the solution to this problem *in the Laplace domain* is

$$\hat{P}(z,s) = \frac{P_i}{s} - \frac{\mu Q\sqrt{D}}{4kLHs^{3/2}} e^{-z\sqrt{s/D}}$$

Flow to an Hydraulically Fractured Well, 6

Finally, we must invert $\hat{P}(z,s)$ to find $P(z,t)$. To keep the analysis simple, we will only invert for the pressure in the fracture, $P(z=0,t)$, which is actually the pressure that we are most interested in. (If the permeability of the fracture is much greater than that of the reservoir - which is the idea behind hydraulic fracturing! - then there will be very little pressure drop in the fracture itself, and the pressure in the fracture will be equal to the pressure in the wellbore.)

In the fracture,

$$\hat{P}(z=0,s) = \frac{P_i}{s} - \frac{\mu Q\sqrt{D}}{4kLHs^{3/2}}$$

We now invert this function, to go back to the time domain, making use of linearity, and using our knowledge of a few specific Laplace transforms:

$$P(z=0,t) = L^{-1}\{\hat{P}(z=0,s)\} = L^{-1}\left\{\frac{P_i}{s} - \frac{\mu Q\sqrt{D}}{4kLHs^{3/2}}\right\}$$

$$= P_i L^{-1}\left\{\frac{1}{s}\right\} - \frac{\mu Q\sqrt{D}}{4kLH} L^{-1}\left\{\frac{1}{s^{3/2}}\right\}$$

$$= P_i - \frac{\mu Q}{2kLH}\sqrt{\frac{Dt}{\pi}} = P_i - \frac{\mu Q}{2kLH}\sqrt{\frac{kt}{\phi\mu c\pi}} = P_i - \frac{Q}{2LH}\sqrt{\frac{\mu t}{\pi\phi ck}}$$

The key point is that the drawdown increases proportional to the square root of t.

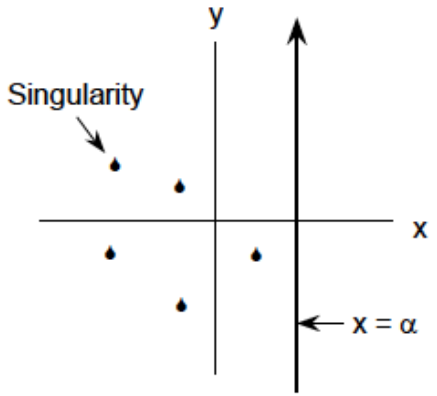
Inversion of Laplace Transforms

In the hydraulic fracturing problem, we were able to easily "invert" the pressure from the Laplace domain, to the time domain, because we recognised the Laplace transforms s^{-1} and $s^{-3/2}$ as ones that we had already encountered.

In general, unless we recognise the functions that we obtain after we solve for $\hat{P}(s)$, or find them in a table of Laplace transforms, we will need to perform an integration to "invert" $\hat{P}(s)$ to find $P(t)$.

This inversion process, first derived by the British mathematician Bromwich in the early 1900s, requires performing an integration in the complex plane, where $s = x + iy$, and the integration path is any vertical line in the complex plane that lies to the *right* of all the singularities of $\hat{P}(s)$:

$$P(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \hat{P}(s) e^{st} ds$$



The proof of this inversion formula can be found in most monographs on transform methods, such as *Operational Mathematics* by H. S. Carslaw and J. C. Jaeger (Oxford University Press, 1949).

Inversion of Laplace Transforms

Many numerical algorithms have been derived that allow us to avoid doing an integration in the complex plane.

The algorithm that is the most commonly used in petroleum engineering is the Stehfest algorithm, which requires evaluating $\hat{P}(s)$ at a finite number of carefully chosen *real* values of s, multiplying these values by carefully chosen weighting functions, and summing up.

The Stehfest algorithm can be written, after correcting two small typographical errors in the textbook, as:

$$P(t) = \frac{\ln 2}{t} \sum_{n=1}^{2N} V_n \hat{P}\left(s = \frac{n \ln 2}{t} \right)$$

where $$V_n = (-1)^{n+N} \sum_{k=\lfloor (n+1)/2 \rfloor}^{\min(n,N)} \frac{k^N (2k)!}{(N-k)!\,k!\,(k-1)!\,(n-k)!\,(2k-n)!}$$

and $\lfloor x \rfloor$ is the largest integer k such that $k \le x$. For example, if n = 2, then then $\lfloor (n+1)/2 \rfloor = 1$, whereas if n = 3, then $\lfloor (n+1)/2 \rfloor = 2$.

In contrast to most numerical integration schemes, the Stehfest algorithm does *not* become more accurate as we add terms to the summation. It is generally observed that N = 9 is the optimum value.

Problems for Lecture 5

FFiPM Problem 7.2. Starting with the basic definition of the Laplace transform, verify the formula $L\{f(at)\} = \hat{f}(s/a)/a$.

FFiPM Problem 7.3. Derive the general expression for the Laplace transform of the function t^n.

FFiPM Problem 7.4. Following the steps that taken to solve the problem of flow to a hydraulic fracture at a *constant flowrate*, use Laplace transforms to solve the problem of flow into a hydraulic fracture with *constant pressure* in the fracture:

$$\text{PDE:} \quad \frac{1}{D}\frac{dP}{dt} = \frac{d^2P}{dz^2}$$

$$\text{IC:} \quad P(z, t=0) = P_i$$

$$\text{far-field BC:} \quad P(z \to \infty, t) = P_i$$

$$\text{fracture BC:} \quad P(z=0, t) = P_f$$

First, find the pressure in the Laplace domain, $\hat{P}(z,s)$. Next, use Darcy's law to find the flowrate into the fracture, in the Laplace domain; call it $\hat{Q}_f(s)$. Lastly, invert $\hat{Q}_f(s)$ back to the time domain to find the flowrate into the fracture as a function of time, $Q_f(t)$.

Imperial College MSc in
GeoEnergy with Machine Learning and Data Science

GEMS 3: Geomechanics and Pressure Transient Analysis

Lecture 6:
Traction, Stress, and Mohr's Circle

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Traction Acting on a Plane

Consider a vertical fault in the subsurface, aligned in a north-south direction, with the two opposing faces of rock being pressed together by tectonic forces. The rock to the east then exerts a force on the rock to the west, and *vice versa*. The most relevant quantify is actually not the total force exerted on the sides of the fault, but rather the *force per unit area*, which is known as the *traction*. Since force is a vector, the traction will also be a vector.

If a net force **F** is acting on a planar surface that has area *A*, the traction vector **p** is defined as the ratio of the resultant force **F** to the surface area *A*:

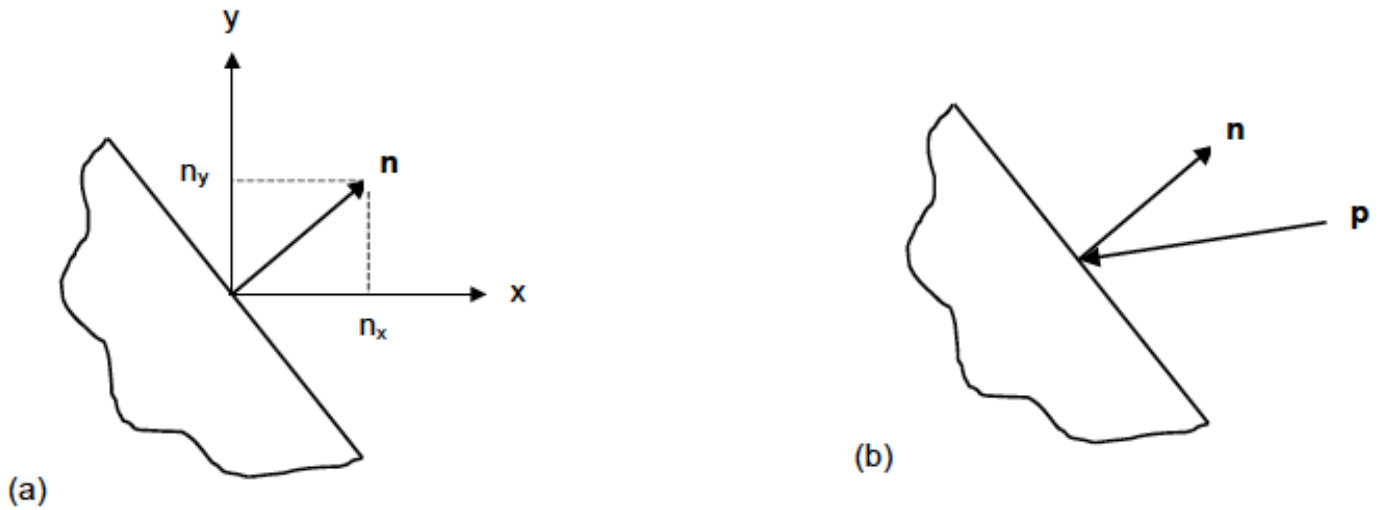
$$\mathbf{p}(\text{averaged over the area}) = \frac{1}{A}\mathbf{F}$$

This is a generalisation of the concept of fluid pressure. However, whereas a static fluid can only exert a force *normal* to a surface, the traction vector in a rock might also have a component that acts *tangentia*l to the surface.

We now introduce a sign convention that is widely used in rock mechanics. The component of the traction vector **p** in any given direction **r** is considered to be a *positive* number if the dot product **p•r** is *negative*. The reason for this sign convention will become clear after we discuss the concept of stress.

Traction Acting on a Plane

Consider now a specific planar surface of a rock, as in (a) below. This surface might be an actual exposed surface in a mine, tunnel, borehole, or outcrop, or it might be a fictitious surface in the interior of the rock mass.



This plane can be uniquely identified by its *outward unit normal vector* **n**, which is oriented normal to its surface, and points in the direction *away* from the rock.

With respect to some (x,y) coordinate system, **n** has components (n_x,n_y). Pythagoras's theorem tells us that $(n_x)^2 + (n_y)^2 = 1$. Note that **n** is *perpendicular* to the plane; it does not lie *within* the plane.

The traction vector **p** acting on this surface, as shown in (b) above, may act at an arbitrary angle with respect to **n**.

Cauchy's Theory of Stress

Since the traction may vary from point to point, it will generally depend on position, **x**. However, at any given point **x**, the traction will also, in general, be different on different planes that pass through that point. In other words, the traction will also be a function of **n**, the outward unit normal vector. The fact that **p** is a function of two vectors, the position vector **x** and the outward unit normal vector **n**, is mathematically awkward. Specifically, if we eventually want to develop a PDE that governs the deformation if the rock, we need to work with variables that are functions of **x**.

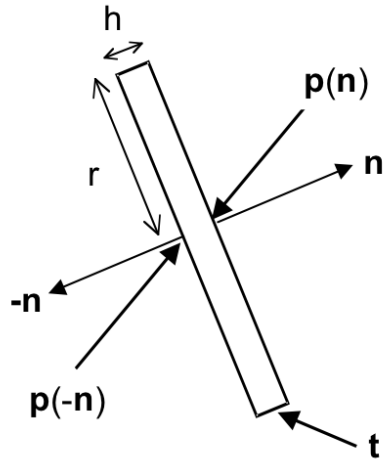
This difficulty is eliminated by appealing to the concept of stress, which was introduced in 1823 by the French civil engineer and mathematician Cauchy. The stress concept allows all possible traction vectors at a point to be represented by a single mathematical entity, called the *stress tensor*, that does not explicitly depend on the unit normal vector of any particular plane.

Cauchy's concept of stress was perhaps the most significant development in the field of continuum mechanics. Within a few decades of this development, a large number of problems in solid and fluid mechanics had been solved.

The keys to Cauchy's theory of stress are his so-called First and Second Laws, which will now be derived.

Cauchy's First Law

Consider a thin coin-shaped piece of rock, as seen in (a) at the right in a side view, and do a force balance.



(a)

The total force acting on the face with outward unit normal vector **n** is $\pi r^2\mathbf{p}(\mathbf{n})$, whereas the total force acting on the opposing face is $\pi r^2\mathbf{p}(-\mathbf{n})$. The total force acting on the outer rim is found by integrating the traction vector over the outer rim, and is given by $2\pi rh\mathbf{t}$, where **t** is the mean traction acting over the outer rim.

Performing a force balance on this thin slab of rock yields

$$\pi r^2\mathbf{p}(\mathbf{n}) + \pi r^2\mathbf{p}(-\mathbf{n}) + 2\pi rh\mathbf{t} = 0$$

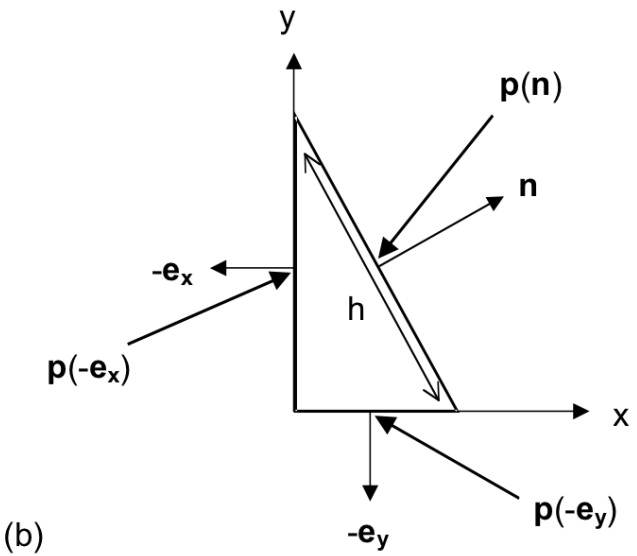
If we let the thickness *h* of this slab become very small, the third term drops out, and the condition for equilibrium becomes

$$\mathbf{p}(-\mathbf{n}) = -\mathbf{p}(\mathbf{n})$$

The above equation, known as *Cauchy's First Law*, essentially represents Newton's 3rd Law: if the material to the left of a given plane exerts a traction **p** on the material on its right, then the material on the right will exert a traction **–p** on the material to its left.

Cauchy's Second Law

Consider a triangular slab of rock, as in (b) at the right, with a uniform thickness w in the third (z) direction. Two faces of this slab have outward unit normal vectors that lie in the $-x$ and $-y$ directions, respectively, whereas the third face has an outward unit normal vector $\mathbf{n} = (n_x, n_y)$. The length of the face with o.u.n. vector $\mathbf{n}$ is h.



(b)

The length of the face that has o.u.n. vector $\mathbf{n}$ is h, and so its area is hwn_x. The traction vector on this face is $\mathbf{p}(-\mathbf{e_x})$, and so the total force acting on this face is $hwn_x\mathbf{p}(-\mathbf{e_x})$. Similar considerations show that the total force acting on the face with outward unit normal vector $-\mathbf{e_y}$ is $hwn_y\mathbf{p}(-\mathbf{e_y})$.

A force balance on this slab leads to

$$hwn_x\mathbf{p}(-\mathbf{e_x}) + hwn_y\mathbf{p}(-\mathbf{e_y}) + hw\mathbf{p}(\mathbf{n}) = 0$$

Canceling out the common terms hw, and utilizing Cauchy's first law, leads to *Cauchy's Second Law*:

$$\mathbf{p}(\mathbf{n}) = n_x\mathbf{p}(\mathbf{e_x}) + n_y\mathbf{p}(\mathbf{e_y})$$

As shown in *FoRM*, this law continues to hold if the rock is also subjected to a body force, such as that due to gravity.

The Stress Tensor

Now recall that traction is a *vector*, and therefore (in 2D) it will have two components. The components of a traction vector are denoted using two indices: the first to indicate the direction of the outward unit normal vector, and the second to indicate the component of the traction vector, *i.e.*,

$$\mathbf{p}(\mathbf{e_x}) = \begin{bmatrix} \tau_{xx} \\ \tau_{xy} \end{bmatrix} \qquad \mathbf{p}(\mathbf{e_y}) = \begin{bmatrix} \tau_{yx} \\ \tau_{yy} \end{bmatrix}$$

Recalling from the previous slide that $\mathbf{p}(\mathbf{n}) = n_x \mathbf{p}(\mathbf{e_x}) + n_y \mathbf{p}(\mathbf{e_y})$, we see that we can write this equation in matrix form as follows:

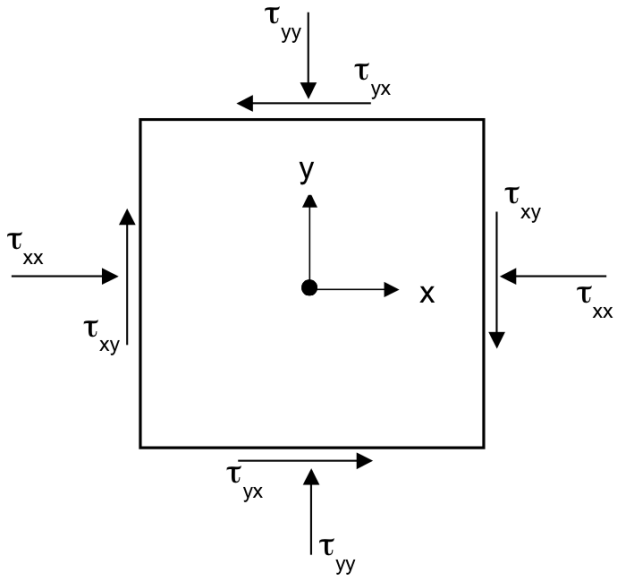
$$\mathbf{p}(\mathbf{n}) = n_x \begin{bmatrix} \tau_{xx} \\ \tau_{xy} \end{bmatrix} + n_y \begin{bmatrix} \tau_{yx} \\ \tau_{yy} \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{yx} \\ \tau_{xy} & \tau_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

The matrix on the right side is essentially the “stress tensor”, with one slight difference. The standard way of writing a matrix is for the first index to denote the *row*, and the second index to denote the *column*. Since *x* is the first index, and *y* is the second index, the matrix in the above equation is actually the transpose of the stress tensor/matrix $\boldsymbol{\tau}$, *i.e.,*

$$\mathbf{p}(\mathbf{n}) = \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \boldsymbol{\tau}^{\mathsf{T}} \mathbf{n}$$

Symmetry of the Stress Tensor

The physical significance of the stress tensor is traditionally illustrated by the cartoon shown at the right. Consider a two-dimensional square element of rock, whose faces are each perpendicular to one of the two coordinate axes.



Consider the right face of this cube, for which $\mathbf{n} = \mathbf{e}_x$. Hence, the traction in this face is given by

$$\mathbf{p}(\mathbf{n}) = \begin{bmatrix} \tau_{xx} & \tau_{yx} \\ \tau_{xy} & \tau_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{yx} \\ \tau_{xy} & \tau_{yy} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tau_{xx} \\ \tau_{xy} \end{bmatrix}$$

Another key point about the stress tensor is that it is always *symmetric*. This can be seen by taking a moment balance about the centre of the cube. The details can be found in the *FoRM* textbook, but it should be obvious that the cube will only be in rotational equilibrium if $\tau_{xy} = \tau_{yx}$.

Hence, it is always the case that $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{T}}$. Consequently, the relationship between traction and stress can be written as

$$\mathbf{p} = \boldsymbol{\tau}\,\mathbf{n}$$

Principal Stresses, 1

The equation, $\mathbf{p} = \boldsymbol{\tau}\mathbf{n}$ gives the traction vector $\mathbf{p}$ that acts on a plane whose outward unit normal vector is $\mathbf{n}$, in terms of stress tensor, $\boldsymbol{\tau}$. An interesting question to ask is: for a given stress tensor, are there any planes on which the traction vector acts *normal* to the plane, with no tangential component?

If $\mathbf{p}$ is normal to the plane, then it is parallel to $\mathbf{n}$, and so $\mathbf{p} = \sigma\mathbf{n}$, where σ is a scalar. But we also know that $\mathbf{p} = \boldsymbol{\tau}\mathbf{n}$. Equating these two expressions for $\mathbf{p}$ gives $\boldsymbol{\tau}\mathbf{n} = \sigma\mathbf{n}$.

The equation $\boldsymbol{\tau}\mathbf{n} = \sigma\mathbf{n}$, where $\boldsymbol{\tau}$ is a matrix, $\mathbf{n}$ is a vector, and σ is a scalar, is a standard algebraic eigenvalue problem. You may already know how to solve this type of problem; if not, we will now develop the solution procedure.

Recall that we can write $\mathbf{n} = \mathbf{I}\mathbf{n}$, where $\mathbf{I}$ is the identity matrix. So, we can write $\sigma\mathbf{n}$ as $\sigma\mathbf{I}\mathbf{n}$. Hence,

$$\boldsymbol{\tau}\mathbf{n} = \sigma\mathbf{I}\mathbf{n}, \quad \text{so} \quad \boldsymbol{\tau}\mathbf{n} - \sigma\mathbf{I}\mathbf{n} = \mathbf{0}, \quad \text{and hence} \quad (\boldsymbol{\tau} - \sigma\mathbf{I})\mathbf{n} = \mathbf{0}$$

We know $\boldsymbol{\tau}$, but at this point we don't know σ or $\mathbf{n}$. The vector $\mathbf{n} = \mathbf{0}$ will satisfy $(\boldsymbol{\tau} - \sigma\mathbf{I})\mathbf{n} = 0$. But $\mathbf{n}$ must be a unit vector, whereas the vector $\mathbf{0}$ has zero length, so $\mathbf{n} = \mathbf{0}$ is not an admissible solution.

Principal Stresses, 2

Recall that in order for the equation **Ax** = **0** to have a *non-zero* solution for **n**, the determinant of **A** must be zero. (Proof: Assume that **Ax** = **0** has a non-zero solution **n**. Now assume that det **A** ≠ 0. If so, then $\mathbf{A}^{-1}$ exists, and so $\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$, which contradicts our assumption that **n** is non-zero. Hence, the assumption that det **A** ≠ 0 must have been false. Therefore, det**A** = **0**. QED)

Therefore, in order to have non-zero solutions **n** for the equation $(\boldsymbol{\tau} - \sigma\mathbf{I})\mathbf{n} = \mathbf{0}$, we need $\det(\boldsymbol{\tau} - \sigma\mathbf{I}) = 0$. Let's set $\det(\boldsymbol{\tau} - \sigma\mathbf{I}) = 0$, and see where this leads us. Recalling that **I** has the property that its diagonal terms are 1, and its off-diagonal terms are 0, and also recalling that $\tau_{yx} = \tau_{xy}$, we have:

$$\det(\boldsymbol{\tau} - \sigma\mathbf{I}) = \det\begin{bmatrix} \tau_{xx} - \sigma & \tau_{xy} \\ \tau_{yx} & \tau_{yy} - \sigma \end{bmatrix} = (\tau_{xx} - \sigma)(\tau_{yy} - \sigma) - (\tau_{xy})^2$$

$$= \sigma^2 - (\tau_{xx} + \tau_{yy})\sigma + [\tau_{xx}\tau_{yy} - (\tau_{xy})^2] = 0.$$

So, we see that setting $\det(\boldsymbol{\tau} - \sigma\mathbf{I}) = 0$ leads to a quadratic equation for σ, which is called the "characteristic equation". There will always be two solutions for σ, although in some special cases the two roots of the quadratic equation might coalesce and become equal.

Principal Stresses, 3

Using the standard solution method for a quadratic equation, we find

$$\sigma_1 = \frac{1}{2}(\tau_{xx} + \tau_{yy}) + \text{sqrt}\left[(\tau_{xy})^2 + \frac{1}{4}(\tau_{xx} - \tau_{yy})^2\right]$$

$$\sigma_2 = \frac{1}{2}(\tau_{xx} + \tau_{yy}) - \text{sqrt}\left[(\tau_{xy})^2 + \frac{1}{4}(\tau_{xx} - \tau_{yy})^2\right]$$

These two values are known as the *principal stresses*. It is traditional to label the larger one as σ_1, and the smaller one as σ_2, and refer to them as the *maximum* and *minimum* principal stresses. Since the terms inside the large square brackets are always non-negative, these two principal stresses will be real numbers, not complex numbers.

Now that we have found the two principal stresses ("eigenvalues" in mathematical jargon), we need to find the *eigenvector* **n** that corresponds to each of these two eigenvalues. Since the vector $\mathbf{n}_1$ represents the outward unit normal vector of plane on which the normal traction is equal to σ_1, we call this vector the *direction of the maximum principal stress*. Similarly, $\mathbf{n}_2$ is the *direction of the minimum principal stress* σ_2.

Principal Stresses, 4

We could proceed in general, to find $\mathbf{n}_1$ and $\mathbf{n}_2$ in terms of the three stress components $\{\tau_{xx}, \tau_{yy}, \tau_{xy}\}$. This is done in detail in *FoRM4*. But the expressions are messy, and it is much easier and more instructive to look at a specific case.

Example: Consider the following stress matrix, with respect to the (x,y) coordinate system, in units of MPa:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix} = \begin{bmatrix} 30 & 20 \\ 20 & 30 \end{bmatrix}$$

From the equations on the previous slide, we can find the two principal stresses:

$$\sigma_1 = \frac{1}{2}(\tau_{xx} + \tau_{yy}) + \text{sqrt}\left[(\tau_{xy})^2 + \frac{1}{4}(\tau_{xx} - \tau_{yy})^2\right]$$

$$= \frac{1}{2}(30 + 30) + \text{sqrt}\left[(20)^2 + \frac{1}{4}(30 - 30)^2\right] = 50$$

$$\sigma_2 = \frac{1}{2}(\tau_{xx} + \tau_{yy}) - \text{sqrt}\left[(\tau_{xy})^2 + \frac{1}{4}(\tau_{xx} - \tau_{yy})^2\right]$$

$$= \frac{1}{2}(30 + 30) - \text{sqrt}\left[(20)^2 + \frac{1}{4}(30 - 30)^2\right] = 10$$

Principal Stresses, 5

Now, let's calculate the eigenvector that corresponds to $\sigma_1 = 50$. We start with $(\boldsymbol{\tau} - \sigma_1 \mathbf{I})\mathbf{n} = \mathbf{0}$. If we write out these two equations explicitly, the first row becomes

$$(\tau_{xx} - \sigma_1)n_x + \tau_{xy}n_y = 0 \Rightarrow (30 - 50)n_x + 20n_y = 0 \Rightarrow -20n_x + 20n_y = 0 \Rightarrow n_x = n_y$$

The second equation becomes

$$\tau_{yx}n_x + (\tau_{yy} - \sigma_1)n_y = 0 \Rightarrow 20n_x + (30 - 50)n_y = 0 \Rightarrow 20n_x - 20n_y = 0 \Rightarrow n_x = n_y$$

Note that the two equations have degenerated into the same equation, $n_x = n_y$, and therefore did not supply us with two independent equations that we would need to uniquely solve for n_x and n_y. This always happens in an eigenvalue problem: the equations are no longer "linearly independent", and so we can find the *direction* of the eigenvector, but not its *magnitude*.

However, if we recall that **n** must be a *unit vector*, this helps us to find specific values for n_x and n_y. Since $n_x = n_y$, the length of **n** is given by

$$|\mathbf{n}| = \sqrt{(n_x)^2 + (n_y)^2} = \sqrt{(n_x)^2 + (n_x)^2} = \sqrt{2(n_x)^2} = \sqrt{2}\,|n_x| = 1$$

which tells us that $n_x = n_y = 1/\sqrt{2}$. Hence, the first principal stress direction is

$$\mathbf{n}_1 = [1/\sqrt{2},\ 1/\sqrt{2}]$$

This vector is rotated by 45° (anti-clockwise) from the *x*-axis.

Principal Stresses, 6

Now, let's calculate the eigenvector that corresponds to $\sigma_2 = 10$. We start with $(\boldsymbol{\tau} - \sigma_2 \mathbf{I})\mathbf{n} = \mathbf{0}$. If we write out these two equations explicitly, the first row becomes

$$(\tau_{xx} - \sigma_1)n_x + \tau_{xy}n_y = 0 \Rightarrow (30 - 10)n_x + 20n_y = 0 \Rightarrow 20n_x + 20n_y = 0 \Rightarrow n_x = -n_y$$

We know from the discussion on the previous slide that the second equation would give us this same result, $n_x = -n_y$, so we don't need to consider it further.

Recalling again that **n** must be a *unit vector*, we see that

$$|\mathbf{n}| = \sqrt{(n_x)^2 + (n_y)^2} = \sqrt{(n_x)^2 + (-n_x)^2} = \sqrt{2(n_x)^2} = \sqrt{2}\,|n_x| = 1$$

which tells us that $n_x = -1/\sqrt{2}$, and $n_y = 1/\sqrt{2}$. Hence, the second principal stress direction is

$$\mathbf{n}_2 = [-1/\sqrt{2},\ 1/\sqrt{2}]$$

This vector is rotated by 135^o (anti-clockwise) from the *x*-axis.

Note that $\mathbf{n}_1 \bullet \mathbf{n}_2 = (-1/\sqrt{2})(1/\sqrt{2}) + (1/\sqrt{2})(1/\sqrt{2}) = 0$, which means that the two eigenvectors are *orthogonal* to each other. This will *always* be the case; it follows mathematically from the fact that the stress matrix is *symmetric.*

Note also that we could have chosen $n_x = +1/\sqrt{2}$, and $n_y = -1/\sqrt{2}$. We chose our signs so that $\mathbf{n}_2$ is rotated by 90^o *anticlockwise* from $\mathbf{n}_1$.

Mohr's Circle, 1

We could study the manner in which the traction vector acting on a plane varies with the orientation of the plane, by using the equations derived above. But it is simpler and more visually instructive to use a graphical construction known as Mohr's circle.

First assume that we start out with a coordinate system oriented with the *x*-axis pointing east, and the *y*-axis pointing north, for example. In this coordinate system, there will generally be four non-zero components in the 2×2 stress matrix.

However, if we instead change to a *new* coordinate system in which the principal stress directions $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit coordinate vectors, there will be no shear stresses appearing in the stress matrix! In other words, in this new coordinate system, the stress matrix has a much simpler form:

$$\boldsymbol{\tau} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

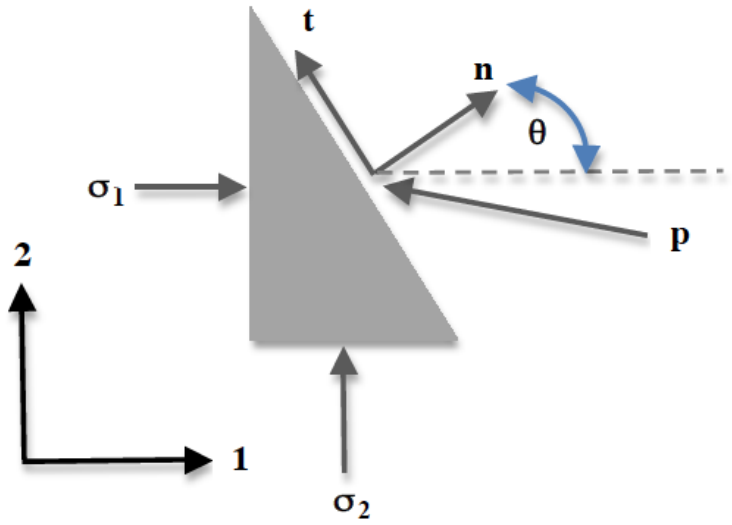
Now, let's calculate the traction on a plane that has an outward unit normal vector $\mathbf{n}$, relative to the principal coordinate system. Recall that $\mathbf{p} = \boldsymbol{\tau}\mathbf{n}$, so

$$\mathbf{p} = \boldsymbol{\tau}\,\mathbf{n} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 n_1 \\ \sigma_2 n_2 \end{bmatrix}$$

Mohr's Circle, 2

We now make the notation more "geometric", by noting that $\mathbf{n}_1 = \cos\theta$, and $\mathbf{n}_2 = \sin\theta$, so that

$$\mathbf{p} = \begin{bmatrix} \sigma_1 \cos\theta \\ \sigma_2 \sin\theta \end{bmatrix}$$



In general, **p** will have a normal component and a shear component. Let's first calculate the normal component, which we will denote by σ.

The value of σ is given by the projection of **p** onto **n**, *i.e.,* $\mathbf{p} \cdot \mathbf{n}$. So:

$$\sigma = \mathbf{p} \cdot \mathbf{n} = \begin{bmatrix} \sigma_1 \cos\theta \\ \sigma_2 \sin\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta$$

We now make use of the following trigonometric identities

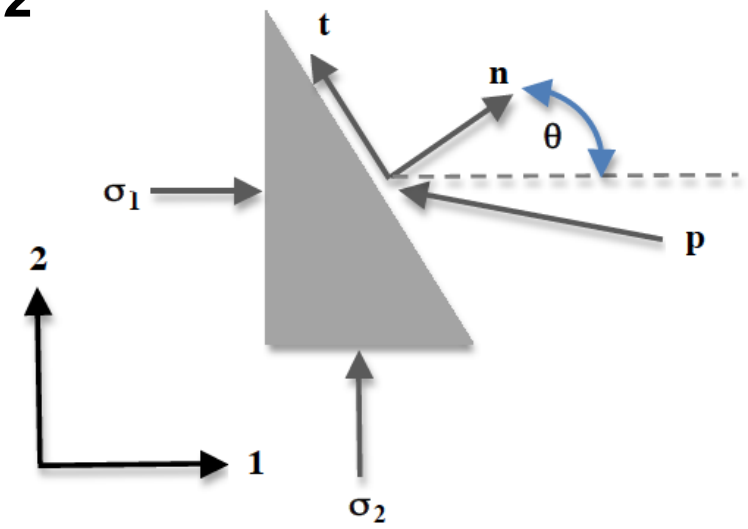
$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta, \quad \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

and thereby rewrite the expression for σ as

$$\sigma = \sigma_1\left[\frac{1}{2} + \frac{1}{2}\cos 2\theta\right] + \sigma_2\left[\frac{1}{2} - \frac{1}{2}\cos 2\theta\right] = \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta$$

Mohr's Circle, 2

We now perform a similar calculation to find τ, the shear component of **p**, which lies parallel to the **t** vector, that lies *along* the plane. Note that



$$\mathbf{t} = \begin{bmatrix} \cos(\theta + 90\circ) \\ \sin(\theta + 90\circ) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

The value of τ is given by the projection of **p** onto **t**, *i.e.,*

$$\tau = \mathbf{p} \cdot \mathbf{t} = \begin{bmatrix} \sigma_1 \cos\theta \\ \sigma_2 \sin\theta \end{bmatrix} \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = -\sigma_1 \sin\theta\cos\theta + \sigma_2 \sin\theta\cos\theta$$

$$= -(\sigma_1 - \sigma_2)\sin\theta\cos\theta = -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta$$

Recall from the previous slide that

$$\sigma = \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta$$

These expressions for σ and τ can be plotted on a graph known as a Mohr diagram, after the German engineering professor Otto Mohr, who popularised the use of this graphical construction in the 1880s.

Mohr's Circle, 3

The two traction components that act on some plane who outward unit normal vector makes an angle θ with the σ_1 direction are therefore given by:

$$\sigma = \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \qquad \tau = -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta$$

Aside from the factor of 2 in the trigonometric terms, and the minus sign in the equation for τ, these equations are essentially the equations of a *circle*.

Recall that the equation of a circle in the $\{\sigma,\tau\}$ plane will have the general form

$$\sigma = \sigma_o + R\cos\beta, \quad \tau = \tau_o + R\sin\beta$$

where $\{\sigma_o,\tau_o\}$ is the centre of the circle, R is its radius, and β is the angle of rotation from the σ-axis. Comparison of the above sets of equations shows that the graph of τ vs. σ will be a circle that has the following properties:

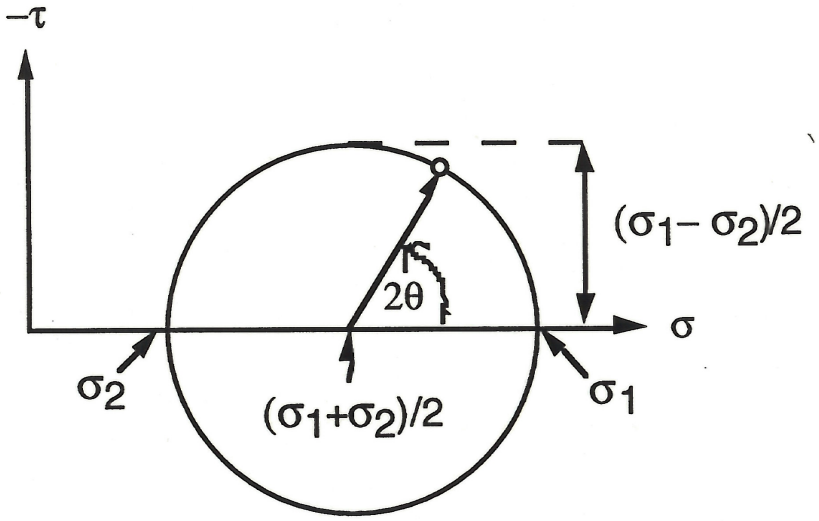
(a) Its centre will lie on the σ-axis, at a point given by $\sigma_o = (\sigma_1+\sigma_2)/2$

(b) Its radius will be given by $R = (\sigma_1-\sigma_2)/2$

(c) The angle β that parameterises the circle will be equal to *two-times* the angle of rotation θ of the outward unit normal vector of the plane

(d) The vertical axis is actually $-\tau$ instead of τ.

Mohr's Circle, 4



A generic Mohr's circle is show on the left.

Many interesting facts can be seen directly from a Mohr's circle. For example:

Since Mohr's circle intersects the σ-axis at the two points $\sigma = \sigma_1$ and $\sigma = \sigma_2$, we see that σ_1 is the maximum possible normal stress that acts on any plane, and σ_2 is the minimum normal stress acting on any plane. This is another sense in which the principal stresses are "special".

The maximum shear stress (in absolute value; forgetting about the signs) that acts along any plane is equal to $(\sigma_1 - \sigma_2)/2$. This occurs at the top and bottom of the Mohr's circle. At the top, $2\theta = 90°$, and so $\theta = 45°$, which corresponds to a plane that has a normal vector that bisects the σ_1 and σ_2 directions. The other plane of maximum shear stress occurs where $2\theta = 270°$, or $\theta = 135°$. This plane has a normal vector that bisects the σ_1 and $-\sigma_2$ directions.

Much more discussion of Mohr's circle can be found in Chapter 2 of *FoRM*.

Problems for Lecture 6

Problem 6.1. Consider the following stress matrix, with respect to the (x,y) coordinate system, in units of MPa:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}$$

Find the maximum and minimum principal stresses, and their associated directions. Verify that the two principal stress directions are perpendicular to each other.

Problem 6.2. The principal stresses can be found without using any knowledge of matrix algebra or determinants. First, write out explicitly the two equations embodied by matrix equation $(\boldsymbol{\tau} - \sigma\mathbf{I})\mathbf{n} = \mathbf{0}$. Now, try to solve these equations using Gaussian elimination, and you should arrive at the same characteristic equation that we found on slide 130, using determinants.

Problem 6.3. Consider a location in an underground ore body at which the two principal horizontal stresses are equal to σ_1 = 60 MPa, and σ_2 = 30 MPa. Draw a Mohr's diagram on a sheet of graph paper, to find the normal and shear tractions that act on a plane whose outward unit normal vector is rotated by 60^o from the σ_1-direction.

Imperial College MSc in
GeoEnergy with Machine Learning and Data Science

GEMS 3: Geomechanics and Pressure Transient Analysis

Lecture 7:
Deformation, Strain, and Elasticity

Prof. Robert Zimmerman

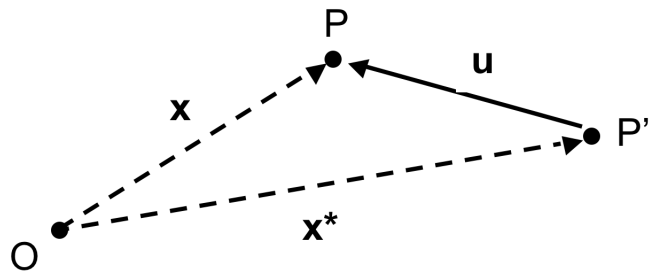
March 2024

Deformation and Displacement

In the Lecture 6, we defined and discussed the stress tensor, which is the mathematical entity that we use to quantify the forces that act within a rock.

These stresses cause the rock to deform. To quantify the deformation of the rock, let vector **x** = (*x*,*y*,*z*) denote the initial position of a particle of rock, and let vector **u** = (*u*,*v*,*w*) be the displacement of this particle of rock, so that vector **x*** = **x** − **u** = (*x*−*u*,*y*−*v*,*z*−*w*) is the new location of this particle of rock after it has deformed.

The minus sign is used so as to be consistent with the convention that positive displacements correspond to a rock that is *compressed*.

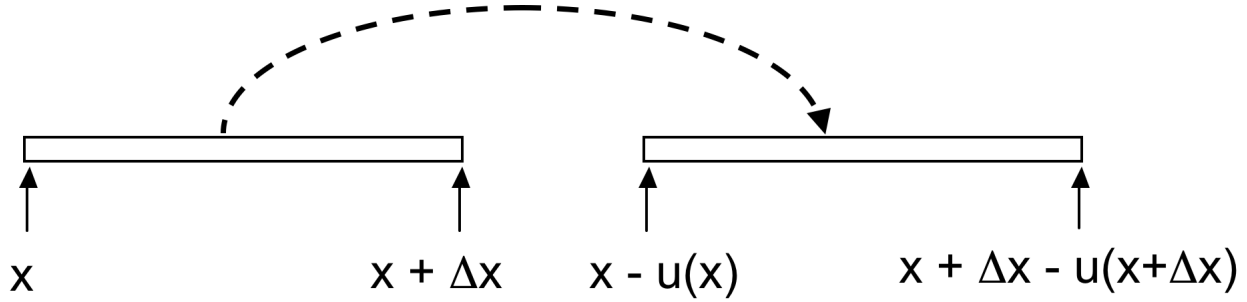


Note that the general process of a rock changing its location or shape is called *deformation*, and the vector that quantifies the motion of a piece of rock is called the *displacement.*

Normal Strains

The stresses acting on the rock are not directly related to the displacement, but rather to the *relative* displacement of particles of rock that are initially close to each other. This relative displacement is quantified by the *strain*.

Consider a short element of rock that initially lies along the *x*-axis, with its left edge at *x*, and it right edge at $x + \Delta x$:



The normal strain of this element is the *fractional decrease in its length*, *i.e.*,

$$\varepsilon = \frac{L - L^*}{L} = \frac{\Delta x - \{[x + \Delta x] - u(x + \Delta x) - [x - u(x)]\}}{\Delta x} = \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

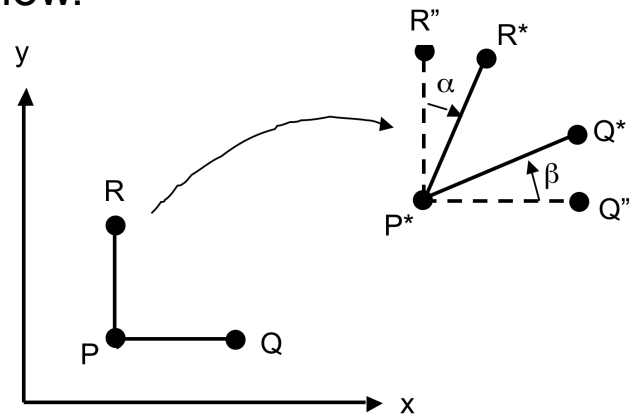
If we let Δx become very small, the normal strain ε_{xx} at point *x* is defined as

$$\varepsilon_{xx}(x) = \lim_{L \to 0} \frac{L - L^*}{L} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} \equiv \frac{du}{dx}$$

Shear Strains

The *normal strains* are the diagonal components of the strain tensor.

The off-diagonal components are called the *shear strains*. The shear strain ε_{xy} is defined as one-half of the increase in the angle initially formed by two small line segments that initially lie parallel to the *x* and *y* axes, *i.e.*, $\varepsilon_{xy} = (\alpha + \beta)/2$, with reference to the figure below.



As shown in Section 2.10 of *FoRM4,* using only simple calculus and trigonometry, the shear strain ε_{xy} is related to the displacements by

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

The other two shear strains can be defined in an analogous manner*, i.e.,*

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Strain Tensor

Taken together, the normal and shear strains form the *strain tensor*:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

The three-dimensional version of the strain tensor follows the same pattern.

Being (necessarily) a symmetric tensor, the strain tensor will always have three principal values, aligned along three mutually orthogonal axes, *etc.*

In vector-matrix notation, the strain tensor can be expressed as:

$$\boldsymbol{\varepsilon} = \text{sym}(\nabla \mathbf{u}) \equiv \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathsf{T}}\right]$$

In polar co-ordinates, the displacement components in the r and θ directions can be denoted by u and v, but the strain components have different expressions than they do in Cartesian co-ordinates:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)$$

Stress-Strain Relationships for Rocks

Rocks can exhibit many different types of stress-strain behavior. *Elastic* behavior can be defined as that in which the current state of strain depends *only* on the current state of stress. Elastic behavior can be linear, or nonlinear, and can also exhibit hysteresis:

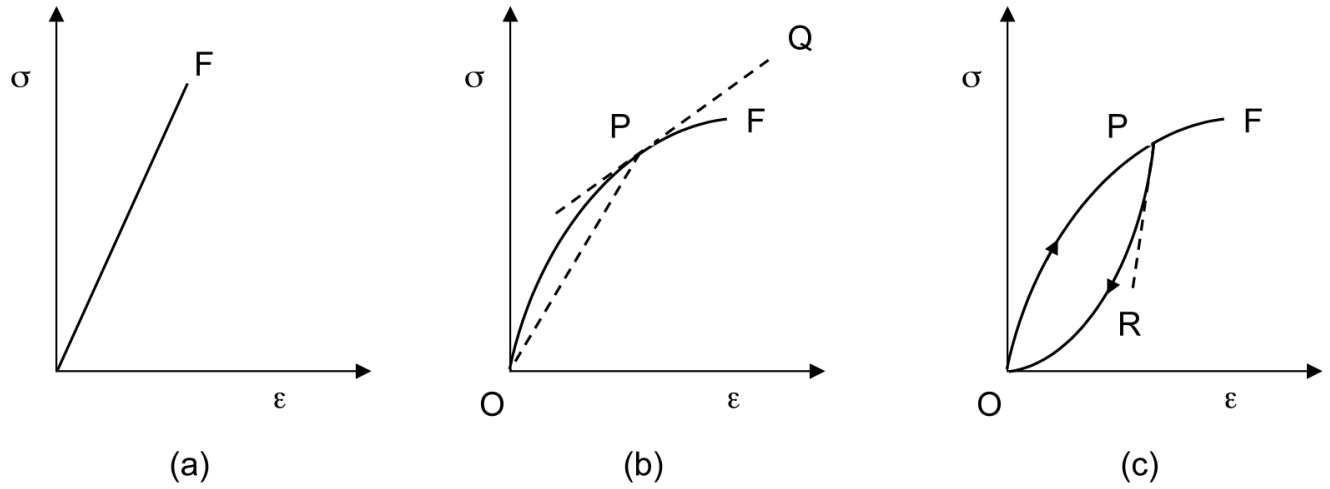


Fig. 4.2.2. (a) Linearly elastic behaviour, with failure at *F*. (b) Nonlinearly elastic behaviour: slope of *OP* is the secant modulus, slope of *PQ* is the tangent modulus. (c) Hysteretic material: unloading modulus at *P* is given by the slope of *PR*.

As discussed in detail in Chapter 9 of *FoRM4*, in *plastic* behavior, a permanent strain will exist even if the stresses return to zero. In *viscoelastic* behavior, the strain depends not only on the stress, but also on the rate of change of the stress. In this module we will focus on the most common model for the stress-strain behavior of a rock: *linear elasticity*.

Hooke's Law for Linear Elastic Behavior, 1

In linear elastic behavior, the stress tensor is linearly related to the strain tensor. This type of linear elastic relationship is referred to as "Hooke's law".

Let's focus first on isotropic rocks, in which "the behavior is the same in all directions". Hooke's law for isotropic rocks can be written in tensor notation as

$$\boldsymbol{\tau} = \lambda \,\mathrm{trace}(\boldsymbol{\varepsilon})\mathbf{I} + 2G\boldsymbol{\varepsilon}$$

where λ is called the *Lamé modulus* (named after the French mathematician Gabriel Lamé, who wrote the first textbook on elasticity in 1854), G is the *shear modulus*, and the trace of the strain tensor, trace($\boldsymbol{\varepsilon}$), is defined as the sum of the diagonal components, *i.e.,* trace($\boldsymbol{\varepsilon}$) = $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$.

When written in explicit component form, Hooke's law takes the following form:

$$\tau_{xx} = (\lambda + 2G)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz}$$

$$\tau_{yy} = \lambda\varepsilon_{xx} + (\lambda + 2G)\varepsilon_{yy} + \lambda\varepsilon_{zz}$$

$$\tau_{zz} = \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda + 2G)\varepsilon_{zz}$$

$$\tau_{xy} = 2G\varepsilon_{xy}, \qquad \tau_{xz} = 2G\varepsilon_{xz}, \qquad \tau_{yz} = 2G\varepsilon_{yz}$$

Hooke's Law for Linear Elastic Behavior, 2

If we want to express the strains as functions of the stresses, Hooke's law can be written as follows, where E is *Young's modulus*, and ν is *Poisson's ratio*:

$$\varepsilon_{xx} = \frac{1}{E}\tau_{xx} - \frac{\nu}{E}\tau_{yy} - \frac{\nu}{E}\tau_{zz} = \frac{1}{E}[\tau_{xx} - \nu(\tau_{yy} + \tau_{zz})]$$

$$\varepsilon_{yy} = \frac{1}{E}\tau_{yy} - \frac{\nu}{E}\tau_{xx} - \frac{\nu}{E}\tau_{zz} = \frac{1}{E}[\tau_{yy} - \nu(\tau_{xx} + \tau_{zz})]$$

$$\varepsilon_{zz} = \frac{1}{E}\tau_{zz} - \frac{\nu}{E}\tau_{xx} - \frac{\nu}{E}\tau_{yy} = \frac{1}{E}[\tau_{zz} - \nu(\tau_{xx} + \tau_{yy})]$$

$$\tau_{xy} = 2G\varepsilon_{xy}, \quad \tau_{xz} = 2G\varepsilon_{xz}, \quad \tau_{yz} = 2G\varepsilon_{yz}$$

The various elastic moduli can be expressed in terms of each other, as follows:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{2G\nu}{(1-2\nu)}, \quad \nu = \frac{\lambda}{2(\lambda+G)}, \quad K = \frac{E}{3(1-2\nu)},$$

where K, the bulk modulus, is the multiplicative inverse of the compressibility.

For a material to be stable, it must be the case that $K > 0$ and $G > 0$, which implies that ν must lie between -1 and 0.5. However, although artificial materials have been created that have negative Poisson ratios, the Poisson ratios of all known natural materials, including rocks, are *positive*.

Hooke's Law for Anisotropic Rocks

Many rocks are elastically anisotropic, in the sense that, for example, the Young's modulus has different values in different directions.

There are many types of elastic anisotropy. The most important type of anisotropy, particularly in petroleum reservoirs, is the case of so-called *transverse isotropy*, where the rock possesses a z-axis of rotational symmetry (usually vertical), and is isotropic within the (x,y) plane normal to this axis.

A commonly used notation for transversely isotropic materials is for the elastic coefficients pertaining to the (x,y) plane of isotropy to be referred to as $\{E,G,\nu\}$, with $E = 2G(1+\nu)$, and those involving the z-direction denoted by $\{E',G',\nu'\}$, where it is *not* necessarily true that $E' = 2G'(1+\nu')$.

The stress-strain relations for a transversely isotropic rock can be written as

$$\varepsilon_{xx} = \frac{1}{E}\tau_{xx} - \frac{\nu}{E}\tau_{yy} - \frac{\nu'}{E'}\tau_{zz}, \quad \varepsilon_{zz} = -\frac{\nu_{zx}}{E_z}\tau_{xx} - \frac{\nu_{zx}}{E_z}\tau_{yy} + \frac{1}{E_z}\tau_{zz}, \quad \varepsilon_{yy} = -\frac{\nu}{E}\tau_{xx} + \frac{1}{E}\tau_{yy} - \frac{\nu'}{E'}\tau_{zz},$$

$$\varepsilon_{yz} = \frac{1}{2G_{xz}}\tau_{yz}, \quad \varepsilon_{xz} = \frac{1}{2G_{xz}}\tau_{xz}, \quad \varepsilon_{xy} = \frac{(1+\nu_{yx})}{E_{xy}}\tau_{xy}.$$

Note that whereas an isotropic rock has *two* independent elastic parameters, a transversely isotropic rock has *five* independent elastic parameters.

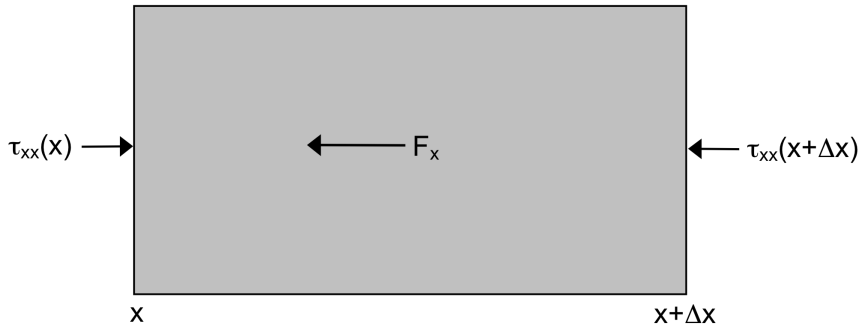
Navier Equations of Equilibrium (or Motion)

Thus far, we have learned how to represent the forces acting within a rock by a mathematical entity called the stress tensor, we have learned how to quantify the deformation of the rock with an entity called the strain tensor, and we have learned that, during linear elastic behavior of a rock, the stress is related to the strain by Hooke's law.

In order to solve a full rock mechanics problem in the subsurface, we need a set of partial differential equations that govern the way in which the stresses and strains vary in time and space. These equations are called the Navier equations, and can be derived by applying Newton's laws of motion to the rock.

We will derive these equations in the one-dimensional context. This will simplify the mathematical details, but still demonstrate the basic ideas involved. Consider a small piece of rock, having cross-sectional area A, lying between x and $x + \Delta x$, as shown below (right):

Note the sign convention used for the body-force F_x, which is considered to be *positive* if it points in the *negative* x-direction. This is done to be consistent with the *compression is positive* rule.



Navier Equations in One Dimension

This piece of rock is acted upon by surface forces acting along its two faces, and by a generic body force F that acts throughout the interior of the body. We now simply apply Newton's law of motion to this piece of rock.

The net surface force acting on the left face of this rock is $A\tau_{xx}(x)$. The net surface force acting on the right face is $-A\tau_{xx}(x+\Delta x)$. If F_x is the body force per unit mass, then the total body force is $-\rho F_x A\Delta x$. The sum of these three forces must equal the mass times the acceleration of this piece of rock:

$$A\tau_{xx}(x,t) - A\tau_{xx}(x+\Delta x,t) - \rho F_x A\Delta x = -\rho A\Delta x \frac{\partial^2 u}{\partial t^2}$$

Now divide both sides by $A\Delta x$, and take the limit as Δx goes to 0, yielding

$$\frac{\partial \tau_{xx}}{\partial x} + \rho F_x = \rho \frac{\partial^2 u}{\partial t^2}$$

We now recall the 1-D version of Hooke's law,

$$\tau_{xx} = E\varepsilon_{xx} = E\frac{\partial u}{\partial x}$$

and insert it into the above PDE, to obtain:

$$E\frac{\partial^2 u}{\partial x^2} + \rho F_x = \rho \frac{\partial^2 u}{\partial t^2}$$

Navier Equations in Three Dimensions, 1

We can see that the left side of this equation contains second derivatives of the displacement with respect to the space variable, multiplied by the elastic moduli, and the right side contains the second derivative of the displacement with respect to time, multiplied by the density.

If we retain the time derivative, this equation governs the dynamic motion of the rock, for example during seismic wave propagation. If the time derivative is zero, the equation governs the static equilibrium of a rock mass.

The Navier equations in 3D are derived in Section 5.4 of *FoRM4*. The basic idea is the same: apply Newton's law of motion, and then use Hooke's law to express the stresses in term of the strains – *i.e.*, in terms of the partial derivatives of the displacements.

Ignoring the acceleration terms, the 3D Navier equations can be written in vector form as follows:

$$(\lambda + G)\nabla(\nabla \cdot \mathbf{u}) + G\nabla^2\mathbf{u} + \rho\mathbf{F} = \mathbf{0}$$

where ∇ is the gradient operator.

The Navier equations are a set of three *coupled* PDEs in three unknowns, the three components of the displacement vector, **u** = (*u*,*v*,*w*).

Navier Equations in Three Dimensions, 2

If we write out the Navier equations explicitly, without using vector notation, they will take the following form:

$$\lambda\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}\right)+G\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho F_x=0$$

$$\lambda\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\partial z}\right)+G\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho F_y=0$$

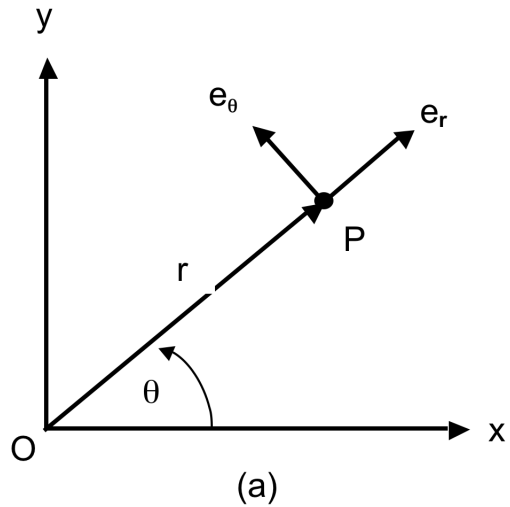
$$\lambda\left(\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 v}{\partial z\partial y}+\frac{\partial^2 w}{\partial z^2}\right)+G\left(\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 v}{\partial z\partial y}+\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho F_z=0$$

These equations are difficult to solve, in part because they are *coupled*, in the sense that all three unknowns, (u,v,w), appear in each of the three equations. Nevertheless, many analytical methods have been developed over the past 200 years, which have allowed these equations to be solved in various geometries and with various boundary conditions. Chapter 8 of *FoRM4* discusses some of these methods.

In the next lecture, we will present the solutions to a few important rock mechanics problems.

Elasticity Equations in Cylindrical Co-ordinates, 1

For problems involving boreholes or circular tunnels, it is convenient to use a cylindrical co-ordinate system, which is a polar co-ordinate system in the {*x*,*y*} plane, with a Cartesian *z*-axis appended, normal to the {*x*,*y*} plane.



(a)

The derivation of the elasticity equations in cylindrical co-ordinates can be found in Chapter 5 of FORM4. These equations are summarised below.

Strain-displacement equations:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}, \qquad \varepsilon_v = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}$$

$$\varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right), \quad \varepsilon_{z\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z}\right), \quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)$$

Stress-strain equations (note: take the same form as in Cartesian co-ordinates):

$$\tau_{rr} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 2G\varepsilon_{rr} \qquad \tau_{r\theta} = 2G\varepsilon_{r\theta}$$

$$\tau_{\theta\theta} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 2G\varepsilon_{\theta\theta} \qquad \tau_{rz} = 2G\varepsilon_{rz}$$

$$\tau_{zz} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 2G\varepsilon_{zz} \qquad \tau_{\theta z} = 2G\varepsilon_{\theta z}$$

Elasticity Equations in Cylindrical Co-ordinates, 2

Equations of elastic equilibrium, in terms of stresses:

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r} + \rho F_z = 0$$

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + \rho F_r = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{2\tau_{r\theta}}{r} + \rho F_\theta = 0$$

Navier equations of equilibrium:

$$(\lambda + G)\frac{\partial \varepsilon_V}{\partial r} + G\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} + \frac{\partial^2 u}{\partial z^2}\right) + \rho F_r = 0$$

$$(\lambda + G)\frac{1}{r}\frac{\partial \varepsilon_V}{\partial \theta} + G\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} + \frac{\partial^2 v}{\partial z^2}\right) + \rho F_\theta = 0$$

$$(\lambda + G)\frac{\partial \varepsilon_V}{\partial z} + G\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho F_z = 0$$

Problems for Lecture 7

Problem 7.1. Invert the equation $\boldsymbol{\tau} = \lambda \text{trace}(\boldsymbol{\varepsilon}) + 2G\boldsymbol{\varepsilon}$, which expresses the stresses in terms of the strains, to find the analogous tensor equation that expresses the strains in terms of the stresses. Then, write out the six resulting stress-strain equations explicitly.

Hint: start by taking the trace of both sides of the equation $\boldsymbol{\tau} = \lambda \text{trace}(\boldsymbol{\varepsilon}) + 2G\boldsymbol{\varepsilon}$, to find an expression for $\text{tr}(\boldsymbol{\tau})$.

Problem 7.2. The result of Problem 7.1 will be an equation that contains λ and G as the two elastic parameters. Equate this expression to the forms of the stress-strain equation shown on the top of slide 148, and thereby derive the relations between λ, G, E, and ν.

Imperial College MSc in
GeoEnergy with Machine Learning and Data Science

GEMS 3: Geomechanics and Pressure Transient Analysis

Lecture 8:
Failure and Fracture of Rock

Prof. Robert Zimmerman

March 2024

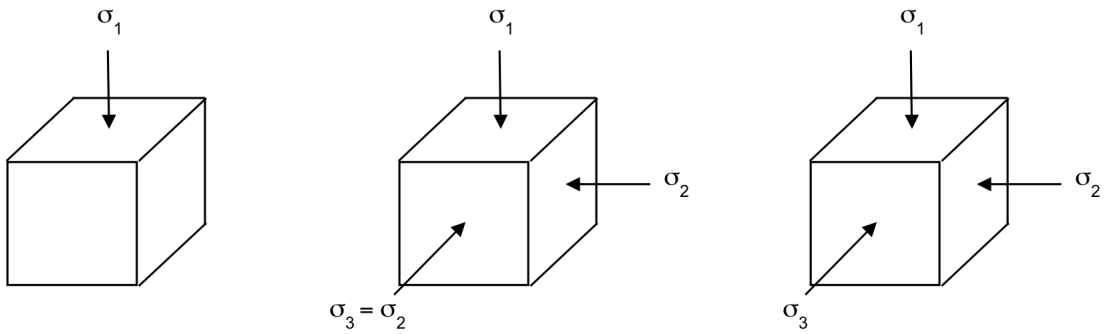
Triaxial Testing of Rocks

The most common method of studying the mechanical properties of rocks is by axial compression of a circular cylinder, with the lateral stresses applied by a pressurised fluid, and the axial stress applied by an axial load.

If the lateral surface of the rock is traction-free, the configuration is referred to as *uniaxial compression*, or unconfined compression (below, a). In this case, the resulting state of stress in the rock is $\{\sigma_1 > 0,\ \sigma_2 = \sigma_3 = 0\}$.

If tractions are applied to the lateral surfaces, the experiment is referred to as *confined compression*. For the standard configuration described above, the stresses applied in the two orthogonal directions perpendicular to the cylinder axis are necessarily equal (below, b), and the resulting state of stress in the rock is $\{\sigma_1 > \sigma_2 = \sigma_3 > 0\}$. This state is traditionally referred to as "triaxial".

The more general state of stress, in which $\{\sigma_1 > \sigma_2 > \sigma_3 > 0\}$, can be achieved with cubical specimens and a special type of apparatus, and is known either as "polyaxial" or "true triaxial" (below, c).

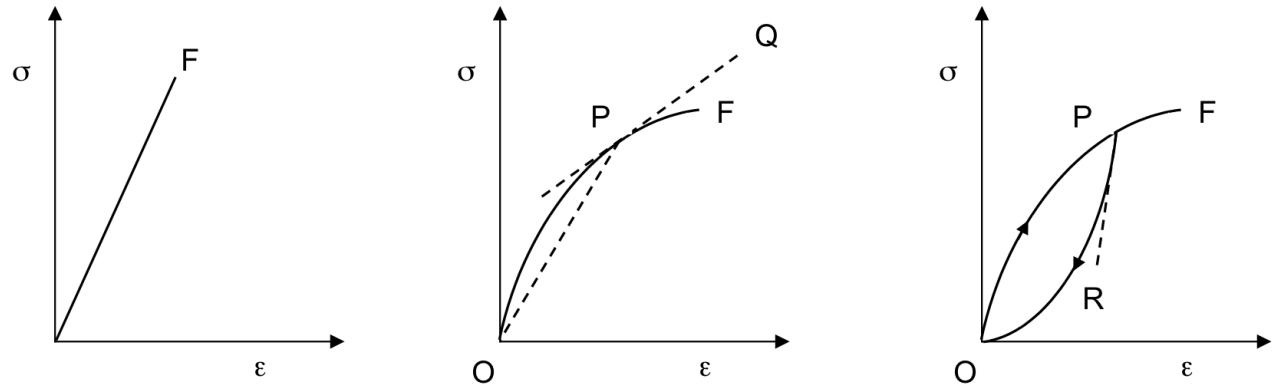


Idealised Stress-Strain Curves of a Rock

Consider now the so-called "triaxial" test. Typically, σ_2 and σ_3 are held constant, while σ_1 is increased. The results can be plotted in the form of a stress-strain curve, in which "σ" (*i.e.*, σ_1) is plotted against "ε" (*i.e.*, ε_1).

The simplest possible behavior is illustrated below in (a), in which the strain increases linearly with stress, until the rock abruptly breaks at some point F. This type of linear relationship between stress and strain corresponds to the elastic behaviour that was discussed in the previous lecture. For this type of behaviour, if the load is removed gradually, the rock moves "back down" on the same stress-strain curve as during loading.

More general, and less idealised, types of behavior are *nonlinear elastic* behaviour (b), in which the the slope of the stress-strain curve varies with the level of stress, but the rock follows the same stress-strain curve during loading as during unloading, and *hysteretic* behaviour (c), in which the rock follows a different stress-strain path during loading than during unloading.

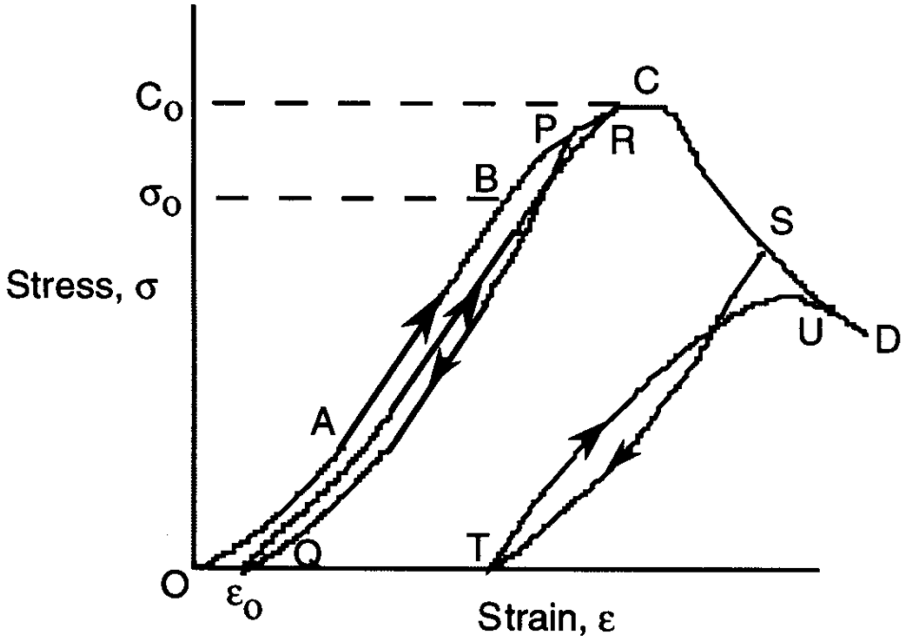


(a) (b) (c)

Realistic Stress-Strain Curve of a Rock, 1

The idealized materials described on the previous slide each deform until F, at which point they fail abruptly if the applied stress is increased further. This type of abrupt failure is observed in materials under tension, but the behavior of a rock under a compressive stress regime is more complicated (see below).

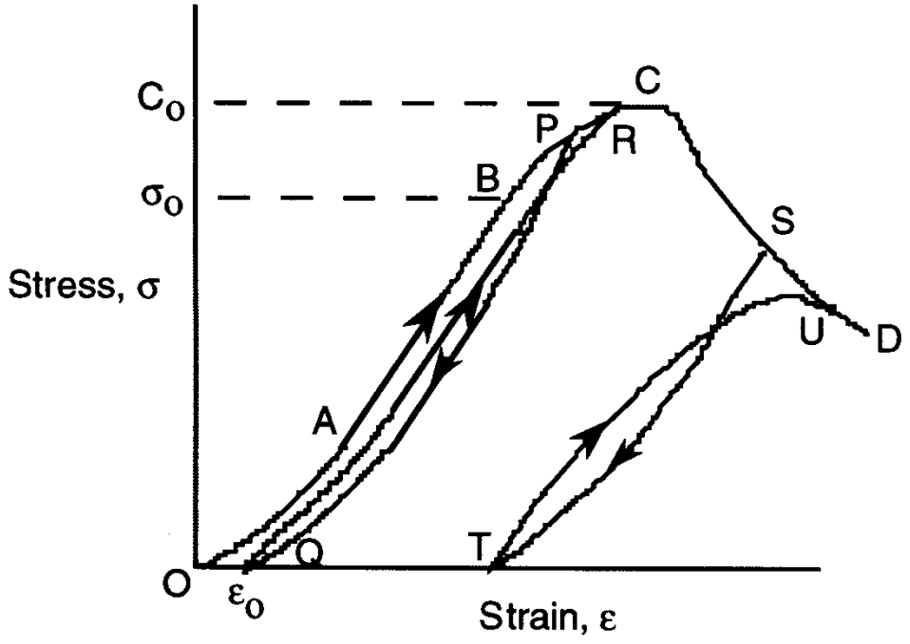
The stress-strain curve for a rock under uniaxial *compression* can be divided conceptually into four regions. In region OA, the curvature, roughly indicated by the second derivative, is positive. In region AB the curve is very nearly linear. The curve continues to rise in region BC, but the curvature is now negative. The strain reaches a maximum at C, after which it falls throughout region CD.



Realistic Stress-Strain Curve of a Rock, 2

In the first two regions, OA and AB, the behavior is nearly elastic. The local elastic modulus increases with stress, as small pre-existing cracks close up and the rock becomes stiffer. Some slight hysteresis may occur in this regime, but loading and unloading in this region will not produce irreversible changes in the structure or properties of the rock.

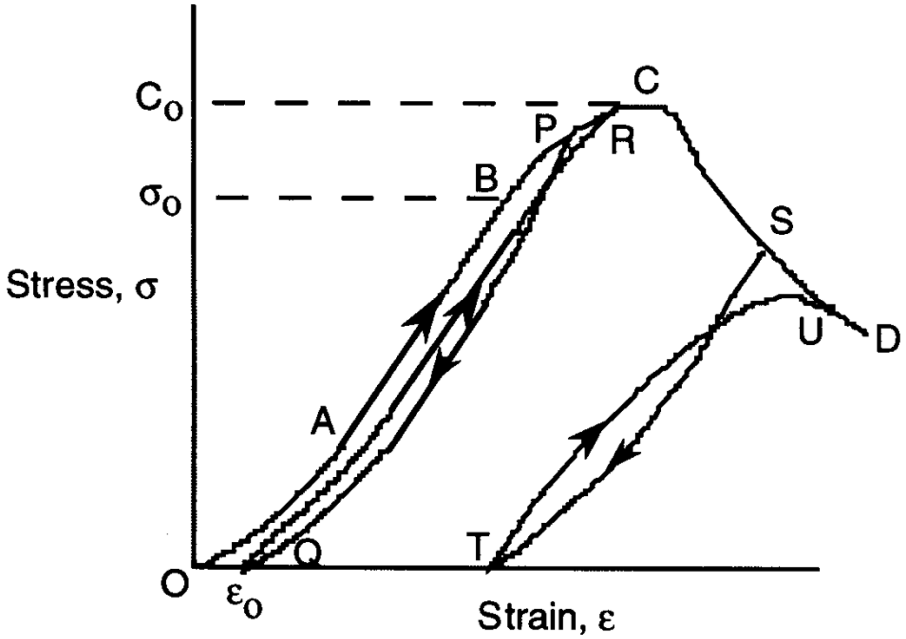
In the third region, BC, which usually begins at a stress of about two-thirds of the maximum stress at C, the slope of the stress-strain curve decreases steadily to zero as the stress increases. In this region, irreversible changes occur in the microstructure of the rock, and an unloading cycle such as PQ that starts in region BC would lead to a permanent strain when the stress returns to zero.



Realistic Stress-Strain Curve of a Rock, 3

In region BC the rock is said to be in a *ductile* state. Ductile behavior is characterised by the ability of the rock to support an increasing load as it deforms. In region CD, on the other hand, the load supported by the rock decreases as the strain increases. A rock exhibiting this behavior is said to be in a brittle state.

The stress at point B, which marks the transition from elastic to ductile behavior, is known as the yield stress of the rock, and is usually denoted by σ_o. The stress at point C, which marks the transition from ductile to brittle behavior, is known as the uniaxial compressive strength, and is usually denoted by C_o.



The process of *failure* is a continuous process that occurs throughout the brittle region CD, in which the rock physically deteriorates, and its ability to support a load decreases. Failure therefore begins at C, and the criteria for failure for a rock subjected to uniaxial compression would simply consist of the condition that "failure occurs when $\sigma = C_o$".

Coulomb Failure Criterion, 1

Failure under triaxial stress states is governed by a more complicated criterion than merely saying "failure occurs when $\sigma = C_o$".

The simplest, and still most widely used, failure criterion is that of Coulomb (1773). Coulomb assumed that failure in a rock or soil takes place along a plane due to the shear stress τ acting along that plane. In analogy with sliding along non-welded surfaces, the sliding motion was assumed to be resisted by a frictional-type traction whose magnitude equals the normal traction σ acting along this plane, multiplied by some constant factor μ.

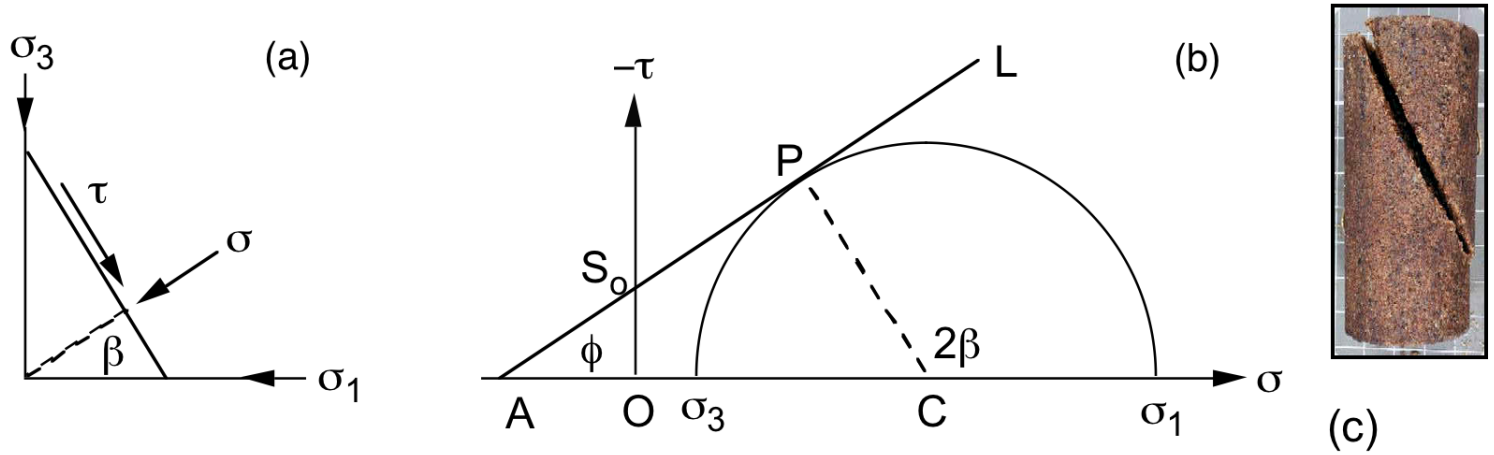
But in contrast to sliding along non-welded surfaces, Coulomb assumes that motion along the initially intact failure plane is also resisted by an internal cohesive force of the material. These considerations lead to the criterion that failure will occur along a plane if the following condition is satisfied:

$$| \tau | = S_o + \mu\sigma$$

The sign of the shear stress only effects the direction of sliding after failure, so the *absolute value* of τ appears in the Coulomb failure criterion. The parameter S_o, sometimes denoted by *c*, is known as the cohesion. The parameter μ is the *coefficient of internal friction*, as it applies along an imaginary surface that is *internal* to the rock *before* failure occurs.

Coulomb Failure Criterion, 2

The mathematical form of the Coulomb criterion suggests that the Mohr's circle construction will be useful. The Coulomb equation defines a straight line on the $\{\sigma, -\tau\}$ plane that intercepts the τ-axis at $-S_o$, and has slope μ. The angle ϕ that this line makes with the σ-axis is given by $\phi = \tan^{-1}\mu$, and is known as the *angle of internal friction*.



(a) Normal and shear tractions on a plane whose outward normal is rotated from the σ_1 direction by some angle β.

(b) Mohr diagram, with the Coulomb failure curve shown as line AL. Failure will occur on a specific plane whose angle β, demarcated by line CP, is given by $\beta = 45^o + (\phi/2)$.

(c) Example of a rock core that has failed in shear under compression.

Coulomb Failure Criterion, 3

When written in terms of the maximum and minimum principal stresses, the Coulomb criterion takes the form (see FoRM4, section 4.5)

$$\sigma_1 = 2S_o\left(\frac{\cos\phi}{1-\sin\phi}\right) + \sigma_3\left(\frac{1+\sin\phi}{1-\sin\phi}\right)$$

Making use of the result that $\beta = 45^o + (\phi/2)$, and after some trigonometric manipulations, the Coulomb criterion can also be written as

$$\sigma_1 = 2S_o \tan\beta + \sigma_3 \tan^2\beta$$

$$\sigma_1 = 2S_o\left[(1+\mu^2)^{1/2} + \mu\right] + \left[(1+\mu^2)^{1/2} + \mu\right]^2 \sigma_3$$

The value of σ_1 that would cause failure under unconfined compression, *i.e.,* when σ_3 = 0, is called the *unconfined compressive strength,* C_o, and the above equations show that

$$C_o = 2S_o\left[(1+\mu^2)^{1/2} + \mu\right] = 2S_o \tan\beta$$

On the other hand, under *unconfined tension,* when $\sigma_3 < 0$ (tension!) and $\sigma_1 = 0$, the value of the tensile stress σ_3 that would cause failure, which is known as *unconfined tensile strength,* T_o, is given by

$$T_o = -\sigma_3 = 2S_o \cot\beta = \frac{2S_o}{(1+\mu^2)^{1/2} + \mu}$$

Deficiencies of the Coulomb Failure Criterion

The equations on the previous slide imply that, according to the Coulomb criterion, the ratio of unconfined compressive strength to unconfined tensile strength is given by

$$\frac{C_o}{T_o} = \tan^2\beta = \left[(\mu^2 + 1)^{1/2} + \mu\right]^2$$

Since μ is often around 0.6, and is never found to exceed 1, the Coulomb criterion predicts that the compressive strength will exceed the tensile tensile by only a factor between about 2 and 6. But experimental data show that C_o is usually at least 10 times greater than C_o.

Another shortcoming of Coulomb's failure criterion is that it predicts that τ will increase linearly with σ, and that σ_3 will increase linearly with σ_1. In reality, curves of τ *vs.* σ (*i.e.*, a Mohr diagram), or σ_3 *vs.* σ_1, are usually somewhat *concave downwards*.

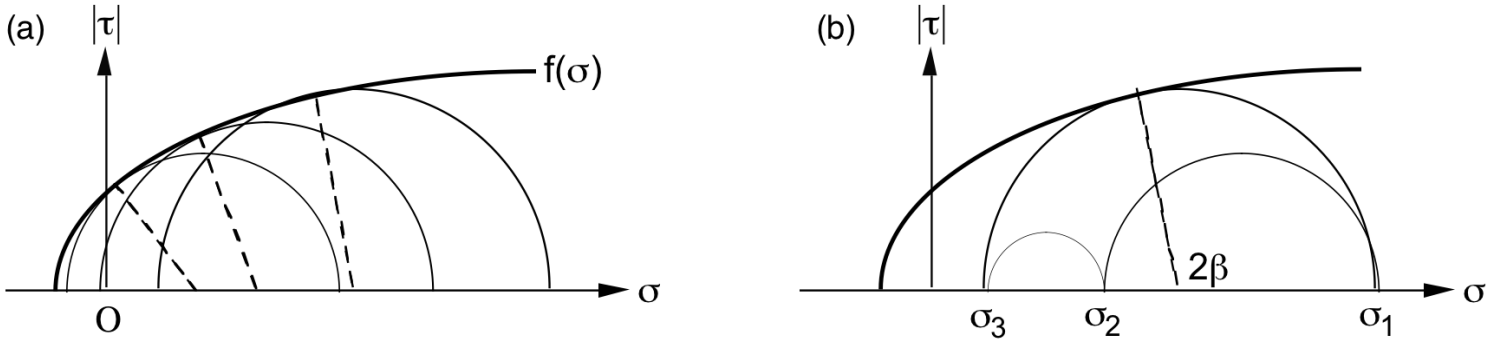
In order to correct these deficiencies, in 1900, Otto Mohr suggested that Coulomb's equation, be replaced by a nonlinear relation of the general form

$$|\tau| = f(\sigma)$$

Many different forms have been proposed for this failure function; some of the more widely used ones are reviewed in Chapter 4 of *FoRM*4.

Mohr's Generalization of the Coulomb Failure Criterion: $|\tau| = f(\sigma)$

Aside from the fact that *f* may be a nonlinear function, Mohr retained the basic ideas of Coulomb's theory. Failure is supposed to occur if one of the Mohr's circles touches the failure curve in $\{\tau, \sigma\}$ space. As shown below, this will necessarily occur for the circle defined by σ_1 and σ_3, and so the intermediate principal stress is not expected to affect the onset of failure.



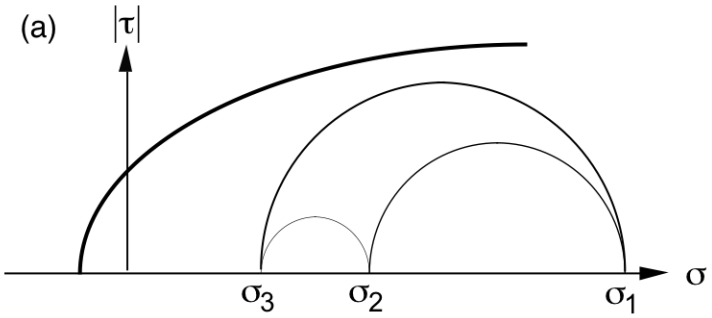
The state of stress at the point of contact of the Mohr's circle and the failure curve represents the stresses acting on the failure plane, and so the generalized Mohr theory predicts that the failure plane passes through the direction of the intermediate principal stress, and its normal vector makes an angle *β* with the direction of maximum principal stress. If the failure criterion is concave downwards, the angle *β* of the failure plane will decrease with increasing confining stress, as indicated in (b) above.

Effect of Pore Fluids on Failure, 1

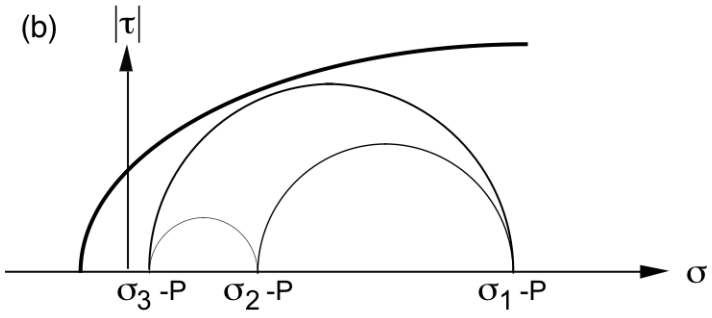
The preceding discussion ignored the fact that crustal rocks are typically porous, with a pore space filled with fluids under pressure. The pore fluid may affect the failure of the rock in two ways: due to the purely mechanical effect of pore pressure, or due to chemical interactions between the rock and the fluid.

It seems plausible that pore pressure, which acts "outwards" from the pore space, would act like a tensile stress. In an isotropic rock, this effect should be the same in any three mutually orthogonal directions. Karl Terzaghi proposed in 1936 that failure of a soil would be controlled by the "effective stresses", defined as the principal stresses minus the pore pressure P, *i.e.,*

$$\sigma_1' = \sigma_1 - P, \quad \sigma_2' = \sigma_2 - P, \quad \sigma_3' = \sigma_3 - P,$$



A stress state that lies below the failure curve.

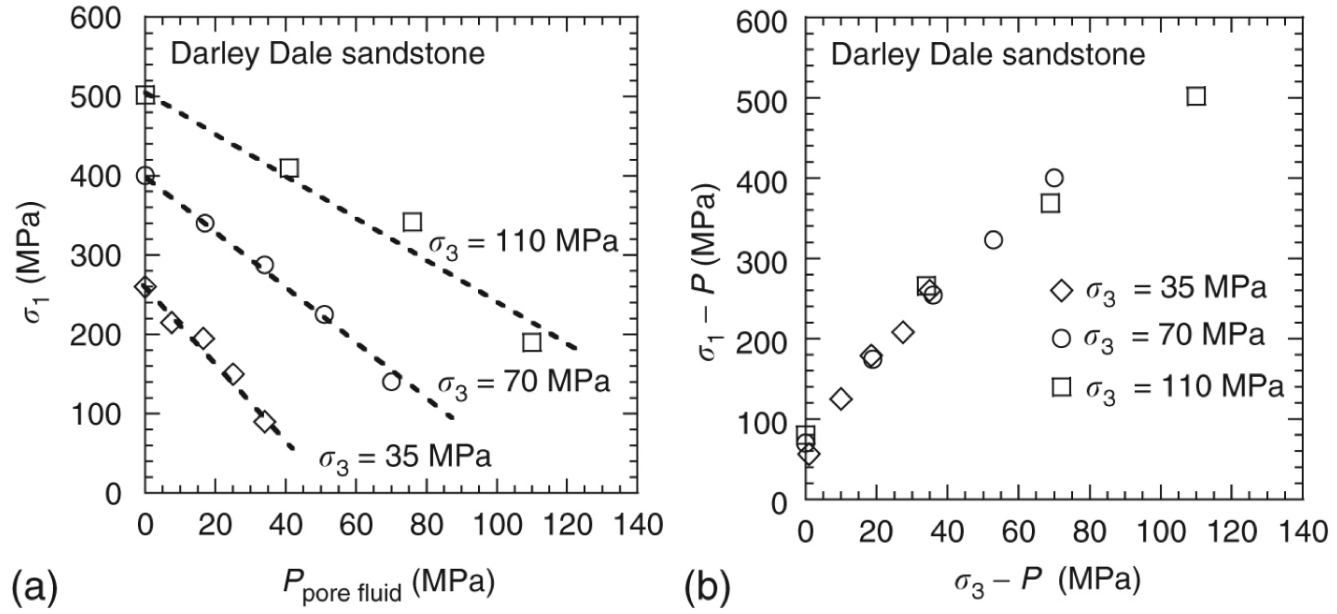


Pore pressure causes the effective stress state to move closer to the failure curve.

Effect of Pore Fluids on Failure, 2

Murrell (1965) conducted standard triaxial compression tests on a Darley Dale sandstone, at several different values of the pore pressure. In each test, the pore pressure and the confining stress were held constant, while the axial stress was increased until failure occurred. When plotted in the {P, σ_1} plane, as in (a) below, the data fall on different curves, for different values of σ_3.

If plotted on the {σ_1, σ_3} plane, the data would form three distinct vertical lines. However, when plotted in the plane of maximum and minimum *effective stresses*, as in (b) below, the failure data nearly form a single failure curve, which in this particular case is slightly concave downwards .



Failure Under True-Triaxial Conditions, 1

Although the Mohr-Coulomb theory assumes that σ_2 has no influence on failure, compression tests conducted under true-triaxial conditions show that, for many rocks, the intermediate principal stress *does* influence the value of σ_1 at failure.

In the context of metal plasticity, Nadai suggested that the "driving force" for failure will be the "octahedral shear stress", which is defined by

$$\tau_{oct} = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

The octahedral shear stress is sort of an "average shear stress" in the rock; specifically, it is equal to $(5/3)^{1/2}$ times the root-mean-square shear stress, with all planes weighted equally.

Nadai further suggested that failure was "opposed" by the mean stress in the material, which is consistent with the concept that confinement *strengthens* the rock. The mean stress is defined by

$$\tau_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

Hence, Nadai's assumption can be written in the form

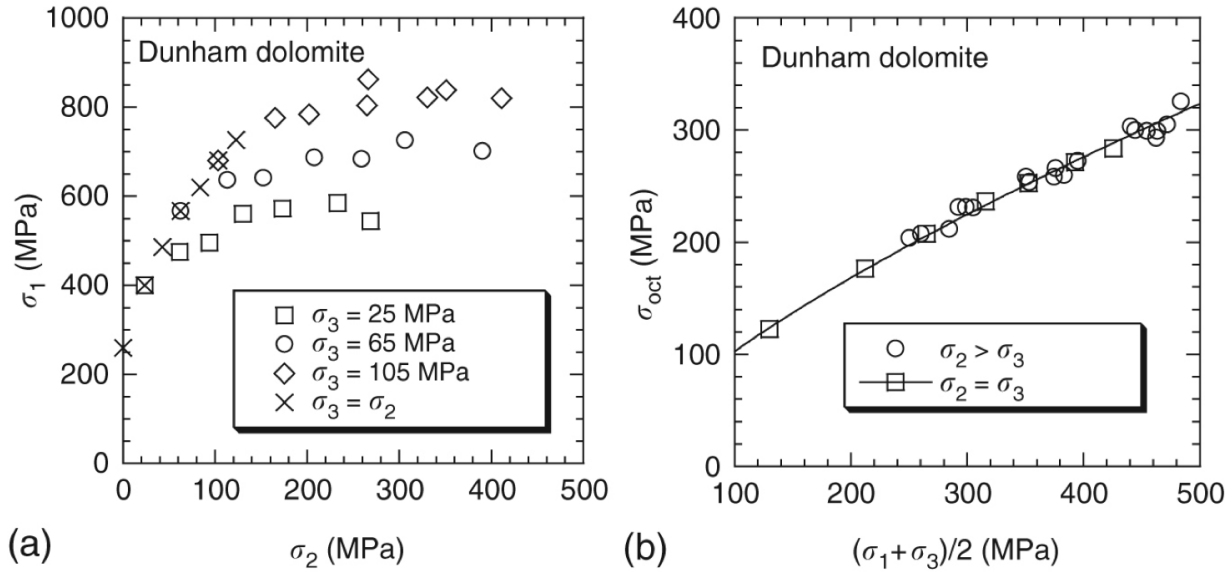
$$\tau_{oct} = f(\tau_m)$$

Failure Under True-Triaxial Conditions, 2

In 1971, Mogi modified Nadai's reasoning, as follows. According to the Mohr hypothesis, the failure plane will lie parallel to the direction of the intermediate principal stress. Hence, it is plausible that fracture is resisted only by the mean normal stress acting on that plane, which is $(\sigma_1 + \sigma_3)/2$, rather than by the total mean normal stress. This suggests a brittle failure criterion of the general form

$$\tau_{oct} = f(\tau_{m2}), \quad \text{where} \quad \tau_{m2} = (\sigma_1 + \sigma_3)/2$$

To test this hypothesis, Mogi conducted true-triaxial compression tests on several rocks. The results showed that, for a fixed value of the minimum stress σ_3, the value of σ_1 at failure at first increases with an increase in σ_2, but eventually decreases slightly as σ_2 increases further. If the stresses at failure are plotted in the $\{\tau_{oct}, \tau_{m2}\}$ plane, the results coalesce to a single curve:



Failure Under True-Triaxial Conditions, 3

Al-Ajmi and Zimmerman (*IJRMMS*, 2005) proposed using a linear expression for Mogi's failure function:

$$\tau_{oct} = a + b\sigma_{m,2}$$

This expression reduces to the Mohr-Coulomb criterion if any two of the principal stresses are equal. Thus, the linear Mogi law, which they referred to as the "Mogi-Coulomb" criterion, is in some sense a natural extension of the Coulomb criterion into the polyaxial (true-triaxial) stress domain.

Moreover, they shows that the parameters *a* and *b* that appear in the Mogi-Coulomb criterion can be expressed in terms of the strength parameters that appear in the Coulomb criterion, as follows:

$$a = (2\sqrt{2}/3)S_o\cos\phi, \quad b = (2\sqrt{2}/3)\sin\phi$$

An advantage of the linear form of the Mogi-Coulomb criterion is that it allowed Al-Ajmi and Zimmerman to develop closed-form analytical solutions for the problem of the stability of a borehole, for vertical and horizontal boreholes.

Application of these solutions to various sets of drilling data showed that the Mogi-Coulomb criteria led to more realistic predictions than does the Coulomb criterion (Al-Ajmi & Zimmerman, *IJRMMS*, 2006).

Problems for Lecture 8

Problem 8.1. Three samples of a rock from the same reservoir are tested under triaxial compression, and the following stresses are measured at "failure":

$$\sigma_1 = 50 \text{ MPa}, \quad \sigma_3 = 10 \text{ MPa},$$

$$\sigma_1 = 90 \text{ MPa}, \quad \sigma_3 = 20 \text{ MPa},$$

$$\sigma_1 = 130 \text{ MPa}, \quad \sigma_3 = 30 \text{ MPa}.$$

In this problem, assume that $\sigma_2 = \sigma_3$ for all of the tests.

(a) Plot these stresses on a Mohr diagram, and find the cohesion, S_o, the coefficient of internal friction, μ, and the angle of internal friction, ϕ.

(b) Draw a picture of one of one of the cylindrical specimens, and show the angle β that the normal to the failure plane makes with the direction of the maximum principal stress, σ_1. What is the numerical value of β?

(c) Another sample of this rock is subjected to the state of stress {$\sigma_1 = 100$ MPa, $\sigma_3 = 40$ MPa}. Will this state of stress cause the rock to fail?

(d) Now imagine that the pore pressure is increased from 0 to *P.* What value of *P* will be sufficient to cause the rock to fail?

Imperial College MSc in
GeoEnergy with Machine Learning and Data Science

GEMS 3: Geomechanics and Pressure Transient Analysis

Lecture 9:
Stresses around Excavations and Boreholes

Prof. Robert Zimmerman

March 2024

Stresses around Excavations in Rock

Some of the most important problems in rock mechanics involve the calculation of the stresses and displacements around subsurface cavities, excavations, and boreholes.

On a macroscopic scale, calculation of the stresses and displacements around boreholes, tunnels, and mine excavations is of paramount importance to petroleum, mining, and civil engineers.

On a microscopic scale, the calculation of stresses around small voids and cracks in a rock is a necessary first step in the development of micromechanically-based theories of rock deformation and failure.

As there is no intrinsic “size effect” in classical linear elasticity, the deformation of both engineering-scale excavations and micro-scale cracks and pores are governed by the same equations.

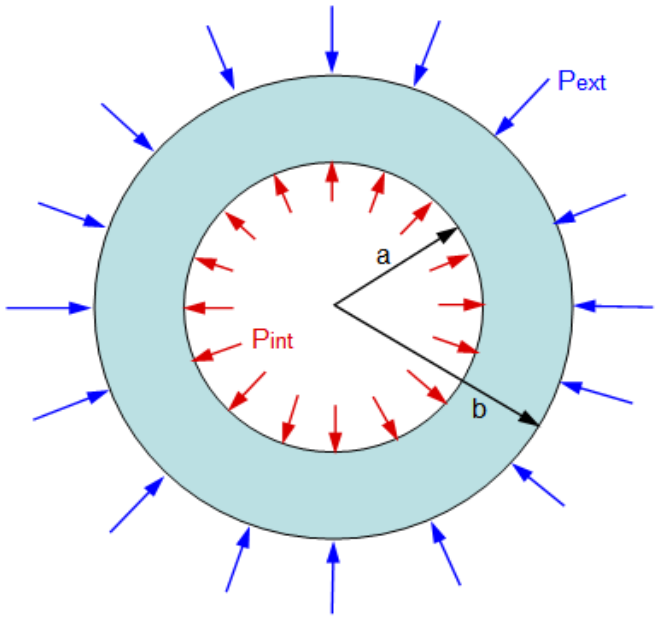
In this lecture we will present analytical solutions to a few important problems involving cavities and voids in rock, in most cases without derivation. Details of the methods used to arrive at these solutions can be found in *FoRM*4, and the references cited therein.

Pressurised Hollow Cylinder, 1

The problem of a circular hole in an infinite rock mass, with a uniform state of stress at infinity, is probably the most important single problem in rock mechanics. We start with the case of hydrostatic stress at infinity. The general case of two unequal far-field principal stresses will be treated later.

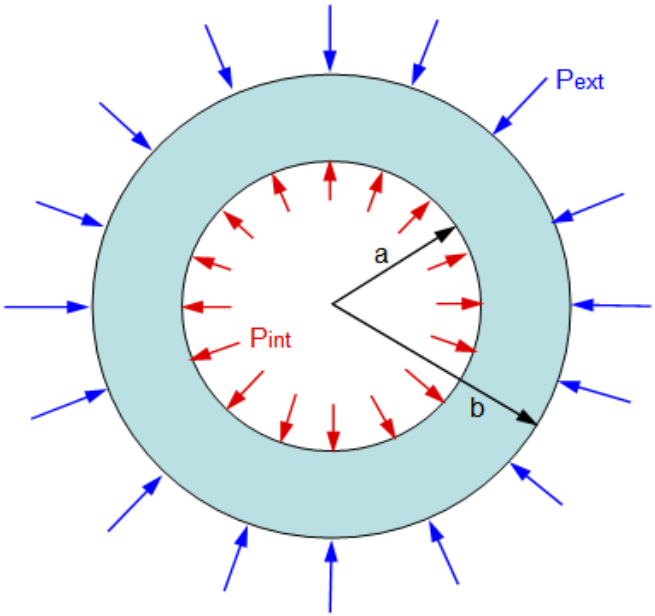
To generate this solution, we start with the problem of a hollow cylinder subjected to uniform pressure at both its inner and outer boundaries. The solution for the infinite region outside of a circular hole can then be found by letting the outer radius of the cylinder become infinite. The hollow cylinder problem is of interest in its own right, in the context of laboratory testing.

Consider a hollow cylinder having inner radius *a*, and outer radius *b*. A uniform pressure (*i.e.*, radial stress) P_o is applied over the outer boundary, and a uniform fluid pressure P_i is applied over the inner boundary.



Pressurised Hollow Cylinder, 2

For a problem such as this, having radial symmetry, the only component of the displacement vector that is not zero is the radial component, u, and u will depend only on r.



Consequently, most terms in the Navier equations, when written in cylindrical coordinates, will be zero, and the Navier equations reduce to the following single ODE:

$$(\lambda + G)\frac{d}{dr}\left(\frac{du}{dr} + \frac{u}{r}\right) + G\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right) = 0$$

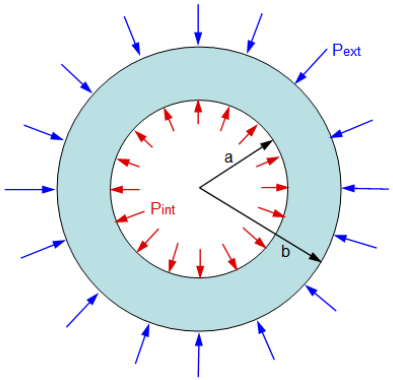
The boundary conditions at the inner and outer surfaces are

$$\tau_{rr}(a) = P_i, \qquad \tau_{rr}(b) = P_o$$

Although the dependent variable in the ODE is the displacement, the boundary conditions involve the stresses. So, we need to relate the stresses to the displacement, using Hooke's law in polar co-ordinates:

$$\tau_{rr} = \lambda\left(\frac{du}{dr} + \frac{u}{r}\right) + 2G\frac{du}{dr}, \qquad \tau_{\theta\theta} = \lambda\left(\frac{du}{dr} + \frac{u}{r}\right) + 2G\frac{u}{r}$$

Pressurised Hollow Cylinder, 3



First, we expand out the derivatives in the ODE, and collect similar terms, to find

$$(\lambda + 2G)\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right) = 0$$

This equation is an Euler-type ODE, whose two independent solutions will be of the form $u = Cr^n$. If we substitute $u = Cr^n$ into this equation, we find

$$n(n-1)Cr^{n-2} + nCr^{n-2} - Cr^{n-2} = 0$$

We now expand out and combine terms to arrive at

$$(n^2 - 1)Cr^{n-2} = 0$$

For this to be true for all r, without requiring $C = 0$, we can have either $n = +1$, or $n = -1$. So, the general solution to this ODE is

$$u(r) = Cr + \frac{D}{r}$$

where C and D are constants, whose values must be found by appealing to the boundary conditions.

Pressurised Hollow Cylinder, 4

If we plug this expression for the displacement into Hooke's law for τ_{rr}, as given on the bottom of slide 177, we find

$$\tau_{rr} = \lambda\left(C - \frac{D}{r^2} + C + \frac{D}{r^2}\right) + 2G\left(C - \frac{D}{r^2}\right) = 2(\lambda + G)C - 2G\frac{D}{r^2}$$

We now force this expression to satisfy the two Boundary Conditions:

$$\tau_{rr}(r = a) = 2(\lambda + G)C - 2G\frac{D}{a^2} = P_i$$

$$\tau_{rr}(r = b) = 2(\lambda + G)C - 2G\frac{D}{b^2} = P_o$$

These are two linear equations for the unknown constants *C* and *D*. These constants are easily found to be

$$C = \frac{b^2P_o - a^2P_i}{2(\lambda + G)(b^2 - a^2)} \qquad D = \frac{(P_o - P_i)a^2b^2}{2G(b^2 - a^2)}$$

If we plug these constants back into the expression $u(r) = Cr + D/r$, and recall that $\lambda = 2G\nu/(1-2\nu)$, we find

$$u(r) = (1 - 2\nu)\frac{(b^2P_o - a^2P_i)r}{2G(b^2 - a^2)} + \frac{a^2b^2(P_o - P_i)}{2G(b^2 - a^2)r}$$

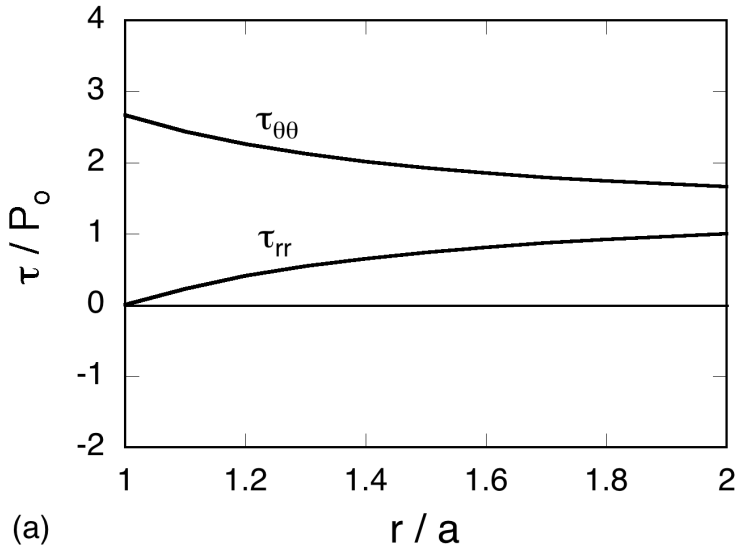
Pressurised Hollow Cylinder, 5

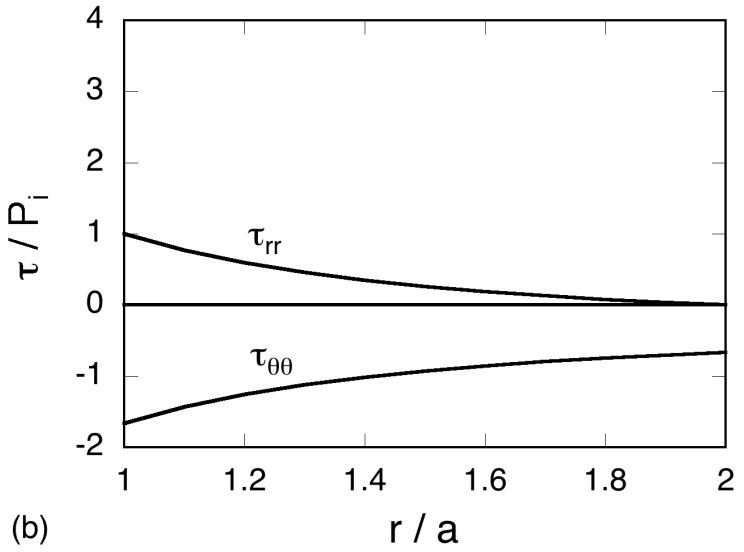
The stresses within the cylinder are found to be

$$\tau_{rr} = \frac{(b^2P_o - a^2P_i)}{(b^2 - a^2)} - \frac{a^2b^2(P_o - P_i)}{(b^2 - a^2)r^2}$$

$$\tau_{\theta\theta} = \frac{(b^2P_o - a^2P_i)}{(b^2 - a^2)} + \frac{a^2b^2(P_o - P_i)}{(b^2 - a^2)r^2}$$

These stresses are plotted below, for the case of a hollow cylinder having $b = 2a$. The left panel (a) shows the case of an external pressure only, and the right panel (b) shows the case of an internal pressure only. Note that in this latter case, $\tau_{\theta\theta}$ is *tensile*!





Circular Hole in an Infinite Rock Mass

The solution for a circular hole in an infinite rock mass, with a far-field hydrostatic stress P_o, and a uniform fluid pressure P_i in the borehole, can be found by letting $b \to \infty$. The results are:

$$2Gu_r(r) = (1-2\nu)P_o r + (P_o - P_i)(a^2 / r)$$

$$\tau_{rr}(r) = P_o - (P_o - P_i)(a/r)^2$$

$$\tau_{\theta\theta}(r) = P_o + (P_o - P_i)(a/r)^2$$

The stresses die off as r^{-2} as we move away from the borehole. This is in contrast to the fluid flow problem, in which the fluid pressure drops off logarithmically with distance from the borehole.

The most important implication of this solution is that, if the borehole pressure is large enough, *tensile* normal stresses in the "theta" direction can be generated at the borehole. If we plug $r = a$ into the equation for $\tau_{\theta\theta}$, we find:

$$\tau_{\theta\theta}(r = a) = P_o + (P_o - P_i) = 2P_o - P_i$$

So, if the borehole pressure P_i is greater than twice as large as the *in situ* stress P_o, the tangential normal stress around the borehole wall will be negative, *i.e.,* tensile! This may give rise to hydraulic fracturing.

Circular Hole in a Rock Mass with a Uniaxial Far-field Stress

The problem of calculating the displacements and stresses outside of a circular hole in an infinite elastic solid, with a non-hydrostatic state of stress far from the hole, was first solved by the German engineer Kirsch in 1898.

Unfortunately, Kirsch did not explain how he found this solution. This problem is solved in *FoRM*4 using complex stress potentials. We will present the solution below, without showing the details of the derivation.

Since we can utilize the principle of superposition, we can start with the case of a single non-zero principal stress at infinity. We align the *x*-axis with this principal stress, the value of which we will denote by σ_1^∞. The expressions for the stresses are as follows:

$$\tau_{rr} = \frac{1}{2}\sigma_1^\infty\left[1-\left(\frac{a}{r}\right)^2\right]+\frac{1}{2}\sigma_1^\infty\left[1-4\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right]\cos 2\theta$$

$$\tau_{\theta\theta} = \frac{1}{2}\sigma_1^\infty\left[1+\left(\frac{a}{r}\right)^2\right]-\frac{1}{2}\sigma_1^\infty\left[1+3\left(\frac{a}{r}\right)^4\right]\cos 2\theta$$

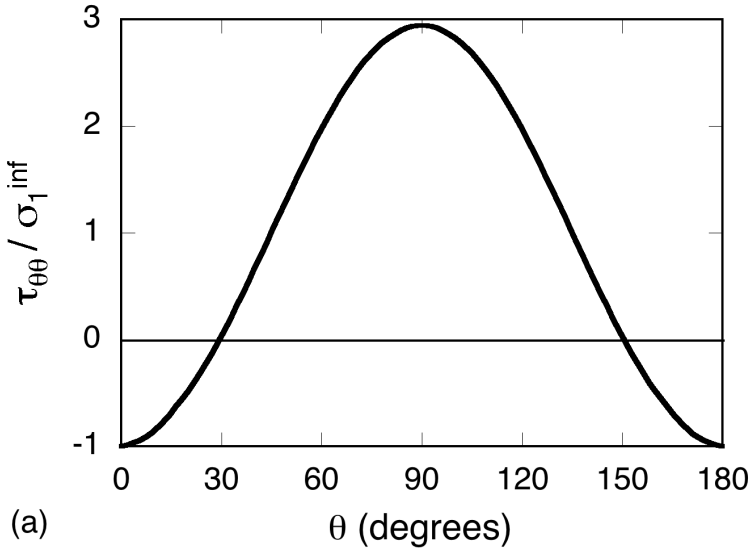
$$\tau_{r\theta} = -\frac{1}{2}\sigma_1^\infty\left[1+2\left(\frac{a}{r}\right)^2-3\left(\frac{a}{r}\right)^4\right]\sin 2\theta$$

Circular Hole in a Rock Mass Subjected to a Far-field Stress

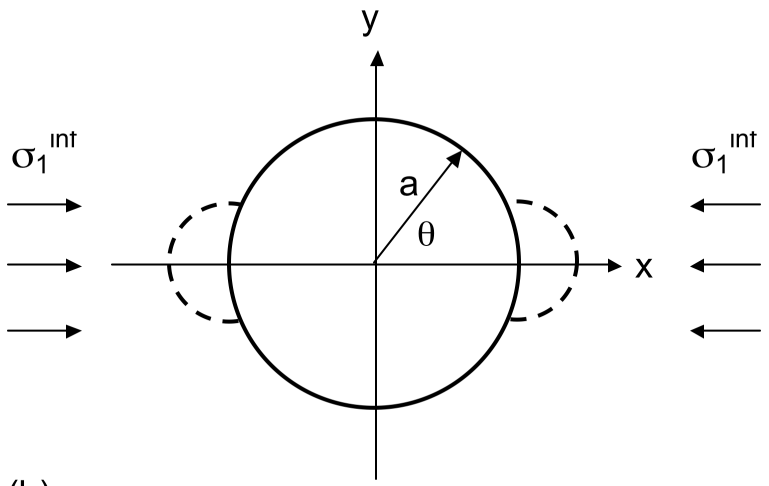
At the surface of the hole, the hoop stress $\tau_{\theta\theta}$ varies with θ, the angle of counterclockwise rotation from the σ_1^{∞} axis, according to:

$$\tau_{\theta\theta}(a,\theta) = \sigma_1^{\infty}[1 - 2\cos 2\theta]$$

This stress therefore varies from a tensile stress of $-\sigma_1^{\infty}$ when $\theta = 0$ or $\theta = \pi$, to a compressive stress $3\sigma_1^{\infty}$ when $\theta = \pi/2$ or $\theta = 3\pi/2$. The hoop stress at the borehole wall is tensile within the regions where $2\cos 2\theta > 1$, which is to say, $-30^o < \theta < 30^o$, and $150^o < \theta < 210^o$.



(a)



(b)

Circular Hole in a Rock Mass with Two Different Far-field Stresses

The general case of two principle stresses σ_1^∞ and σ_2^∞ at infinity, and a fluid pressure P in the borehole, can be found by superposition:

$$\tau_{rr} = \frac{1}{2}(\sigma_1^\infty + \sigma_2^\infty)\left[1-\left(\frac{a}{r}\right)^2\right] + \frac{1}{2}(\sigma_1^\infty - \sigma_2^\infty)\left[1-4\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right]\cos 2\theta + P\left(\frac{a}{r}\right)^2$$

$$\tau_{\theta\theta} = \frac{1}{2}(\sigma_1^\infty + \sigma_2^\infty)\left[1+\left(\frac{a}{r}\right)^2\right] - \frac{1}{2}(\sigma_1^\infty - \sigma_2^\infty)\left[1+3\left(\frac{a}{r}\right)^4\right]\cos 2\theta - P\left(\frac{a}{r}\right)^2$$

$$\tau_{r\theta} = -\frac{1}{2}(\sigma_1^\infty - \sigma_2^\infty)\left[1+2\left(\frac{a}{r}\right)^2-3\left(\frac{a}{r}\right)^4\right]\sin 2\theta$$

At the surface of the hole, $\tau_{\theta\theta}$ varies with θ, according to

$$\tau_{\theta\theta}(a,\theta) = (\sigma_1^\infty + \sigma_2^\infty) - 2(\sigma_1^\infty - \sigma_2^\infty)\cos 2\theta - P$$

This stress varies from a minimum of $3\sigma_2^\infty - \sigma_1^\infty - P$ when $\theta = 0$ or $\theta = \pi$, to a maximum of $3\sigma_1^\infty - \sigma_2^\infty - P$ when $\theta = \pi/2$ or $\theta = 3\pi/2$. A region of tensile hoop stresses will exist if

$$P > 3\sigma_2^\infty - \sigma_1^\infty$$

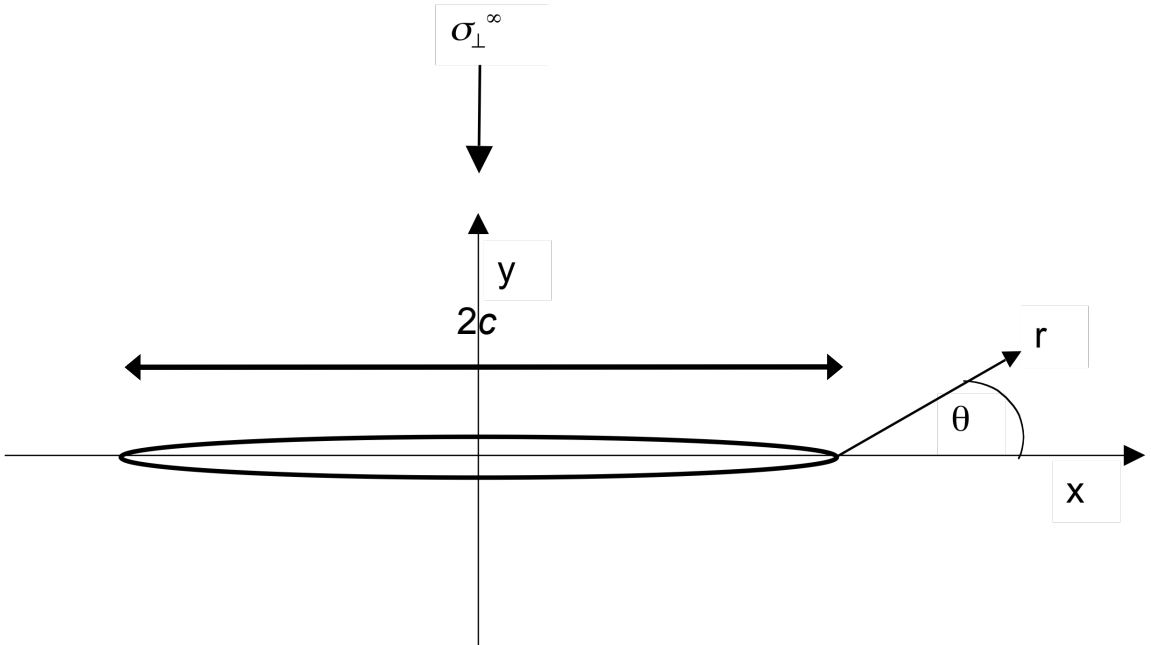
If we assume that the rock has zero strength under tension, this yields the simplest criterion for hydraulic fracturing due to fluid pressure in the borehole.

Stresses near the Tip of a Two-dimensional Crack, 1

The stresses around the boundary of any hole in a rock mass will generally be greater than the *in situ* stresses that existed before the hole is created. An important type of hole is a thin crack. The stresses around the tips of these cracks will be very large, often large enough to cause the cracks to grow.

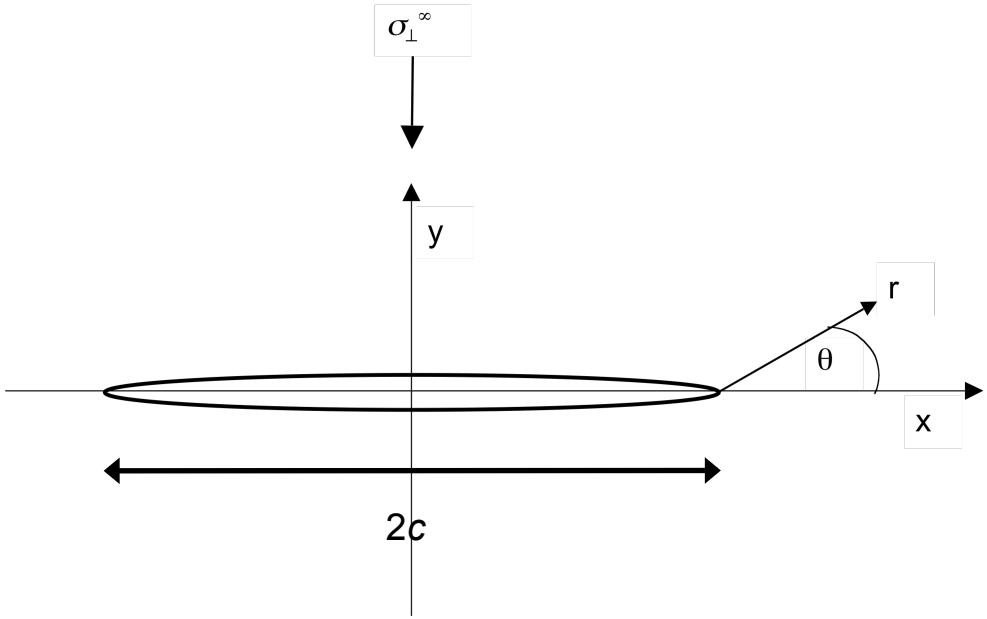
Knowledge of the stresses around the tip of a microcrack is important for understanding the micromechanical processes that cause rocks to deform and fail. On a larger scale, these stresses will determine how faults grow, or how a hydraulic fracture propagates from a pressurised wellbore.

Consider a thin 2D crack, lying along the *x*-axis between $-c < x < c$, in the *x*-*y* plane. The crack extends infinitely far "into the page" in the *z*-direction. A far-field stress τ_{yy}^{∞} of magnitude $\sigma_{\perp}^{\infty}$ acts normal to the crack plane.



Stresses near the Tip of a Two-dimensional Crack, 2

If far-field stress of magnitude $\sigma_{\perp}^{\infty}$ acts normal to the crack plane, the dominant terms in the equations for the stresses will be as shown below, where $\{r,\theta\}$ is a polar co-ordinate system centred on the tip of the crack, *i.e.*, at $\{x = c, y = 0\}$:



$$\tau_{rr} = \sigma_{\perp}^{\infty}\sqrt{\frac{c}{2r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin^2\left(\frac{\theta}{2}\right)\right]$$

$$\tau_{\theta\theta} = \sigma_{\perp}^{\infty}\sqrt{\frac{c}{2r}}\cos^3\left(\frac{\theta}{2}\right)$$

$$\tau_{r\theta} = \sigma_{\perp}^{\infty}\sqrt{\frac{c}{2r}}\cos^2\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

If the crack is filled with a pore fluid at pressure *P*, the stresses are given by these same expressions, with $\sigma_{\perp}^{\infty}$ replaced by $-P$.

Stresses near the Tip of a Two-dimensional Crack, 3

These stresses are often written as follows:

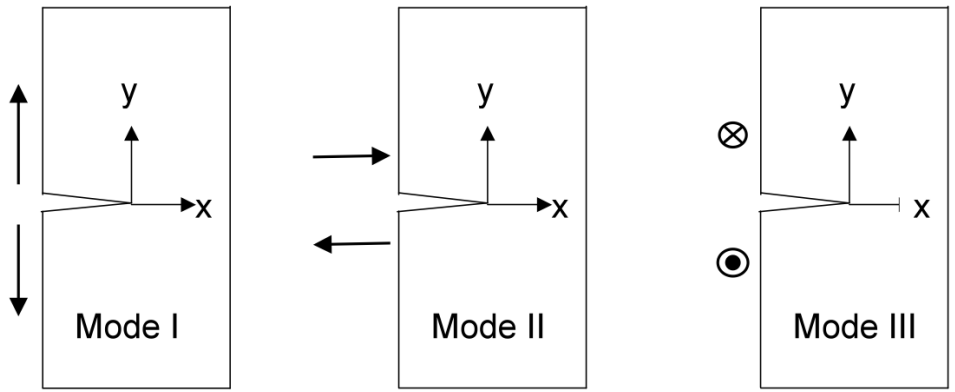
$$\tau_{rr} = K_{\mathrm{I}}(1/2\pi r)^{1/2}\cos(\theta/2)[1+\sin^2(\theta/2)]$$

$$\tau_{\theta\theta} = K_{\mathrm{I}}(1/2\pi r)^{1/2}\cos^3(\theta/2)$$

$$\tau_{r\theta} = K_{\mathrm{I}}(1/2\pi r)^{1/2}\cos^2(\theta/2)\sin(\theta/2)$$

where $K_{\mathrm{I}} = \sigma_{\perp}^{\infty}(\pi c)^{1/2}$ is called the *mode I stress intensity factor*.

The three "modes" of deformation at a crack tip are illustrated below, with mode I being the "crack-opening" mode, mode II being the "sliding" mode, and mode III being the "tearing" mode (in which the top half of figure moves into page, and the bottom half moves out of page):



Stresses near the Tip of a Two-dimensional Crack, 3

If the far-field state of stress is a shear stress τ_{yx} that has magnitude τ^{∞}, then the dominant terms in the stresses near the crack tip will be:

$$\tau_{rr} = K_{II}(1/2\pi r)^{1/2}\sin(\theta/2)[1-3\sin^2(\theta/2)]$$

$$\tau_{\theta\theta} = -3K_{II}(1/2\pi r)^{1/2}\sin(\theta/2)\cos^2(\theta/2)$$

$$\tau_{r\theta} = K_{II}(1/2\pi r)^{1/2}\cos(\theta/2)[1-3\sin^2(\theta/2)]$$

where $K_{II} = \tau^{\infty}(\pi c)^{1/2}$ is called the *mode II stress intensity factor*.

The third important case, mode III, which is caused by a far-field out-of-plane shear stress τ^{∞}_{yz}, gives rise to the following stresses near the crack tip:

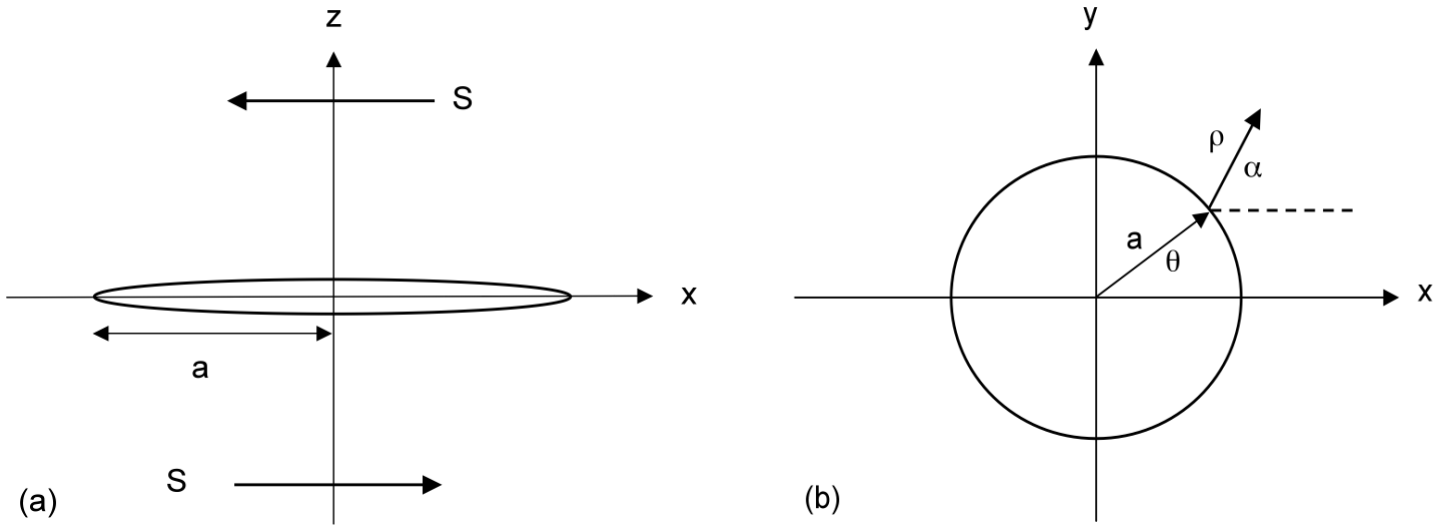
$$\tau_{xz} = -\tau^{\infty}_{yz}(c/2r)^{1/2}\sin(\theta/2) = -K_{III}(1/2\pi r)^{1/2}\sin(\theta/2)$$

$$\tau_{yz} = \tau^{\infty}_{yz}(c/2r)^{1/2}\cos(\theta/2) = K_{III}(1/2\pi r)^{1/2}\cos(\theta/2)$$

where $K_{III} = \tau^{\infty}_{yz}(\pi c)^{1/2}$ is the *mode III stress intensity factor*.

Stresses near a 3D "Penny-Shaped" Crack, 1

Consider a very thin "penny-shaped" crack of radius *a*, that lies in (*x*,*y*) plane, as shown below, where (a) is the side view, and (b) is the planview:



If the crack is filled with a fluid at pressure *p*, the leading terms in the stresses near the edge of the crack are:

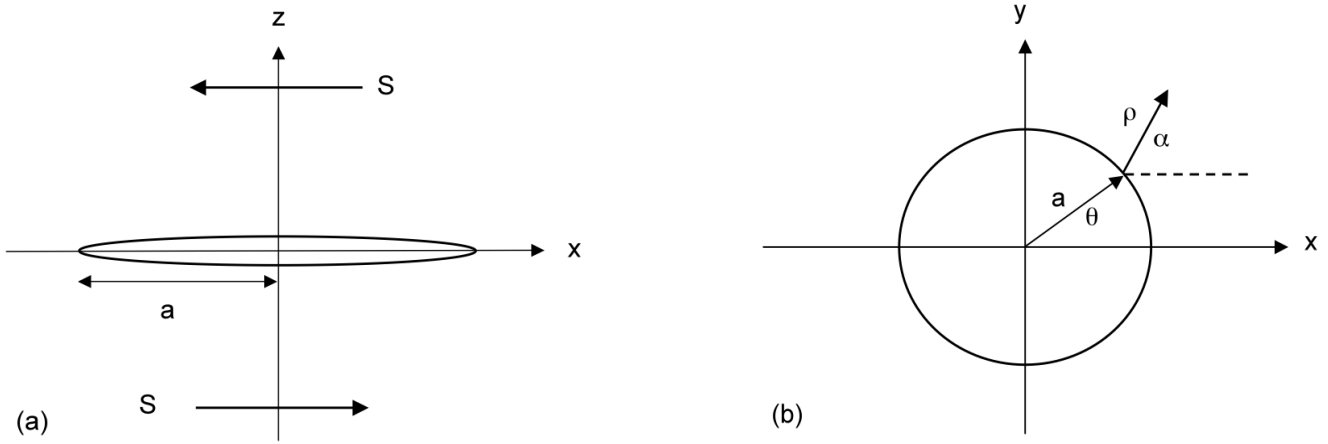
$$\tau_{\rho\rho} = \frac{p}{2\pi}\left(\frac{a}{2\rho}\right)^{1/2}\left[5\cos(\alpha/2) - \cos(3\alpha/2)\right] \qquad \tau_{\alpha\alpha} = \frac{p}{2\pi}\left(\frac{a}{2\rho}\right)^{1/2}\left[3\cos(\alpha/2) + \cos(3\alpha/2)\right]$$

$$\tau_{\rho\alpha} = \frac{p}{2\pi}\left(\frac{a}{2\rho}\right)^{1/2}\left[\sin(\alpha/2) + \sin(3\alpha/2)\right] \qquad \tau_{zz} = \frac{4\nu p}{\pi}\left(\frac{a}{2\rho}\right)^{1/2}\cos(\alpha/2) \qquad \tau_{\rho z} = \tau_{\alpha z} = 0$$

where (ρ,α) is a local polar coordinate system lying in the (*x*,*y*) plane, centred on a point on the edge of the crack.

Stresses near a 3D "Penny-Shaped" Crack, 2

If the crack faces are subjected to a shear traction of magnitude *S*, then the stresses in the vicinity of the edge of the crack will be, to leading order:



$$\tau_{\rho\rho} = \frac{S}{(2-\nu)\pi}\left(\frac{a}{2\rho}\right)^{1/2}\left[3\sin(3\alpha/2) - 5\sin(\alpha/2)\right]\cos\theta$$

$$\tau_{\alpha\alpha} = -\frac{S}{(2-\nu)\pi}\left(\frac{a}{2\rho}\right)^{1/2}\left[3\sin(3\alpha/2) + 3\sin(\alpha/2)\right]\cos\theta$$

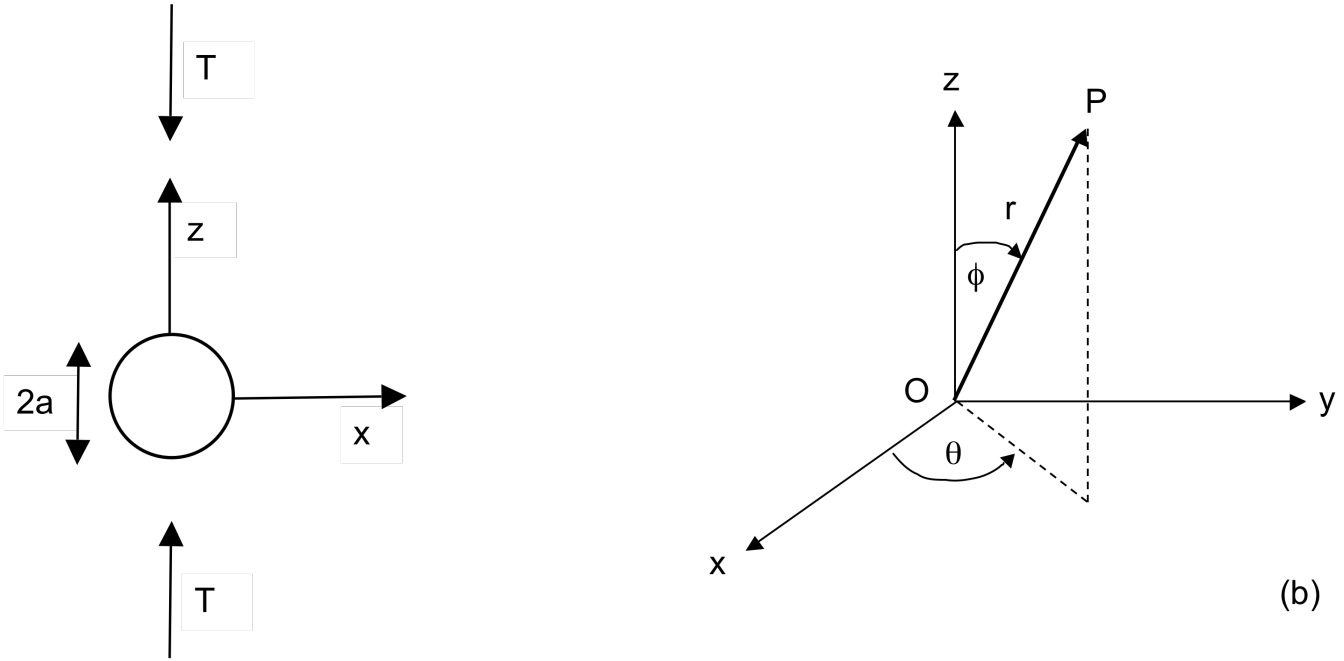
$$\tau_{\rho\alpha} = \frac{S}{(2-\nu)\pi}\left(\frac{a}{2\rho}\right)^{1/2}\left[\cos(3\alpha/2) + (1/3)\cos(\alpha/2)\right]\cos\theta$$

The problem of a traction-free crack subject to far-field shear stress *S* can easily be found from the above solution, using superposition.

Stresses around a Spherical Cavity

Spherical cavities are not as common or important in rock mechanics as cylindrical cavities, but they are still of some interest, particularly on a small scale, where they can represent small pores in a rock.

Imagine a rock mass containing a spherical cavity of radius *a*, subjected to a uniform stress $\tau_{zz} = T$ at infinity (below left). A spherical co-ordinate system is shown below on the right. The cavity surface is assumed to be traction-free.



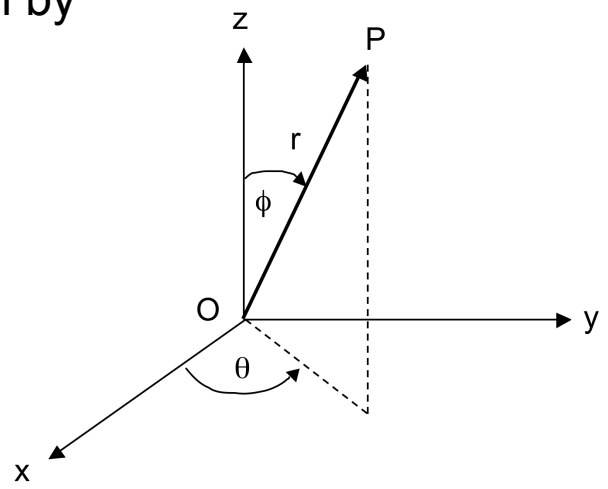
The full solution to this problem is derived in Section 8.12 of *FORM*4, using Papkovich-Neuber displacement functions. The key outcome of the solution is the stress concentration at the surface of the cavity (see next slide).

Stresses around a Spherical Cavity, 2

The stresses at the surface of the cavity are given by

$$\frac{\tau_{\phi\phi}(a)}{T} = \frac{27-15\nu}{2(7-5\nu)} - \frac{15}{(7-5\nu)}\cos^2\phi$$

$$\frac{\tau_{\theta\theta}(a)}{T} = \frac{-3(1-5\nu)}{2(7-5\nu)} - \frac{15\nu}{(7-5\nu)}\cos^2\phi$$



The maximum value of $\tau_{\phi\phi}$ occurs along the equator ($\phi = \pi/2$), and equals $2T$ for a rock with $\nu = 0.2$. For this value of ν, $\tau_{\theta\theta}$ varies between 0 and $-0.5T$, and attains its extreme value of $-0.5T$ at the north ($\phi = 0$), and south ($\phi = \pi$) poles.

The maximum stress concentration factor $\tau_{\phi\phi}/T$ along the equator varies slightly with ν, albeit only within a narrow range from 1.93 when $\nu = 0$, to 2.17 when $\nu = 0.5$. This is in contrast to the stress concentration at the boundary of a two-dimensional circular cavity under far-field uniaxial stress, which equals 3, regardless of the value of the Poisson ratio.

As a general rule, stress concentrations around three-dimensional cavities are less severe than those around two-dimensional cavities.

Pressurised Spherical Cavity in an Infinite Rock Mass

Another interesting and important problem involving a spherical geometry is that of a spherical cavity in an infinite rock mass, subjected to an internal fluid pressure of magnitude *P.*

Due to the radial symmetry of this problem, the only non-zero displacement component is the radial component, u_r, which is given by

$$\frac{u_r(r)}{a} = -\frac{P}{4G}\left(\frac{a}{r}\right)^2$$

Also due to the symmetry of this problem, no shear stresses will be developed anywhere within the rock mass. The normal stress are given by

$$\tau_{rr}(r) = P\left(\frac{a}{r}\right)^3, \qquad \tau_{\phi\phi} = \tau_{\theta\theta} = -\frac{P}{2}\left(\frac{a}{r}\right)^3$$

The displacement dies off as r^{-2}, and the stresses decay as r^{-3}. This is in contrast to a pressurised two-dimensional circular cavity, for which the displacement varies as r^{-1}, and the stresses vary as r^{-2}. As a general rule, stresses around three-dimensional cavities die off more rapidly with distance from the cavity surface than do stresses around two-dimensional cavities.

Imperial College MSc in
GeoEnergy with Machine Learning and Data Science

GEMS 3: Geomechanics and Pressure Transient Analysis

Lecture 10:
Wave Propagation in Rocks

Prof. Robert Zimmerman

March 2024

Introduction to Elastic Wave Propagation

It is usually assumed that rock is at rest under the action of static stresses, and most problems in rock mechanics are treated as problems of *statics*.

However, there are a number of important situations in which the stresses are of a dynamic nature, and the propagation of these stresses through the medium as a wave must be considered.

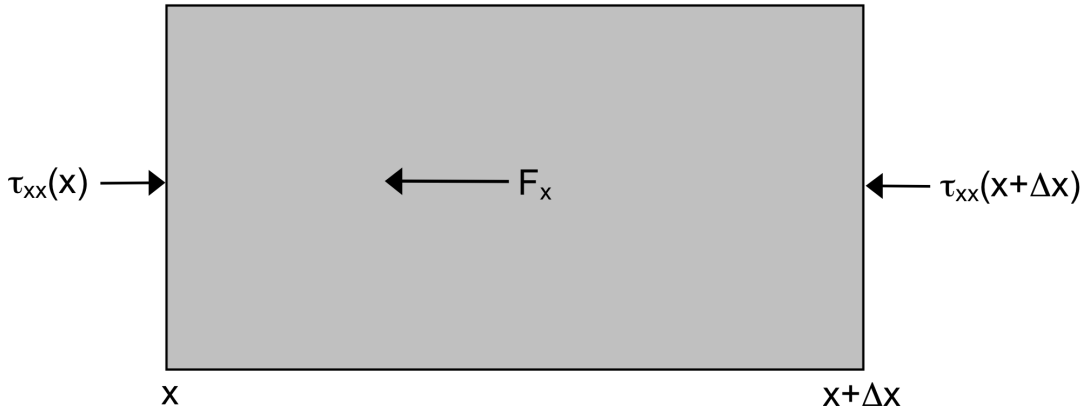
Such situations may arise naturally, for example, in earthquakes. Dynamic stresses in rock may also be the result of man-made activities, such as explosive blasting. In other cases they are inadvertent consequences of human activities, such as the rockbursts that occur in underground mines due to the redistribution of stresses caused by the excavations.

The amplitudes of the dynamic stresses are usually small compared to the compressive strength of the rock, except perhaps in the immediate vicinity of the source, and the time of application is generally short. In such situations, the resulting stresses and displacements can be analyzed using the dynamic theory of linear elasticity.

Waves traveling through rock, governed by the laws of linear elasticity, are known as *seismic waves*.

One-dimensional Elastic Wave Propagation

The propagation of elastic waves in a rock is essentially a three-dimensional problem. However, many of the concepts of elastic wave propagation can be understood within the context of a simplified, one-dimensional model.



Imagine a thin elastic rod of a given cross-sectional shape that is uniform along the length of the rod. If the rod is acted upon only by longitudinal forces in the *x* direction, and its outer boundary is traction-free, then the only non-zero stresses within the rod will be τ_{xx}.

Ignoring the body force, a force balance in the *x*-direction taken on the infinitesimal segment of rod between *x* and *x* + Δ*x* then yields

$$[\tau_{xx}(x,t) - \tau_{xx}(x+\Delta x,t)]A = -\rho(A\Delta x)\frac{\partial^2 u}{\partial t^2}$$

One-dimensional Elastic Wave Propagation

Dividing both sides of this equation by $A\Delta x$, and taking the limit as Δx goes to 0, yields

$$\frac{\partial \tau_{xx}}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}$$

This equation is independent of the stress-strain relation of the rock. Assuming linear elastic behavior, and noting that there are no forces acting in the directions perpendicular to the x-axis, then $\tau_{xx} = E\varepsilon_{xx} = E(\partial u / \partial x)$, and so

$$\frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

This is a one-dimensional wave equation. It describes disturbances that can propagate along the bar, in either the +x or -x direction, at a speed c that is given by

$$c = (E/\rho)^{1/2}$$

How do we know that the solutions to this equation will be “travelling waves”?

One hint of this can be found by looking at the units of $(E/\rho)^{1/2}$, which are

$$[(\mathrm{N/m^2})/(\mathrm{kg/m^3})]^{1/2} = [(\mathrm{kg\,m/s^2\,m^2})(\mathrm{m^3/kg})]^{1/2} = [\mathrm{m^2/s^2}]^{1/2} = \mathrm{m/s}$$

which seems to imply that $(E/\rho)^{1/2}$ is a velocity of some sort.

d'Alembert Solution to the Wave Equation, 1

To actually prove that the solutions to this equation are indeed travelling waves with velocity $c = (E/\rho)^{1/2}$, we first define a new variable, $\eta = x - ct$, and assume that u is a function of this new variable η (often called the "phase").

Applying the chain rule shows that

$$\frac{\partial u}{\partial x} = \frac{du}{d\eta}\frac{\partial \eta}{\partial x} = \frac{du}{d\eta}$$

and so applying this rule twice shows that

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 u}{d\eta^2}$$

Similarly,

$$\frac{\partial u}{\partial t} = \frac{du}{d\eta}\frac{\partial \eta}{\partial t} = -c\frac{du}{d\eta}$$

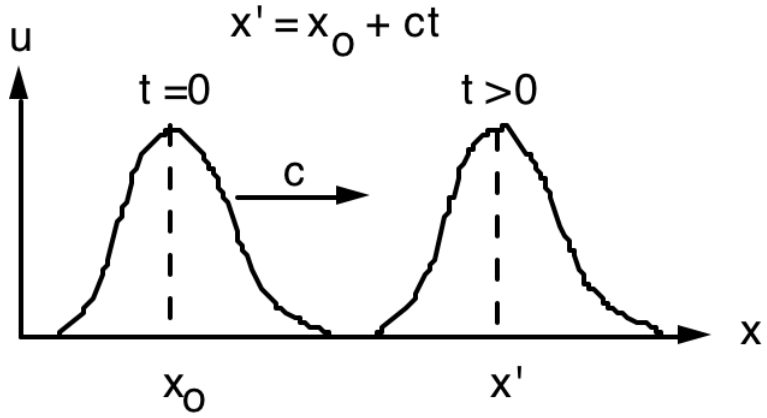
and so

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{d^2 u}{d\eta^2} = \frac{E}{\rho}\frac{d^2 u}{d\eta^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$$

which shows that *any* function $u(\eta)$ will satisfy the PDE on the previous slide.

d'Alembert Solution to the Wave Equation, 2

To show that this function $u(\eta)$ represents a wave that travels with speed c, consider the graph of the displacement, for fixed values of t, as shown below:



To be concrete, consider the crest of the disturbance, which is located at $x = x_o$ at time $t = 0$. The magnitude of the disturbance at this crest is given by $u(\eta = x_o - c*0) = u(x_o)$.

At some time t later, this crest will move to a location at which the variable η has the same value as it did at $t = 0$, *i.e.*, $x' - ct = x_o$, which is to say

$$x' = x_o + ct, \qquad \text{so} \qquad \left(\frac{\partial x}{\partial t}\right)_{\eta=\text{constant}} = c$$

Hence, the crest moves to the right at speed c. But this *same* argument will hold for *any* point on the wave, so we see that $u(\eta) = u(x - ct)$ represents a disturbance that propagates, without distortion, to the right at speed c.

d'Alembert Solution to the Wave Equation, 3

An identical analysis, using the variable $\xi = x + ct$, shows that $u(\xi)$ will be a solution to the wave equation for any function u, and will represent a disturbance moving, without distortion, to the left at speed c.

The general solution to the wave equation, first derived by the French scientist and philosopher d'Alembert in 1747, can be written as

$$u(x,t) = f(x - ct) + g(x + ct)$$

Imagine a thin elastic bar extending infinitely far in both directions. At time $t = 0$, assume that the displacement and velocity of each point along the bar are each given by some known function, *i.e.*,

$$u(x,t=0) = U(x), \quad \frac{\partial u}{\partial t}(x,t=0) = V(x)$$

The solution to the 1D wave equation, subject to these initial conditions, can be written as follows (see Problem 10.2)

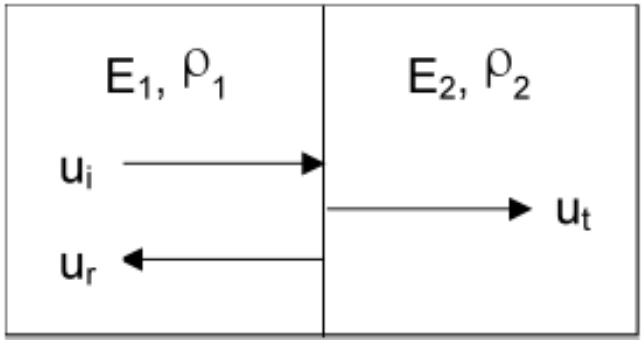
$$u(x,t) = \frac{1}{2}[U(x - ct) + U(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(s)ds$$

The integral term in this expression can also be expressed in terms of left- and right-traveling waves by defining a function H that is the indefinite integral of V:

$$\int_{x-ct}^{x+ct} V(s)ds \equiv H(x + ct) - H(x - ct)$$

Reflection and Transmission of One-Dimensional Waves, 1

Whenever an elastic wave traveling through a given medium 1 impinges on an interface with another medium 2, a “transmitted wave” will pass into medium 2, and a “reflected wave” will reflect off of the boundary and return into medium 1; see figure below:



Consider an incident wave $u_i(x\text{-}c_1t)$ traveling to the right though medium 1, which occupies the region x < 0. The transmitted wave $u_t(x\text{-}c_2t)$ travels to the right through medium 2, and the reflected wave $u_r(x\text{+}c_1t)$ travels to the left through medium 1.

At the interface between the two media, defined by $x = 0$, the displacements and tractions must each be continuous; these are the so-called “welded interface” boundary conditions.

By mathematical arguments given in Section 11.2 of *FoRM*4, we find that the amplitude of the reflected and transmitted waves are related to the amplitude of the incident wave, via the transmission and reflection coefficients, as follows:

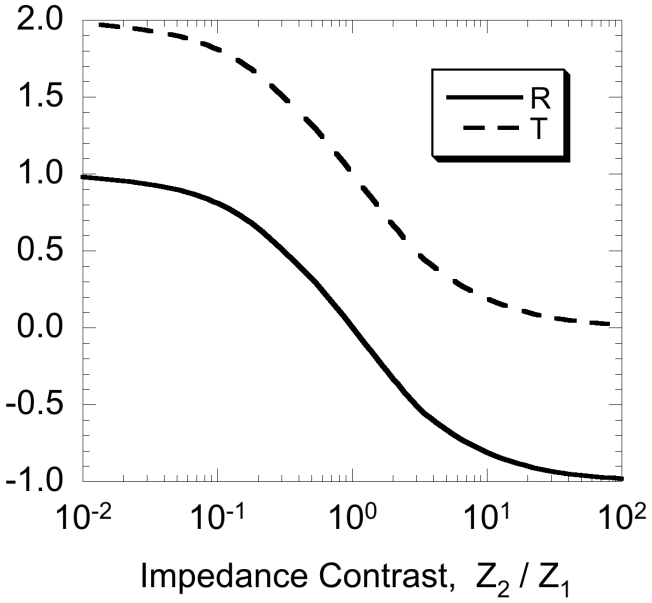
Reflection and Transmission of One-Dimensional Waves, 2

$$T = \frac{\text{amplitude of transmitted wave}}{\text{amplitude of incident wave}} = \frac{2Z_1}{Z_1 + Z_2}$$

$$R = \frac{\text{amplitude of reflected wave}}{\text{amplitude of incident wave}} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

where Z is called the *acoustic impedance*, and is defined by $Z = \rho c$, the product of the density and the wavespeed.

The transmission and reflection coefficients are plotted below, as functions of the impedance ratio:



Harmonic Waves and Group Velocity, 1

The previous discussion treated the general case of a wave of arbitrary shape. But the most important type of wave, which is used as the basis for most mathematical analyses of waves, is a harmonic wave in which the displacement, and hence also the strain and the stress, oscillates in a sinusoidal manner.

The reason for the importance of harmonic waves is that, according to Fourier's theorem, a wave of arbitrary time-variation can be decomposed into a combination of harmonic waves of different frequencies, each with its own amplitude.

Consider a displacement described by

$$u(x,t) = U_o \mathbf{Re}\{e^{ik(x-ct)}\} = U_o \cos[k(x-ct)] \equiv U_o \cos(kx - \omega t)$$

where $\omega = kc$. The form $u = U_o \cos(kx\text{-}ct)$ is often convenient, although it obscures the fact that k and ω are not independent, but are related by $\omega = kc$.

As the argument of an exponential function must be dimensionless, the parameter k, called the *wave number*, must have dimensions of [1/L].

Similarly, the parameter ω must have dimensions of [1/T].

Harmonic Waves and Group Velocity, 2

At a fixed value of t, the wavelength λ of this wave is determined by the condition $k(x+\lambda) = kx + 2\pi$, which shows that $\lambda = 2\pi/k$. The relation shows that the wave number is essentially the inverse of the wavelength.

Similarly, at a fixed location x, the period T of the wave is determined by the condition $\omega(t+T) = \omega t + 2\pi$, which shows that $T = 2\pi/\omega$.

The period T represents the number of "seconds per cycle". Hence, $1/T$ would be the number of "cycles per second", which shows that $\omega = 2\pi/T$ represents the number of "radians per second", *i.e.,* the *angular frequency*.

It is often more natural to refer to the frequency in terms of the number of "cycles per second", rather than "radians per second". This frequency, the so-called *cyclic frequency*, or simply "the frequency", is usually denoted by either f or ν, and is related to ω by $\nu = \omega/2\pi = 1/T$. The units of "cycles per second" are also known as Hz, in honor of the 19th century German physicist and acoustician Heinrich Hertz.

A few of the more useful of the various relations between the parameters of a harmonic wave are

$$\omega = kc, \quad \omega = 2\pi f, \quad T = 2\pi/\omega, \quad \lambda = 2\pi/k, \quad \lambda = 2\pi c/\omega, \quad \lambda = c/f$$

Elastic Waves in 3D Unbounded Media, 1

Elastic waves travel at the speed $c = (E/\rho)^{1/2}$ *only* in the special case of a long-wavelength disturbance traveling along an elastic bar of constant cross-section.

Waves that travel in three-dimensional, unbounded, isotropic elastic media can be studied by starting with the full three-dimensional equations of motion, as derived in Lecture 7, with the inertia term included:

$$(\lambda + G)\nabla(\nabla \cdot \mathbf{u}) + G\nabla^2\mathbf{u} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

Consider a planar wave traveling at speed c along some direction **n**, which can be represented by

$$\mathbf{u} = f(\mathbf{x} \cdot \mathbf{n} - ct)\mathbf{d}$$

where **d,** the particle displacement vector, is taken to be a constant. (In the 1D theory, the particle displacement must necessarily be in the same direction as the wave motion, but this need not be true in three dimensions. We will say more about this point later).

Since $\mathbf{x} \cdot \mathbf{n}$ is the projection of the vector **x** onto the direction **n**, the phase $\mathbf{x} \cdot \mathbf{n} - ct$ will have the same value for all points **x** that lie on a given plane perpendicular to **n**. Hence, the displacement $\mathbf{u} = f(\mathbf{x} \cdot \mathbf{n} - ct)\mathbf{d}$ represents a *plane wave*.

Elastic Waves in 3D Unbounded Media, 2

Since the phase will have the same value for all points **x** that lie on a given plane perpendicular to **n**, for simplicity, we can consider a point lying on the vector **n**, *i.e.,* $\mathbf{x} = \zeta\mathbf{n}$.

For this point, the phase is equal to

$$\eta = (\zeta\mathbf{n})\cdot\mathbf{n} - ct = \zeta(\mathbf{n}\cdot\mathbf{n}) - ct = \zeta - ct$$

The velocity at which the wavefront moves along the direction **n** is then given, in analogy with the discussion on slide 199, by

$$\mathbf{v}(\textit{wavefront}) = \left(\frac{\partial\mathbf{x}}{\partial t}\right)_\eta = \left(\frac{\partial(\zeta\mathbf{n})}{\partial t}\right)_\eta = \left(\frac{\partial\zeta}{\partial t}\right)_\eta \mathbf{n} = c\mathbf{n}$$

This proves that this disturbance does indeed propagate as a wave in the **n** direction, and *c* is the phase velocity of this wave.

At this stage, we don't actually know what value *c* has, and we haven't specified the direction of **d**. To go further, we need to insert the expression $\mathbf{u} = f(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}$ into the governing PDE on the previous slide.

The required time derivatives of **u** are easily found by applying the chain rule:

$$\mathbf{u}_t = -cf'(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}, \quad \mathbf{u}_{tt} = c^2 f''(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}$$

Elastic Waves in 3D Unbounded Media, 3

The phase of this wave is explicitly given by $\eta = xn_x + yn_y + zn_z - ct$, so the spatial derivatives of **u** are:

$$\frac{\partial \mathbf{u}}{\partial x} = f'(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}\frac{\partial \eta}{\partial x} = f'(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}n_x$$

$$\frac{\partial \mathbf{u}}{\partial y} = f'(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}\frac{\partial \eta}{\partial y} = f'(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}n_y$$

$$\frac{\partial \mathbf{u}}{\partial z} = f'(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}\frac{\partial \eta}{\partial z} = f'(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}n_z$$

It follows that

$$\nabla\cdot\mathbf{u} = f'(\eta)\mathbf{d}\cdot\mathbf{n}$$

$$\nabla(\nabla\cdot\mathbf{u}) = f''(\eta)(\mathbf{d}\cdot\mathbf{n})\mathbf{n}$$

$$\nabla^2\mathbf{u} = f''(\eta)\mathbf{d}$$

in which case the governing equation

$$(\lambda + G)\nabla(\nabla\cdot\mathbf{u}) + G\nabla^2\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

reduces to

$$(\lambda + G)f''(\eta)(\mathbf{d}\cdot\mathbf{n})\mathbf{n} + Gf''(\eta)\mathbf{d} = \rho c^2 f''(\eta)\mathbf{d}$$

Elastic Waves in 3D Unbounded Media, 4

Aside from the trivial case $f''(\eta) = 0$, which leads to either a rigid-body motion or a state of uniform strain that is independent of time (see Problem 10.3), this latter equation is equivalent to

$$(\lambda + G)(\mathbf{d} \cdot \mathbf{n})\mathbf{n} + (G - \rho c^2)\mathbf{d} = 0$$

If the particle motion is *perpendicular* to the direction of wave propagation, then $\mathbf{d} \cdot \mathbf{n} = 0$, and the above equation can only be satisfied for non-zero $\mathbf{d}$ if

$$c \equiv c_T = \sqrt{G / \rho}$$

where c_T is the velocity of *transverse waves,* in which the particle motion is *transverse* to the direction of wave propagation. Hence, in an isotropic elastic medium, transverse plane waves can only travel at speed given by $(G/\rho)^{1/2}$.

If, on the other hand, the particle displacement vector $\mathbf{d}$ is *parallel* to, rather than *perpendicular* to, the direction of wave propagation, then $\mathbf{d} = d\mathbf{n}$, where d is a scalar, and the equation at the top of this slide reduces to

$$(\lambda + 2G - \rho c^2)d\mathbf{n} = 0$$

which can only be satisfied by non-zero values of d if

$$c \equiv c_L = \sqrt{(\lambda + 2G) / \rho}$$

Hence, in an isotropic elastic medium, longitudinal plane waves can only travel at a speed given by $[(\lambda + 2G)/\rho]^{1/2}$.

P-Waves and S-Waves, 1

In rock mechanics and geophysics, the transverse wave velocity is usually denoted by V_P, and the longitudinal velocity by V_S. The subscript *S* can be thought of as signifying a *shear wave*, or it can be thought of as signifying a *secondary wave*, as these waves arrive at a receiver later than the faster-moving longitudinal waves. The subscript *P* can similarly be thought of as standing for *primary wave*, or *pressure wave.* These two types of waves are often referred to as P-waves and S-waves.

Use of the relations given in Lecture 7, between the various elastic moduli parameters, allows the longitudinal wavespeed to be written in the following forms

$$c_L \equiv V_P = \sqrt{\frac{K + (4G/3)}{\rho}} = \sqrt{\frac{(1-\nu)E}{(1+\nu)(1-2\nu)\rho}} = \sqrt{\frac{2(1-\nu)G}{(1-2\nu)\rho}} = \sqrt{\frac{2(1-\nu)}{1-2\nu}}\, V_S$$

from which it follows that the ratio of the two wavespeeds is

$$\frac{V_P}{V_S} = \sqrt{\frac{2(1-\nu)}{1-2\nu}}$$

In any isotropic rock or soil, P-waves always travel faster than S-waves. The ratio of the two wavespeeds is 1.41 when ν = 0, increases with increasing ν, and becomes unbounded as ν approaches 0.5.

P and S Waves, 2

Rock Type	V_P (m/s)	V_S (m/s)	ρ (kg/m^3)
Vegetal soil	300-700	100-300	1700-2400
Dry sands	400-1200	100-500	1500-1700
Wet sands	1500-2000	400-600	1900-2100
Saturated shales and clays	1100-2500	200-800	2000-2400
Marls	2000-3000	750-1500	2100-2600
Saturated shale/sand sections	1500-2200	500-750	2100-2400
Porous saturated sandstones	2000-3500	800-1800	2100-2400
Limestones	3500-6000	2000-3300	2400-2700
Chalk	2300-2600	1100-1300	1800-2300
Salt	4500-5500	2500-3100	2100-2300
Anhydrite	4000-5500	2200-3100	2900-3000
Dolomite	3500-6500	1900-3600	2500-2900
Granite	4500-6000	2500-3300	2500-2700
Basalt	5000-6000	2800-3400	2700-3100
Gneiss	4400-5200	2700-3200	2500-2700
Coal	2200-2700	1000-1400	1300-1800
Water	1450-1500	-	1000
Ice	3400-3800	1700-1900	900
Oil	1200-1250	-	600-900

Attenuation, 1

When a wave travels through an *elastic* medium, the total energy contained in the wave, is conserved. Since the energy of a plane wave is related to the square of its amplitude, a plane elastic wave will propagate without any change in amplitude.

However, rocks do not behave entirely elastically under transient conditions. There are numerous mechanisms, such as friction along crack faces and grain boundaries, which cause the kinetic energy of seismic waves to be transformed into internal energy.

This energy is not lost, but rather serves to slightly raise the temperature of the rock. But from a purely mechanical point of view, this energy appears to be “lost”, or “dissipated”, and the amplitude of a plane wave will become attenuated as the wave travels.

A simple model of wave attenuation can be developed by assuming that the rock obeys the “Kelvin-Voigt” stress-strain equation:

$$\tau_{xx} = E\varepsilon_{xx} + \eta \frac{\partial \varepsilon_{xx}}{\partial t}$$

which is essentially a combination of elastic-like and viscous-fluid-like behavior, and where η is a viscosity-like parameter.

Attenuation, 2

If we insert this stress-strain law into the general form of the wave equation as listed on slide 197, $\partial\tau_{xx}/\partial x = \rho(\partial^2 u/\partial t^2)$, we arrive at the following PDE:

$$E\frac{\partial^2 u}{\partial x^2} + \eta\frac{\partial^3 u}{\partial t \partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

As shown in Section 11.9 of *FoRM*4, for "small" values of η, the plane-wave solution to this equation is

$$u(x,t) = U_o \exp(-\alpha x)\exp\{i[(\omega/c_0)x - \omega t]\}$$

where $c_0 = (E/\rho)^{1/2}$ is the wavespeed in the absence of viscous effects, and α is the *attenuation coefficient*, given by

$$\alpha = \frac{\eta\omega^2}{2Ec_0}$$

The total mechanical energy in a plane wave is proportional to the square of the amplitude, so the fractional loss of energy over one wavelength is

$$\frac{\Delta\mathcal{T}}{\mathcal{T}} = \frac{\exp(-2\alpha x) - \exp(-2\alpha\{x+\lambda\})}{\exp(-2\alpha x)} = 1 - \exp(-2\alpha\lambda) \approx 2\alpha\lambda$$

The *quality factor* Q is defined in terms of the fractional energy loss as follows:

$$\frac{1}{Q} = \frac{2\alpha c}{\omega} = \frac{2c\eta\omega^2}{2Ec\omega} = \frac{\eta\omega}{E}$$

Attenuation, 3

Measured values of Q for P-waves in various rocks are shown in the table below.

Porous rocks such as sandstones and limestones tend to have Q values in the range of 10-100, whereas igneous and metamorphic rocks will have values in the range 100-1000.

Rock	Condition	f (Hz)	Q	Source
Tennessee marble	dry	$0\text{-}2\times10^{4}$	480	Wyllie *et al*. (1962)
Quincy granite	air dry	$2\text{-}45\times10^{2}$	125	Birch & Bancroft (1938)
Solenhofen limestone	air dry	$3\text{-}15\times10^{6}$	112	Peselnick & Zeitz (1959)
Amherst sandstone	oven dry	$1\text{-}13\times10^{3}$	52	Born (1941)
Pierre shale	*in situ*	$5\text{-}45\times10^{1}$	32	McDonal *et al*. (1958)
Berea sandstone	brine saturated	$2\text{-}8\times10^{5}$	10	Toksöz *et al*. (1979)

Problems for Lecture 10

Problem 10.1. In a one-dimensional elastic wave, prove that the stress and the strain satisfy the same partial differential wave equation as does the displacement, *i.e.*, $c^2(\partial^2 f / \partial x^2) = (\partial^2 f / \partial t^2)$, where f can represent u, τ, or ε.

Problem 10.2. Starting with the general solution and the initial conditions given on slide 200, derive the d'Alembert solution of the wave equation. If you can't derive it, then try to work backwards from the d'Alembert solution, and *verify* that it satisfies the wave equation and the initial conditions. Hint:

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(s)ds\right] = f(b)b'(x) - f(a)a'(x)$$

Problem 10.3. Prove the assertion made on slide 208, that choosing $f''(\eta) = 0$ will lead to a rigid-body motion, or a steady-state uniform strain, but *not* to a wave-like motion.

Problem 10.4. Using the relations between the various elastic moduli that are given on slide 148 of Lecture 7, and the expressions for the compressional wavespeed given on slide 208 of this lecture, show that the compressional wavespeed can also be written in the forms given on slide 209.