1. Introduction

In this module, we will cover the basic governing equations of seismic waves, and the relationship between seismic wavespeeds and rock microstructure. We start in §2 with an introduction to elastic wave propagation in a simple one-dimensional context. This discussion presupposes some elementary knowledge of one-dimensional concepts of stress, strain and Hooke's law. In §3 we discuss some important concepts related to harmonic waves, which will also be relevant in a three-dimensional context. In §4 we derive the governing equations of three-dimensional elastic wave propagation. This will require us to first define the three-dimensional stress and strain tensors, and the three-dimensional Hooke's law. In §5 we show how, for isotropic rocks, the elastodynamic equations predict the existence of two types of waves, P-waves and S-waves. In §6 we study the reflection, refraction and transmission of waves across an interface between two rock layers. In §7 we discuss Hooke's law for anisotropic rocks, and present some approximate expressions for the wavespeeds in the case of weak anisotropy. In §8 we present some models that account for the effect of pore fluids on elastic wavespeeds. Although ideally elastic waves do not lose energy as they propagate, waves in real rocks are not perfectly elastic, and so in §9 we discuss mechanisms for wave attenuation. Finally, in §10 we discuss the effect that pores and cracks have on the elastic moduli and wavespeeds.

Most of the material in these notes is taken from *Fundamentals of Rock Mechanics,* 4th edition, by J. C. Jaeger, N. G. W. Cook, and R. W. Zimmerman (Wiley-Blackwell, 2007). Other good sources of information relevant to this module are *The Rock Physics Handbook*, by G. Mavko, T. Mukerji, and J. Dvorkin (2nd ed., Cambridge, 2009), and *Quantitative Seismic Interpretation*, by P. Avseth, T. Mukerji and G. Mavko (Cambridge, 2005). Additional references are given in the reference list at the end of these notes, following §10.

2. One-Dimensional Elastic Wave Propagation

The propagation of elastic waves through rock is governed by the three-dimensional equations of elastodynamics, which will be presented in §4. Because of the physical coupling between the different stresses and strains due to the Poisson effect, the governing equations for three-dimensional elastodynamics are mathematically coupled, and the mathematical analysis of elastic waves consequently becomes complicated. However, many of the concepts of elastic wave propagation can be understood within the context of a simplified, one-dimensional model. This model, which is developed and discussed below, actually applies rigorously to long-wavelength waves propagating along a thin elastic bar.

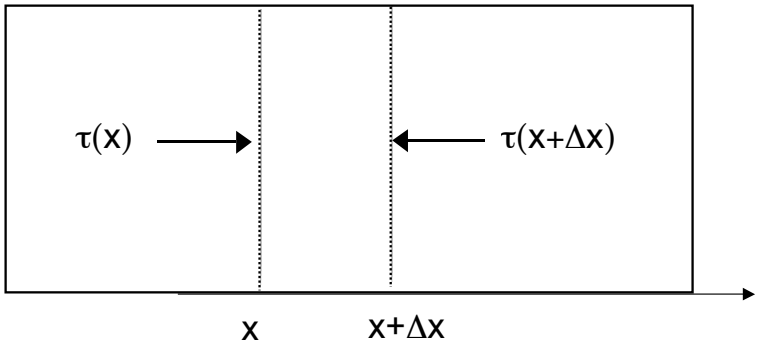

Fig. 2.1. Force balance on a small element of rock that is part of a long thin bar.

Imagine a thin elastic rod of a given cross-sectional shape that is uniform along the length of the rod (Fig. 2.1). The axial coordinate is x. The precise shape of the cross-section is not relevant, but it is convenient to think of it as circular, with radius a. If the rod is thin, we can assume that the normal stress τ, defined as the normal force divided by the area over which the force acts, does not vary over the cross-section, in which case τ varies only with x and t. (This assumption is strictly true for waves whose wavelengths are greater than about ten times the radius of the rod; Graff, 1975, p. 471). If the rod is acted upon only by longitudinal forces in the x-direction, and its outer lateral boundary is traction-free, then the only non-zero stress within the rod will be the axial normal stress, τ.

A force balance in the x-direction taken on the infinitesimal segment of rod between x and $x + \Delta x$ yields

$$[\tau(x,t) - \tau(x + \Delta x,t)]A = -\rho(A\Delta x)\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where the – sign appears on the right because, according to the sign convention used in rock mechanics, the displacement u is considered to be *positive* if the particle moves in the *negative* x-direction. Dividing both sides of (1) by $A\Delta x$, and taking the limit as $\Delta x \to 0$, yields

$$\frac{\partial \tau}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}. \tag{2}$$

This equation is independent of the stress-strain equation that applies to the rock. If we assume linear elastic behaviour, and note that there are no forces acting in the two directions perpendicular to the x-axis, then $\tau = E\varepsilon = E(\partial u / \partial x)$, where E is Young's modulus and ε is the longitudinal strain, and (2) becomes

$$\frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}. \tag{3}$$

if we differentiate both sides of (3) with respect to x, and invoke the fact that the various partial differentiation operators commute with each other, we see that the strain ε and the stress τ also satisfy this same differential equation, *i.e.,*

$$\frac{E}{\rho}\frac{\partial^2 \varepsilon}{\partial x^2} = \frac{\partial^2 \varepsilon}{\partial t^2}, \qquad \frac{E}{\rho}\frac{\partial^2 \tau}{\partial x^2} = \frac{\partial^2 \tau}{\partial t^2}. \tag{4}$$

Eq. (3) is the one-dimensional wave equation. This equation describes disturbances that propagate along the bar, in either the $+x$ or $-x$ direction, at a speed c that is given by

$$c = (E/\rho)^{1/2}. \tag{5}$$

We can prove that the solutions to (3) are "travelling waves" by utilising the following analysis, first given by the French mathematician and philosopher Jean d'Alembert in 1747. Consider any differentiable

function f of one variable, and let the argument of f be the new variable $\eta = x - ct$, where c is given by (5). Applying the chain rule gives

$$\frac{\partial f}{\partial x} = \frac{df}{d\eta}\frac{\partial \eta}{\partial x} = \frac{df}{d\eta}, \quad \text{so} \quad \frac{\partial^2 f}{\partial x^2} = \frac{d^2 f}{d\eta^2}, \tag{6}$$

$$\frac{\partial f}{\partial t} = \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = -c\frac{df}{d\eta}, \quad \text{so} \quad \frac{\partial^2 f}{\partial t^2} = c^2\frac{d^2 f}{d\eta^2} = \frac{E}{\rho}\frac{d^2 f}{d\eta^2}. \tag{7}$$

Inserting the right-hand terms of (6) and (7) into (3) shows that the function $f(x - ct)$ satisfies the wave equation. Hence, any differentiable function that depends on the two variables x and t only through the combination $x - ct$ will be a solution to the wave equation.

The function $f(x - ct)$, which according to (3) and (4) may stand for displacement, stress or strain, represents a disturbance that moves to the right at speed c. To be concrete, consider the peak of the disturbance shown in Fig. 2.2, which is located at $x = x_o$ when $t = 0$. The magnitude of the disturbance at this peak is given by $f(\eta = x_o - c0) = f(x_o)$. At some time $t > 0$ later, this peak will move to a location at which the variable η has the same value as it did at $t = 0$, *i.e.,* $x' - ct = x_o$, which is to say

$$x' = x_o + ct, \quad \text{so} \quad \left(\frac{\partial x}{\partial t}\right)_{\eta=\text{constant}} = c. \tag{8}$$

This argument holds for any point on the wave, not just the peak, and so entire wave moves to the right with speed c, without any distortion.

The waveform shown in Fig. 2.2 can be interpreted as the graph of the displacement as a function of x, for a fixed value of t, but it can also be thought of as the time variation (with a scaling factor c) of the displacement at a fixed location x. The variable η is called the *phase* of the wave, and c is the *phase velocity.*

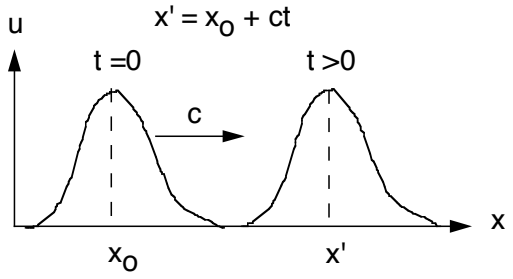

Fig. 2.2. Elastic disturbance moving to the right at velocity *c*. During a time increment *t* the pulse moves to the right by a distance *ct*, without altering its shape.

An identical analysis, using the variable $\xi = x + ct$, would show that $g(x+ct)$ is also a solution to the wave equation, for an "arbitrary" function *g*, and represents a disturbance moving, without distortion, to the *left* at speed *c*. The general solution to (3) is given by

$$u(x,t) = f(x-ct) + g(x+ct), \tag{9}$$

in the sense that *any* solution to (3) can be written in this form, as a combination of a left-moving and a right-moving wave (Bers *et al.*, 1964).

Imagine a thin elastic bar extending infinitely far in both directions. At time $t = 0$, assume that the displacement and velocity of each point along the bar are each given by some known function, *i.e.,*

$$u(x,t=0) = U(x), \quad \frac{\partial u}{\partial t}(x,t=0) = V(x). \tag{10}$$

Eq. (3), along with the two initial conditions given in (10), form a well-posed initial value problem whose solution is (Fetter and Walecka, 1980, p. 213)

$$u(x,t) = \frac{1}{2}[U(x-ct) + U(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(s)ds. \tag{11}$$

Eq. (11) shows clearly that the initial disturbance $U(x)$ splits into two parts, half propagating to the left, half to the right.

The influence of the initial velocity $V(x)$ on the resulting wave, which is given by the integral term, is not as easy to visualise, although it can also be written in terms of left-travelling and right-travelling waves by defining a function *H* that is the indefinite integral of *V*:

$$\int_{x-ct}^{x+ct} V(s)ds \equiv H(x+ct) - H(x-ct). \tag{12}$$

The foregoing analysis applies to waves travelling in an infinite, unbounded bar. Such waves will travel indefinitely, without changing their shape. But whenever an elastic wave travelling through a given medium 1 reaches a boundary with another medium 2, the waveform will be altered. A "transmitted wave" will pass into medium 2, and, in general, a "reflected wave" will reflect off of the boundary and return into medium 1 (Fig. 2.3a). The amplitudes of the transmitted and reflected waves, relative to that of the incident wave, can be shown to depend on the elastodynamic properties of the two media, in the following manner.

Consider an incident wave, $u_i(x - c_1 t)$, travelling to the right though medium 1, which occupies the region $x < 0$. The transmitted wave $u_t(x - c_2 t)$ travels to the right through medium 2, and the reflected wave $u_r(x + c_1 t)$ travels to the left through medium 1. At the interface between the two media, $x = 0$, the displacements and stresses must be continuous; these are the so-called "welded interface" boundary conditions. [If the interface is a fracture or fault, different boundary conditions may be appropriate (see Jaeger *et al.*, 2007, §12.7)]. The variable $x - ct$ can be multiplied by any constant k without altering the fact that $u(x - ct)$ satisfies the wave equation, so without loss of generality we can take $k = -1/c$, and thereby denote the solutions as $u(t - x/c)$. At the interface, $x = 0$, and so the displacement due to the incident wave is $u_i(t)$, for example. As usual, all displacements are considered positive if the motion is in the *negative* x-direction, regardless of the direction of propagation of the wave. The condition of continuity of the displacement at the interface then can be expressed as

$$u_i(t) + u_r(t) = u_t(t). \tag{13}$$

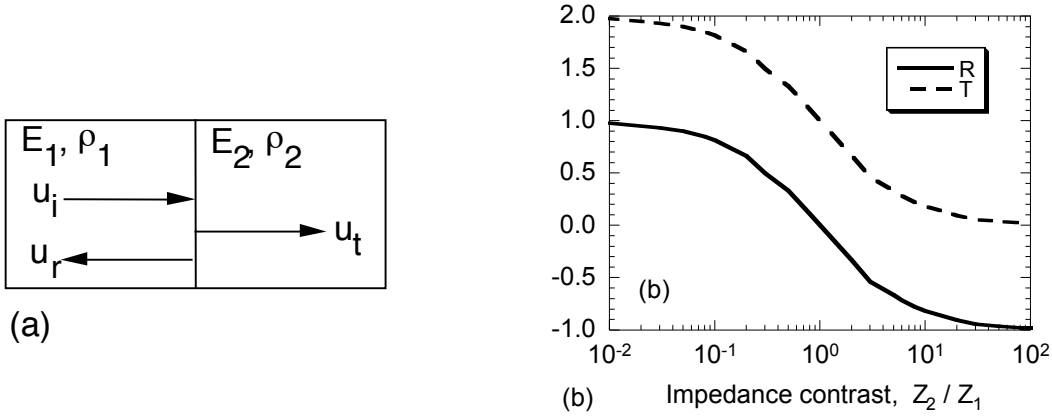

Fig. 2.3. (a) Incident wave, transmitted wave and reflected wave at a welded interface; (b) Reflection and transmission coefficients as functions of the impedance contrast, from (18) and (19).

The stress is related to the displacement by $\tau = E\varepsilon = E(\partial u/\partial x) = \rho c^2(\partial u/\partial x)$. By the chain rule, $\partial u/\partial x = -(1/c)u'$ for a right-travelling wave, and $\partial u/\partial x = (1/c)u'$ for a left-travelling wave, where the prime denotes differentiation with respect to the argument of the function. The condition of stress continuity at the interface therefore can be written as

$$-\rho_1 c_1 u_i'(t) + \rho_1 c_1 u_r'(t) = -\rho_2 c_2 u_t'(t)\,. \tag{14}$$

Integration of (14) yields

$$-\rho_1 c_1 u_i(t) + \rho_1 c_1 u_r(t) = -\rho_2 c_2 u_t(t) + B\,, \tag{15}$$

where B is a constant of integration. If the incoming wave is of finite duration, then after a sufficiently long time has elapsed, the stresses associated with each of the three waves must be zero, implying that the integration constant must be zero. Simultaneous solution of (13) and (15) yields

$$u_t(t - x/c_2) = \frac{2\rho_1 c_1}{\rho_1 c_1 + \rho_2 c_2} u_i(t - x/c_1), \tag{16}$$

$$u_r(t + x/c_1) = \frac{\rho_1 c_1 - \rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2} u_i(t - x/c_1). \tag{17}$$

The product of the density and the wavespeed of a material, $\rho c = (\rho E)^{1/2}$, is called the *acoustic impedance*, and is usually denoted by *Z.* The results (16) and (17) can also be expressed as

$$T = \frac{\text{amplitude of transmitted wave}}{\text{amplitude of incident wave}} = \frac{2Z_1}{Z_1 + Z_2}, \tag{18}$$

$$R = \frac{\text{amplitude of reflected wave}}{\text{amplitude of incident wave}} = \frac{Z_1 - Z_2}{Z_1 + Z_2}. \tag{19}$$

According to the sign convention that we use for displacements, $u < 0$ corresponds to *compression*, and $u > 0$ corresponds to *extension*. If $Z_2 < Z_1$, the transmitted wave has the same "sense" as the incident wave (*i.e.,* compression or rarefaction), and larger amplitude; the reflected wave has the same sense as the incident wave, but smaller amplitude. If $Z_2 > Z_1$, the transmitted wave has the same sense as the incident wave, but smaller amplitude, whereas the reflected wave has the opposite sense, and smaller amplitude.

The reflection and transmission coefficients *R* and *T* are plotted in Fig. 2.3b as functions of the impedance ratio. If these coefficients are defined in terms of stresses rather than displacements, the expressions and curves would be somewhat different. If the displacements are reckoned relative to the direction of wave propagation, rather than with respect to a co-ordinate system fixed in space, the expression for *R* in (19) should be multiplied by -1 (Mavko *et al*., 2009, p. 94).

The special case of a wave impinging on a free surface can be obtained by letting $\rho_2, E_2 \to 0$, in which case $Z_2 \to 0$. The reflected wave then has the same sense and same magnitude as the incident wave, whereas the transmitted wave has the same sense, but twice the magnitude of the incident wave. The prediction of a transmitted wave travelling through a medium with zero stiffness and zero density may seem paradoxical, but it must be remembered that media with zero density and stiffness do not exist. The above result is can be thought of as an asymptotic result that holds in the limit in which the impedance of medium 2 is very small relative to that of medium 1.

The other extreme case is a wave impinging on an interface with a medium of infinitely large impedance. If $Z_2 \to \infty$, (18) shows that there

will be no transmitted wave, and (19) shows that the incident wave will be totally reflected back into medium 1, but with the opposite sense. Hence, a compressive wave will be reflected as a rarefaction wave, and *vice versa*. An example of this is a sound wave in air hitting a hard wall. A third interesting case is when the two media have identical values of the acoustic impedance. In this case, the wave will be fully transmitted into medium 2, with no change in amplitude, and there will be no reflected wave.

An elastic wave carries energy with it, in the form of both elastic strain energy and kinetic energy. For a wave described by $u(x-ct)$, the strain is

$$\varepsilon = \frac{\partial u}{\partial x} = f'(x-ct). \tag{20}$$

The stress is then given by $\tau = Ef'(x-ct)$, and so the elastic strain energy density is

$$\mathcal{E} = \frac{1}{2}\tau\varepsilon = \frac{1}{2}\rho c^2 (f')^2, \tag{21}$$

where (5) has been used to write $E = \rho c^2$.

The local particle velocity of the rock is *not* equal to the phase velocity of the wave, *c.* Instead, we can use the chain rule to find that the velocity is given by

$$\dot{u} = \frac{\partial u}{\partial t} = -cf'(x-ct). \tag{22}$$

The kinetic energy density per unit volume is therefore

$$\mathcal{K} = \frac{1}{2}\rho(\dot{u})^2 = \frac{1}{2}\rho c^2 (f')^2, \tag{23}$$

and is exactly equal, at each location *x* and at each time *t*, to the elastic strain energy density. The total energy density contained in the wave is

$$\mathcal{T} = \mathcal{K} + \mathcal{E} = \rho c^2 (f')^2. \tag{24}$$

The total energy contained in any region of the bar could be found by multiplying (24) by the cross-sectional area A, and integrating along the length.

Comparison of (20) and (22) shows that the particle velocity is related to the phase velocity by $\dot{u} = -c\varepsilon$ for a right-travelling wave, and by $\dot{u} = c\varepsilon$ for a left-travelling wave. Although the particle of rock moves at speed $|\dot{u}| = c|\varepsilon|$, the "wave" travels at speed c. As the strain must by necessity be very small in order for Hooke's law to apply, it follows that the particle velocity in an elastic wave is but a small fraction of the phase velocity.

Use of Hooke's law and (5) in the expression $\dot{u} = c\varepsilon$ shows that the particle velocity can also be expressed as

$$|\dot{u}| = \tau / \rho c , \tag{25}$$

which shows that the acoustic impedance can also be interpreted as the coefficient of proportionality that relates the stress to the particle velocity.

3. Harmonic Waves and Group Velocity

The analysis in §2 treated the general case of a wave of arbitrary shape. But the most important type of wave, and the one that is used as the basis for most seismic analyses, is a *harmonic* wave in which the displacement (and hence also the strain and the stress) oscillates *sinusoidally*. The reason for the importance of harmonic waves is that, by use of Fourier's theorem, a wave of any time-variation can be decomposed into a combination of harmonic waves of different frequencies, each with its own amplitude. In this section we discuss some of the properties of harmonic waves, using the same nomenclature of longitudinal wave propagation in a thin elastic bar as was used in §2.

Consider a displacement described by

$$u(x,t) = U_o \mathbf{Re}\{e^{ik(x-ct)}\} = U_o \cos[k(x-ct)] \equiv U_o \cos(kx-\omega t), \qquad (1)$$

where $\omega = kc$, and we recall that $e^{i\theta} = \cos\theta + i\sin\theta$. The form $U_o \cos(kx-\omega t)$ is convenient, although it obscures the fact that k and ω are not independent, but are related by $\omega/k = c$. Note that c is a property of the rock, whereas k and ω can be different for different waves. Although it is less ambiguous to use only the real part of the exponential as the displacement, it is usually simpler to carry out mathematical manipulations using the complex exponential form, and then to take the real part at the end. As the argument of an exponential must be dimensionless, the parameter k, called the *wave number*, must have dimensions of [1/L]. The parameter ω therefore has dimensions of [1/T].

At a fixed value of t, the wavelength λ of this displacement wave is determined by the condition $k(x+\lambda) = kx + 2\pi$, which shows that $\lambda = 2\pi/k$; the wave number therefore is a sort of inverse wavelength. Similarly, at a fixed location, the period T of the wave is determined by the condition $\omega(t+T) = \omega t + 2\pi$, so $T = 2\pi/\omega$. The period T represents the number of "seconds per cycle", so $1/T$ would be the number of "cycles per second"; it is called the *cyclic frequency*, and is usually denoted by either f or ν. The units of "cycles per second" are also known as *Hz*, in honour of the 19th century German physicist and acoustician

Heinrich Hertz. The parameter $\omega = 2\pi / T = 2\pi f$ is the radial frequency, and represents the number of "radians per second".

Some useful relations between the parameters of a harmonic wave are

$$\omega = kc, \quad \omega = 2\pi f, \quad T = 2\pi / \omega, \quad \lambda = 2\pi / k, \quad \lambda = 2\pi c / \omega \ . \tag{2}$$

The strain associated with a harmonic displacement wave is found by differentiating the displacement:

$$\varepsilon = \partial u / \partial x = -kU_o \sin(kx - \omega t), \tag{3}$$

and the stress is then found by using Hooke's law:

$$\tau = E\varepsilon = -kEU_o \sin(kx - \omega t). \tag{4}$$

Since a sine function is 1/4-cycle out of phase with a cosine, we see that the stress and strain also vary sinusoidally, with the same frequency as the displacement, but 1/4-cycle out of phase with the displacement.

The stored elastic strain energy density, per unit volume, is

$$\mathcal{E} = \frac{1}{2}\varepsilon\tau = \frac{k^2 E U_o^2}{2}\sin^2(kx - \omega t). \tag{5}$$

Using the relations $k = \omega / c$ and $c^2 = E / \rho$, this can be written as

$$\mathcal{E} = \frac{\rho\omega^2 U_o^2}{2}\sin^2(kx - \omega t) \ . \tag{6}$$

The kinetic energy density is

$$\mathcal{K} = \frac{1}{2}\rho(\dot{u})^2 = \frac{\rho\omega^2 U_o^2}{2}\sin^2(kx - \omega t) \ , \tag{7}$$

and so the total energy density is

$$\mathcal{T} = \mathcal{K} + \mathcal{E} = \rho\omega^2 U_o^2 \sin^2(kx - \omega t) \ . \tag{8}$$

At any location x, the energy density varies in time between 0 and $\rho\omega^2 U_o^2$, with an average value of $\rho\omega^2 U_o^2 / 2$. Likewise, at any time t the energy density varies in space between 0 and $\rho\omega^2 U_o^2$, with an average value of $\rho\omega^2 U_o^2 / 2$. At all times, the total energy contained in the wave,

at any location x, is equally partitioned between elastic strain energy and kinetic energy, as was shown in a more general context in §2.

The time-averaged flux of energy through a given cross-section located at x can be calculated as follows. Consider the total energy contained in the region between $x-\lambda$ and x:

$$\Delta \mathcal{T} = \int_{x-\lambda}^{x} \mathcal{T} A dx = \int_{x-\lambda}^{x} \rho\omega^2 U_0^2 \sin^2(kx-\omega t) A dx = \frac{\rho\omega^2 U_0^2 A\lambda}{2}. \quad (9)$$

The entire wave travels to the right at speed c, so all of the energy in the region between $x-\lambda$ and x at time t will pass through the plane at x within an elapsed time of $\Delta t = \lambda / c$. The time-averaged *power flux*, which is the rate at which energy flows past location x, per unit area, is equal to

$$\mathcal{P} = \frac{\Delta \mathcal{T}}{A\Delta t} = \frac{\rho c \omega^2 U_0^2}{2}. \quad (10)$$

As discussed in §2, the actual velocity of the material particles is not the same as the velocity at which the "wave" propagates. The particle velocity is found from (1) by differentiation:

$$\dot{u} = \partial u / \partial t = \omega U_0 \sin(kx-\omega t) = ckU_0 \sin(kx-wt) \ . \quad (11)$$

Comparison of (11) and (3) shows that, in accord with the general results of §2, $\dot{u} = -c\varepsilon$, and so $|\dot{u}|/c << 1$.

Although the theory of wave propagation along a thin elastic bar was presented in §2 to provide a simplified, one-dimensional context in which to develop the basic ideas of elastic wave propagation, it has great importance in its own right, because most laboratory measurements of wave propagation in rocks are made on cylindrical core samples. According to this simplified low-frequency, long-wavelength theory, the wavespeed c is independent of the wavelength or wave number. This is also the case for wave propagation in three-dimensional, unbounded elastic media (see §4 below). Waves for which the speed is independent of wavelength are called *non-dispersive*. This term refers to the fact that, since the various frequency components of the wave each travel at the same speed, the waveform retains its shape as it travels through the medium.

The fully three-dimensional theory of wave propagation in an elastic bar (Graff, 1975, pp. 468-74) predicts that the wavespeed actually depends on wavelength, and hence on frequency. The wavespeed asymptotically approaches $(E/\rho)^{1/2}$ as the wavelength becomes infinite, and decreases as the wave number increases. Moreover, there are additional, higher "modes", each with their own complicated $c(k)$ relationship, that correspond to motions in which the stress is not uniform across a given cross-section at any given time. It is generally thought that these higher modes are not excited in most laboratory measurements. Nevertheless, the wavespeed in an elastic bar, as would be measured in the laboratory, does vary with wave number. Wavespeed will vary with wave number whenever an elastic wave travels through a "waveguide"; for example, when a wave travels along a layer of rock that is bounded above and below by strata having different elastic properties. Waves whose speed varies with frequency are called "dispersive", because, as the different components of the total wave travel at different speeds, the waveform will *not* retain its overall shape as it moves through the medium. Waves that travel through viscoelastic media also exhibit dispersion, although in this case they do so not for geometrical reasons, but because the waves lose energy as they travel.

The phenomenon of dispersion leads to the concept of the *group velocity* of a wave, which was first analysed in the following manner by the British mathematical physicist George Stokes in 1876. Consider a wave that consists of two harmonic components, with the same amplitude, but with slightly different wave numbers, k_1 and $k_2 = k_1 + \Delta k$, and slightly different frequencies, ω_1 and $\omega_2 = \omega_1 + \Delta\omega$:

$$u = U_o \cos(k_1 x - \omega_1 t) + U_o \cos(k_2 x - \omega_2 t). \qquad (12)$$

Using standard trigonometric identities, this can be written as

$$u = 2U_o \cos\{\tfrac{1}{2}(k_1 + k_2)x - \tfrac{1}{2}(\omega_1 + \omega_2)t\}\cos\{\tfrac{1}{2}\Delta k x - \tfrac{1}{2}\Delta\omega t\}. \qquad (13)$$

Now denote the mean wave number by k, and the mean frequency by ω, so that (13) can be written as

$$u = 2U_o \cos\{k(x - \frac{\omega}{k}t)\}\cos\{\tfrac{1}{2}\Delta k(x - \frac{\Delta\omega}{\Delta k}t)\}. \qquad (14)$$

The first cosine term is a high-frequency "carrier" wave that travels at a speed $c = \omega / k$. Since k_1 and k_2 each differ only slightly from k, and ω_1 and ω_2 each differ only slightly from ω, both of the two individual waves in (11) travel essentially at velocity c, the so-called *phase velocity*. However, (14) shows that these two waves combine in such a way that the wave travelling at velocity c is *modulated* by the second cosine term, which represents a low-frequency wave that travels at a speed given by $c_g = \Delta\omega / \Delta k$, the so-called *group velocity*.

This modulated wave is illustrated schematically in Fig. 3.1. If one focuses on the detailed motion of the medium, one observes the carrier wave travelling at speed c. But if one ignores the detailed motion and focuses attention on the "macroscopic" motion, one observes the amplitude curve moving forward at speed c_g. If $c_g > c$, the individual wavelets seem to appear at the front of the group, and disappear at the rear, as they are overtaken by the modulator wave. If $c_g < c$, the individual wavelets seem to appear at the rear of the group, travel forward through it, and disappear at the front (Brillouin, 1960). An easily observable example of this phenomenon is the radially-diverging wave pattern produced by dropping a small object into a still lake or pond. In this case $c_g > c$, and the individual ripples start at the outer edge of the ring, and eventually disappear at the inner edge.

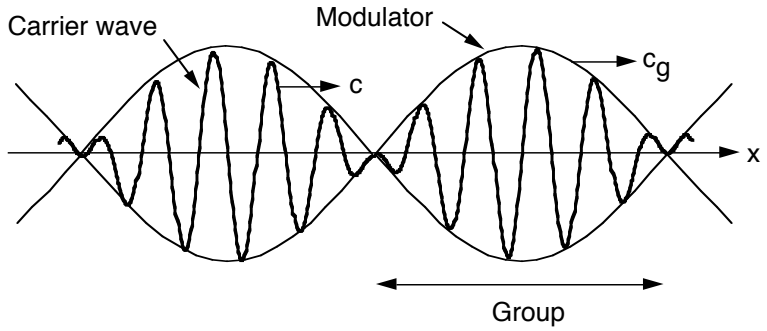

Fig. 3.1. Superposition of two waves of slightly differing frequencies, which combine to yield a high-frequency carrier wave travelling at the phase velocity c, modulated by a low-frequency modulator wave travelling at the group velocity, c_g, as described by (12)-(14).

In the more general context in which a wavepacket may contain a range of frequencies, the group velocity is defined by $c_g = d\omega / dk$. Using the relationships given in (2), the group velocity can also be expressed as

$$c_g = \frac{d\omega}{dk} = c + k\frac{dc}{dk} = c - \lambda\frac{dc}{d\lambda}. \tag{15}$$

Another useful form of (15) is

$$\frac{1}{c_g} = \frac{1}{c} - \frac{\omega}{c^2}\frac{dc}{d\omega}, \tag{16}$$

which is expressed in terms of the *slowness*, 1/*c*.

Dispersion is termed *normal* if the group velocity decreases with frequency (Bourbié *et al*., 1987, p. 111), and is termed *anomalous* or *inverse* if c_g increases with frequency. Dispersion due to geometrical effects, such as the dispersion of waves in an elastic layer, is typically of the normal type, whereas dispersion due to viscoelastic or other energy-dissipative effects is usually anomalous (Mavko *et al*., 2009, p. 84).

For a non-dispersive wave, $c_g = d\omega/dk = c = \text{constant}$, and the group velocity coincides with the phase velocity. In this case, the energy travels with velocity *c*. But for dispersive waves in an elastic medium, it can be shown, by a lengthy mathematical argument (Achenbach, 1973, pp. 211-215), that the energy actually travels through the medium at the group velocity, c_g. The group velocity is therefore also equal to the velocity of energy propagation. Waves in inelastic media are also dispersive, but they lose energy as they propagate, and consequently the relationship between the various velocities is not so simple or meaningful as for elastic media (Mavko *et al*., 2009, pp. 83-86).

An arbitrary wave can be thought of as being composed of a superposition of a (possibly infinite) number of waves, each with its own frequency and amplitude. Mathematically, this is accomplished by the Fourier transform (Bracewell, 1986). Given a time-varying function $f(t)$, its Fourier transform $F(\omega)$ can be defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\ . \tag{17}$$

Other notations for the Fourier transform of $f(t)$ are $f^*(\omega)$ or $\hat{f}(\omega)$. The mathematical conditions under which the integral in (17) exists are discussed by Bracewell (1986). In practice, a Fourier transform exists for

all waveforms arising in wave propagation in rocks, in the laboratory or the field. The function $F(\omega)$ can be thought of as representing that portion of the total wave $f(t)$ that has frequency ω. This is seen from the Fourier inversion integral, in which the function $f(t)$ is represented as a superposition of the various "components" $F(\omega)$:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \ . \tag{18}$$

A more symmetric form of these relationships is obtained by utilising the cyclic frequency, $\omega = 2\pi\nu$. A change of variables applied to (17) and (18) shows that the Fourier transform can also be defined as follows:

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \nu t} dt \ , \quad f(t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i \nu t} d\nu \ . \tag{19}$$

Fourier transforms are useful in solving wave propagation problems (Miklowitz, 1978), and are ubiquitous in the analysis of seismic data (Berkhout, 1987).

4. Equations of Elastodynamics

In previous sections, we assumed knowledge of the simple one-dimensional definitions of stress, strain, and Hooke's law. But in order to derive the partial differential equations that govern three-dimensional seismic wave propagation in rocks, we must define and discuss the concepts of stress and strain in more detail. Stress and strain are related to each other by *constitutive relations*, specifically, Hooke's law of linear elasticity. Combination of Newton's law of motion and Hooke's law then leads to the equations of elastodynamics, which are the equations that govern seismic wave propagation.

4.1. Stress

The concept of stress was introduced by the French civil engineer and mathematician Cauchy in 1823. The concept of stress is a generalisation of the concept of pressure in a fluid. When a static fluid is at rest and in contact with a solid, the fluid will exert a compressive force on the surface, and this force will act normal to the surface. The *force per unit area* is defined as the *pressure*, and it has units of Newtons per square meter, which are also known as *Pascals*. For reference, note that standard atmospheric pressure is 101,325 Pa, and 1 psi = 6895 Pa. If a fluid is under pressure, it will not only exert a pressure on the walls of any container that encloses it, but it will also exert a pressure on any imaginary surface within the fluid. [In these notes we will discuss stress and strain only as far as necessary to be able to derive the equations of elastodynamics. A more detailed discussion of stress and strain can be found in *Fundamentals of Rock Mechanics* by Jaeger *et al.* (2007).]

Consider an infinitesimally small cubical piece of rock, as in Fig. 4.1 below. It may be thought of as a small laboratory specimen, or a cubical region that is part of a larger rock mass, in which case the outer surface is an imaginary surface. An arbitrary force may act on each of the six faces of the cube. Since the cube is small, we can assume that each of these forces is uniformly distributed across the face on which it acts.

We set up a Cartesian co-ordinate system aligned with the axes of the cube. Consider the face whose outward unit normal vector lies along the positive x-axis. The force that acts on this surface has components $\{f_x, f_y, f_z\}$. If we divide each of these force components by the area of the

surface, *dydz*, we will convert the forces into *stress components*, which we label as $\{\tau_{xx}, \tau_{xy}, \tau_{xz}\}$.

In this notation, the first subscript denotes the fact that this surface is normal to the *x*-direction, and the second subscript indicates the direction of the force component. The stresses with matching subscripts are *normal stresses*, because they represent stresses that act normal to the surface. Stresses with un-matched subscripts are *shear stresses*, which act tangentially to the surface.

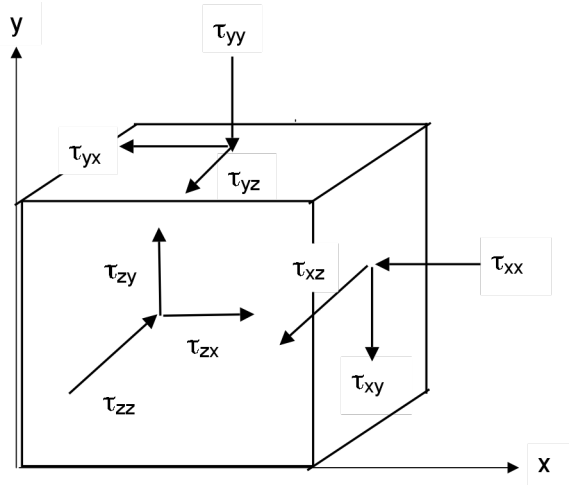

Fig. 4.1. Small cube of rock showing the stress components that act on the three faces that face rightwards (*x*), upwards (*z*), and forward (*z*). Similar stresses act on the three hidden faces. The arrows point in the directions that nominally correspond to positive values of stress.

Since there are six faces of the cube, and three force components acting on each face, it might seem that there would be eighteen different stress components. But Cauchy showed that the stress components that act on the face that faces the +*x* direction must be equal to the stress components that act on the opposite face that faces the –*x* direction. This fact follows from a simple force balance on the cube. So, there are really only nine stress components, and they form the *stress matrix*, which is denoted by $\boldsymbol{\tau}$, and which can be written as

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \tag{1}$$

Cauchy also showed, by taking a moment balance on the small cube of rock, that $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, and $\tau_{zy} = \tau_{yz}$. Hence, the stress matrix is always symmetric, and there are really only six independent stress components *at any point the rock*. Note that the result $\tau_{xy} = \tau_{yx}$ holds at a given point; there is no reason that the value of τ_{xy} at one point should equal τ_{yx} at some other point, for example.

Now let's consider a cubical element of rock that is not infinitesimally small, and not necessarily in static equilibrium. In this case, the stresses that act on the surface of the cube that faces the +*x* direction will not necessarily be equal to the stresses that act on the surface that faces the -*x* direction. Any imbalance in these stresses will in fact cause the rock to accelerate. The relation between the spatial variation in stresses, and the acceleration, can be found by doing a force balance on the cube of rock.

According to Newton's law of motion, the net force acting on the cube in the *x*-direction will equal the product of the mass of this cube times the acceleration of the cube. To keep the drawings and algebra simple, we will consider a two-dimensional version of this problem; the generalisation to three dimensions will be obvious.

Consider a small cube of rock having side lengths {*dx,dy,dz*}, with a Cartesian co-ordinate system aligned with the edges of the cube, as in Fig. 4.2 below. The dimension *dz* is into the page, and so is not explicitly shown. In the figure, each arrow points in the direction that corresponds to the direction of *positive* stresses.

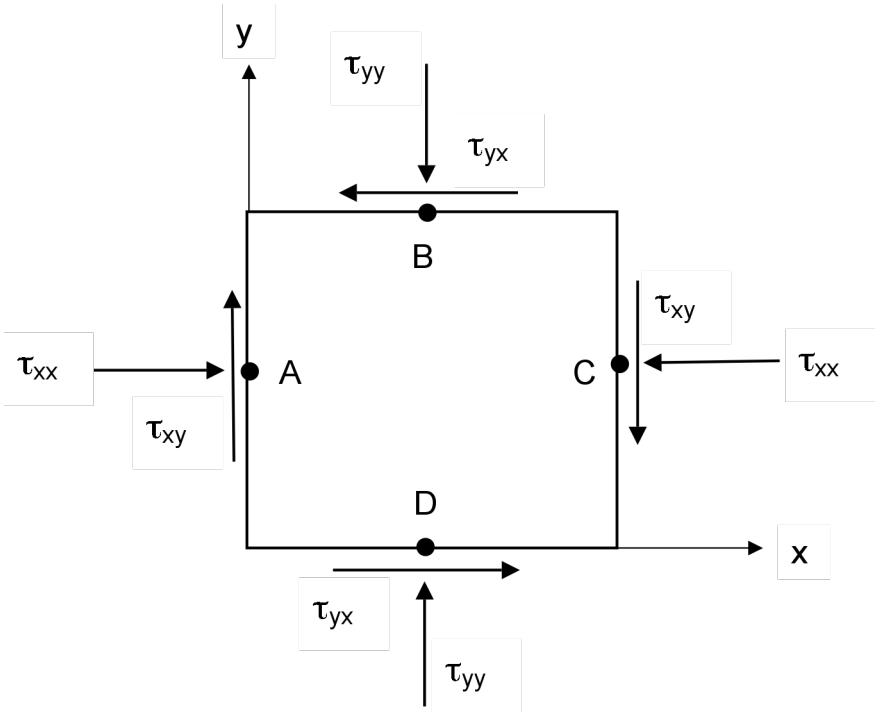

Fig. 4.2. Small cube of rock showing the stress components that act on four of its faces. The stress components acting at A are not necessarily equal to the stress components acting at C. The arrows point in the directions that nominally correspond to positive values of stress.

Let's now do a force balance in the *x*-direction. First, consider the face on the left. Only the stress τ_{xx} acts in the *x*-direction. Although this stress may vary from point to point, the best estimate of its *average* value would be given by its value *at point A*, which is to say, at (x = 0, y = dy/2). If we convert the stress to a force by multiplying by area, we have a force component $\tau_{xx}(0,dy/2)dydz$. If we add up all the forces that act in the x-direction, we find

$$\begin{aligned}\sum F_x &= \tau_{xx}(0,dy/2)dydz - \tau_{xx}(dx,dy/2)dydz \\ &+ \tau_{yx}(dx/2,0)dxdz - \tau_{yx}(dx/2,dy)dxdz.\end{aligned} \tag{2}$$

If we do a Taylor series on the term $\tau_{xx}(dx,dy/2)$, we can say that

$$\tau_{xx}(dx,dy/2) = \tau_{xx}(0,dy/2) + \left[\frac{\partial \tau_{xx}}{\partial x}(0,dy/2)\right]dx + \ldots \tag{3}$$

Similarly, the term $\tau_{yx}(dx/2,dy)$ can be expressed as

$$\tau_{yx}(dx/2,dy) = \tau_{yx}(dx/2,0) + \left[\frac{\partial \tau_{yx}}{\partial y}(dx/2,0)\right]dy + \ldots \qquad (4)$$

Substituting expressions (3) and (4) into eq. (2) yields

$$\sum F_x = -\left[\frac{\partial \tau_{xx}}{\partial x}(0,dy/2)\right]dxdydz - \left[\frac{\partial \tau_{yx}}{\partial y}(dx/2,0)\right]dxdydz. \qquad (5)$$

By Newton's law, this must equal the product of the mass of the cube, which is $\rho dxdydz$, and its acceleration in the x-direction, which is $-(\partial^2 u/\partial t^2)$, where u is the x-component of the displacement. The peculiar minus sign arises from the "compression is positive" sign convention used in rock mechanics. In order to be consistent with that convention as used for stresses, the displacement must be considered positive if the rock moves in the negative x-direction. So, equating the net force to the mass × acceleration leads to

$$-\left[\frac{\partial \tau_{xx}}{\partial x}(0,dy/2)\right]dxdydz - \left[\frac{\partial \tau_{yx}}{\partial y}(dx/2,0)\right]dxdydz = -\rho dxdydz\frac{\partial^2 u}{\partial t^2}. \qquad (6)$$

The acceleration in (6) must be the average value over the entire small cube; the most appropriate place to evaluate it would be at the centre of the cube, *i.e.*, at the point (dx/2, dy/2, dz/2).

If we cancel out the minus signs and the volume term $dxdydz$, we have

$$\frac{\partial \tau_{xx}}{\partial x}(0,dy/2) + \frac{\partial \tau_{yx}}{\partial y}(dx/2,0) = \rho\frac{\partial^2 u}{\partial t^2}(dx/2,dy/2,dz/2). \qquad (7)$$

The last step in this derivation is to note that since dx and dy are small, $(\partial \tau_{xx}/\partial x)$ evaluated at the point $(0,dy/2)$ is equal (to first order) to $(\partial \tau_{xx}/\partial x)$ evaluated at the point (0,0). Likewise for the other terms. Hence, we can actually evaluate all the derivatives in (7) at the point (0,0,0). But the location of our origin was arbitrary, so the result actually holds at any point:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = \rho\frac{\partial^2 u}{\partial t^2}. \qquad (8)$$

If we had taken into account the two faces of the cube that are parallel to the page, we would also find a term $\partial\tau_{zx}/\partial z$. So, the equation of motion in the *x*-direction is

$$\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}=\rho\frac{\partial^2 u}{\partial t^2}. \tag{9}$$

Likewise, the equations of motion in the other two directions are

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}=\rho\frac{\partial^2 v}{\partial t^2}. \tag{10}$$

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}=\rho\frac{\partial^2 w}{\partial t^2}, \tag{11}$$

where *v* and *w* are the displacement components in the *y* and *z* directions. Note that when writing these equations, we haven't made use of the facts that $\tau_{xz}=\tau_{zx}$, *etc*. So, there are really only six independent stresses appearing in these equations, not nine.

Most of the steps in the above derivation involved approximations, so it might seem that the results are only approximately true. However, if we had consistently applied Taylor series to all terms, and then let *dx*, *dy* and *dz* all get very small, we would have found that the resulting equations (9-11) are *exact*.

A much shorter derivation that involves no seemingly arbitrary approximations is given in *Fundamentals of Rock Mechanics* by Jaeger *et al.* (2007). However, that derivation requires some properties of the stress matrix that we have not covered in this course, and also makes use of the divergence theorem, which might not be familiar.

Note that eqs. (9-11) are a set of *three* equations, containing *nine* unknowns: six stresses and three displacements. In order to solve these equations, we therefore need to find more equations. These equations are found by relating the stresses to the displacements, which requires first that we introduce the concept of strain.

4.2. Displacement and Strain

In rock mechanics, as in the mechanics of particles and rigid bodies, the fundamental kinematic variable is the *displacement*, which is the vector that quantifies the change in the position of a given particle of rock. The position of each rock particle can be labelled by its location, relative to some coordinate system, in some state that is taken to be the "initial" state of the rock. This position can be denoted by $\mathbf{x} = (x,y,z)$. If a load is applied to the rock, or a stress wave passes through the rock, then the rock particle that was initially located at point **x** will be displaced to a new position, $\mathbf{x}^* = (x^*,y^*,z^*)$**.** The vector that connects the original position **x** and the final position $\mathbf{x}^*$ is known as the "displacement of the particle that was initially at point **x**", or simply the "displacement at **x**". This vector is denoted in vector notation by **u**, and its components are (u,v,w). To be consistent with the sign convention used for tractions, in which a traction component is represented by a positive number if it points in the *negative* coordinate direction, the displacement vector must be defined according to

$$\mathbf{x}^* = \mathbf{x} - \mathbf{u}, \quad i.e., \quad x^* = x - u, \quad y^* = y - v, \quad z^* = z - w. \qquad 1)$$

The displacement **u** can be interpreted as a vector that points *from* the new position, **x***, *towards* the original position, **x** (Fig. 4.3). In general, the displacement will vary from point to point, so that each component (u,v,w) will vary with all three position coordinates, x, y, and z.

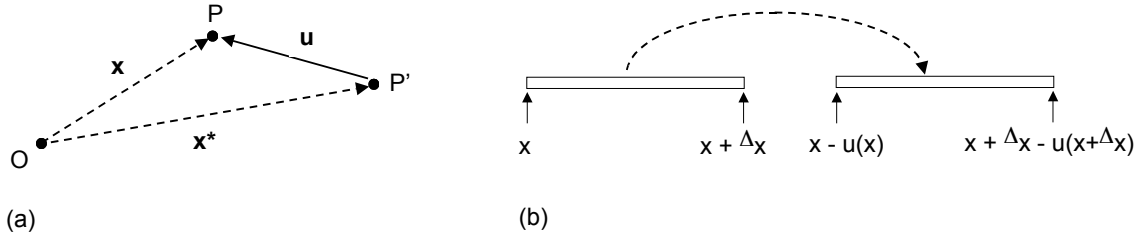

Fig. 4.3. (a) Displacement vector **u** of the piece of rock that is initially located at point **x**. (b) Generic displacement of a one-dimensional bar, used to define the normal strain.

Ultimately, we need to relate the *stresses* to the *displacements*. But to do this, it is first necessary to introduce a set of intermediate quantities known as the *strains*. Strain is essentially a measure of the relative displacement of nearby particles, rather than a measure of their absolute displacement.

The basic concept behind the strain can be introduced in a one-dimensional context (Fig. 4.3b). Consider a short one-dimensional bar, initially of length L, whose left edge is initially located at point x, and whose right edge is located at point $x+\Delta x$. The initial length of this bar is given by $L=\Delta x$. This bar is now assumed to be deformed, such that the left edge of the bar moves to the location $x-u(x)$, and the right edge of the bar moves to the position $[x+\Delta x]-u(x+\Delta x)$. The new length of the bar is therefore equal to $L^*=\{[x+\Delta x]-u(x+\Delta x)\}-[x-u(x)]$. We now define the mean strain undergone by this bar as the *fractional decrease in the length of the bar*, *i.e.*,

$$\varepsilon=\frac{L-L^*}{L}=\frac{\Delta x-\{[x+\Delta x]-u(x+\Delta x)-[x-u(x)]\}}{\Delta x}=\frac{u(x+\Delta x)-u(x)}{\Delta x}. \quad (2)$$

According to this definition, the strain will be a positive number if the bar becomes shorter, and *vice versa*. Hence, positive strains represent contractions, and negative strains represent extensions.

Eq. (2) gives the mean strain over the region between x and $x+\Delta x$. The strain *at the point x* is found by taking the limit of an infinitesimally short bar, which is mathematically equivalent to letting the initial length of the bar go to zero:

$$\varepsilon(x)=\lim_{L\to 0}\frac{L-L^*}{L}=\lim_{\Delta x\to 0}\frac{u(x+\Delta x)-u(x)}{\Delta x}\equiv\frac{du}{dx}. \quad (3)$$

The strain is therefore defined in terms of the spatial derivative of the displacement.

The type of strain described above, which is called a *normal strain*, can be generalised in an obvious way to two or three dimensions. However, in two or more dimensions there are other types of strains, called the *shear strains*, which measure angular distortion, rather than stretching. Together, the normal and shear strains form a second-order tensor, mathematically analogous to the stress tensor.

Consider a particle in a “two-dimensional” rock that is initially located at point $P=(x,y)$, as in Fig. 4.4. Now consider a second particle that is initially located at $Q=(x+\Delta x,y)$, and a third particle located at $R=(x,y+\Delta x)$. The rock is then assume to be deformed, such that these three particles move to positions P^*, Q^*, and R^*. The coordinates of these new locations are

$$P^* = P - \mathbf{u}(P) = (x,y) - [u(x,y), v(x,y)] = [x - u(x,y), y - v(x,y)], \tag{4}$$

$$\begin{aligned} Q^* &= Q - \mathbf{u}(Q) = (x + \Delta x, y) - [u(x + \Delta x, y), v(x + \Delta x, y)] \\ &= [x + \Delta x - u(x + \Delta x, y), y - v(x + \Delta x, y)], \end{aligned} \tag{5}$$

$$\begin{aligned} R^* &= R - \mathbf{u}(R) = (x, y + \Delta y) - [u(x, y + \Delta y), v(x, y + \Delta y)] \\ &= [x - u(x, y + \Delta y), y + \Delta y - v(x, y + \Delta y)]. \end{aligned} \tag{6}$$

Also shown in Fig. 4.4 are the points Q'', which is the projection of point Q^* onto an x-axis that passes through point P^*, and R'', which is the projection of point R^* onto a y-axis that passes through point P^*. For example, Q'' will have the same x-component as Q^*, and the same y-component as P^*, and likewise for R''.

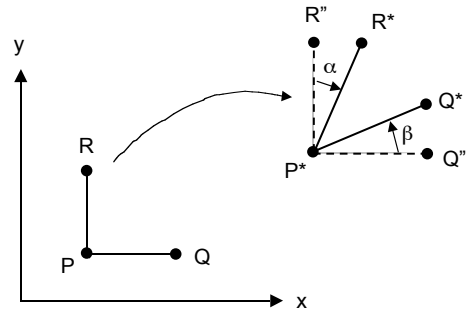

Fig. 4.4. Displacement of two small line segments PQ and PR that are initially at right angles to each other.

We now express each of the displacements that appear in (4)-(6) as Taylor series taken about the point (x,y). As the increments Δx and Δy are infinitesimally small, all terms in the Taylor series higher than the first-order terms can be ignored. For example, we can say that $u(x,y + \Delta y) = u(x,y) + (\partial u / \partial y)\Delta y$, *etc.*, where the partial derivative is understood to be evaluated at (x,y). The positions of the five points shown in Fig. 4.4 can therefore be expressed as

$$P^* = (x - u, y - v), \tag{7}$$

$$Q^* = (x + \Delta x - u - \frac{\partial u}{\partial x}\Delta x, y - v - \frac{\partial v}{\partial x}\Delta x), \tag{8}$$

$$R^* = (x - u - \frac{\partial u}{\partial y}\Delta y, y + \Delta y - v - \frac{\partial v}{\partial y}\Delta y) \tag{9}$$

$$Q'' = (x + \Delta x - u - \frac{\partial u}{\partial x}\Delta x, y - v), \tag{10}$$

$$R'' = (x - u, y + \Delta y - v - \frac{\partial v}{\partial y}\Delta y), \tag{11}$$

where it is understood that u, v, and all of their partial derivatives are evaluated at the point (x,y).

The normal strain in the x direction, which is denoted by ε_{xx}, is now defined, as previously, as the fractional shortening of a line element that is initially oriented along the x-axis. In other words, the strain ε_{xx} at point (x,y) is equal to the fractional contraction undergone by the element PQ, in the limit as $\Delta x \to 0$. Initially, the length of element PQ, which we denote by $|PQ|$, is Δx. After deformation, the length $|P^*Q^*|$ is found using the Pythagorean theorem:

$$|P^*Q^*|^2 = |P^*Q''|^2 + |Q^*Q''|^2 = \left(\Delta x - \frac{\partial u}{\partial x}\Delta x\right)^2 + \left(\frac{\partial v}{\partial x}\Delta x\right)^2$$

$$= (\Delta x)^2\left[1 - 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]. \tag{12}$$

The crucial assumption is now made that all partial derivatives of the displacements are much smaller than unity, which is equivalent to assuming that the displacements not to vary too abruptly from point to point. This assumption leads to the theory of *infinitesimal strain*, within which theory the strains are linear functions of the partial derivatives of the displacement components. This assumption is usually acceptable in seismic wave propagation problems, and also has the advantage of leading to a set of mathematically linear equations that are solvable by classical techniques.

Under the assumption that the partial derivatives of the displacement are small, we can neglect the squares of these derivatives in (12), and use the approximation that $(1-2\delta)^{1/2} \approx 1-\delta$ when $\delta << 1$, to find

$$|P^*Q^*| = \Delta x\left(1 - \frac{\partial u}{\partial x}\right). \tag{13}$$

Hence, the normal strain in the *x*-direction at the point (*x,y*) is given by

$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \frac{|PQ| - |P^* Q^*|}{|PQ|} = \lim_{\Delta x \to 0} \frac{\Delta x - \Delta x(1 - \partial u / \partial x)}{\Delta x} = \frac{\partial u}{\partial x}. \qquad (14)$$

A similar analysis shows that the normal strain in the *y*-direction, ε_{yy}, is given by

$$\varepsilon_{yy} = \lim_{\Delta x \to 0} \frac{|PR| - |P^* R^*|}{|PR|} = \lim_{\Delta x \to 0} \frac{\Delta y - \Delta y(1 - \partial v / \partial y)}{\Delta y} = \frac{\partial v}{\partial y}. \qquad (15)$$

There are other types of distortion, other than stretching or contraction, that can also be quantified. Fig. 4.4 shows that the three points {*P,Q,R*} initially form a right angle, but, in the case shown, form an acute angle after deformation. The change in this angle is known as the *shear strain*. Specifically, the shear strain ε_{xy} is defined as one-half of the *increase* in the angle initially formed by two perpendicular infinitesimal line segments that initially lie parallel to the *x* and *y* axes, *i.e.*,

$$\varepsilon_{xy} = \frac{1}{2} \lim_{\Delta x, \Delta y \to 0} \left(\angle R^* P^* Q^* - \angle RPQ \right). \qquad (16)$$

From Fig. 4.4 we see that $\angle R^* P^* Q^* = 90^\circ - \alpha - \beta$, whereas $\angle RPQ = 90^\circ$, by construction. The angle α is calculated from

$$\tan\alpha = \frac{|R^* R''|}{|P^* R''|} = \frac{-\left(\frac{\partial u}{\partial y}\right)\Delta y}{\Delta y\left(1 - \frac{\partial v}{\partial y}\right)} \approx -\left(\frac{\partial u}{\partial y}\right), \qquad (17)$$

where, due to the smallness of the partial derivatives, we ignore the term $\partial v / \partial y$ in the denominator. But the smallness of the partial derivatives also allows us to approximate the angle α by $\tan\alpha$, so that $\alpha = -(\partial u / \partial y)$. Similarly, it can be shown that $\beta = -(\partial v / \partial x)$. Combining these results with (16) leads to

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \qquad (18)$$

The pattern embodied in (18) is that the shear strain ε_{xy} is equal to the mean of the partial derivative of the displacement in the *x* direction with

respect to *y*, and the partial derivative of the displacement in the *y* direction with respect to *x*. By this definition, we see that if a shear strain ε_{yx} is defined, it will necessarily be equal to ε_{xy}.

These four strains, $\{\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{xy},\varepsilon_{yx}\}$, can be thought of as the four components of the *strain matrix*, $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} \end{bmatrix}. \tag{19}$$

The strain matrix is equal to the symmetric part of a matrix that is known as the *displacement gradient*, $\nabla\mathbf{u}$, whose components are the partial derivatives of the displacements with respect to the two coordinates, *i.e.,*

$$\boldsymbol{\varepsilon} = \text{sym}(\nabla\mathbf{u}) \equiv \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}\right], \tag{20}$$

where

$$\nabla\mathbf{u} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}. \tag{21}$$

The displacement gradient, and hence also the strain, is a second-order tensor.

It is clear from (20) that any deformation whose displacement gradient is anti-symmetric will lead to zero infinitesimal strain. One such type of deformation is an *infinitesimal rigid-body rotation*. To verify this, consider a rigid rotation of the rock in the anti-clockwise direction, by some small angle φ. This rigid-body rotation is described by

$$x^* = x\cos\varphi - y\sin\varphi, \quad y^* = x\sin\varphi + y\cos\varphi. \tag{22}$$

The coordinates (*x*,y**) are the coordinates of the particle that was originally located at (*x,y*), *after* the rock has been rotated, but still referred to the *original* coordinate system. Since $\mathbf{x}^* = \mathbf{x} - \mathbf{u}$, the displacement components are given by

$$u = x - x^* = x(1-\cos\varphi) + y\sin\varphi, \tag{23}$$

$$v = y - y^* = -x\sin\varphi + y(1-\cos\varphi). \tag{24}$$

The displacement gradient can be calculated as

$$\nabla\mathbf{u} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-\cos\varphi & \sin\varphi \\ -\sin\varphi & 1-\cos\varphi \end{bmatrix}. \tag{25}$$

As the angle of rotation is small, we can expand out the trigonometric terms in Taylor series, and ignore all terms that are higher than first-order in φ, yielding

$$\nabla\mathbf{u} = \begin{bmatrix} 0 & \varphi \\ -\varphi & 0 \end{bmatrix}. \tag{26}$$

The displacement gradient corresponding to an infinitesimal rigid-body rotation is therefore *anti-symmetric*, in the sense that $(\nabla u)^{\mathsf{T}} = -(\nabla u)$. This type of rotation leads to no strain, since $2\varepsilon = [(\nabla u) + (\nabla u)^{\mathsf{T}}] = \mathbf{0}$.

The above analysis applied to a rigid-body rotation of the entire rock. But for a general deformation that varies from point to point, we can, at each point, define a local tensor ω as the anti-symmetric part of the displacement gradient, *i.e.,*

$$\omega = \text{asym}(\nabla\mathbf{u}) \equiv \frac{1}{2}\left[\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathsf{T}}\right] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \\ -\frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & 0 \end{bmatrix}, \tag{27}$$

in which case it follows from (20) and (27) that $\nabla\mathbf{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$.

The definitions of the strain components in three dimensions follow exactly as for two-dimensions. The formal relations between $\boldsymbol{\varepsilon}$, $\boldsymbol{\omega}$ and $\nabla\mathbf{u}$ given by (20) and (27) continue to hold, where $\nabla\mathbf{u}$, as defined by (21), is generalised in an obvious way into a 3×3 matrix. The explicit definitions of the strains in terms of the displacement derivatives are:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}, \tag{28}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \tag{29}$$

with $\varepsilon_{xy} = \varepsilon_{yx}$, $\varepsilon_{yz} = \varepsilon_{zy}$, and $\varepsilon_{zx} = \varepsilon_{xz}$.

4.3. Hooke's Law

Eqs. §4.1(9-11) are equations of motion expressed in terms of the stresses and the displacements. Eqs. §4.3(19,20) relate the displacements to the strains. Ultimately, we would like to eliminate the stresses from the equation of motion, and express this equation solely in terms of the displacement. In order to do this, the stresses and strains must be related to each other by a *constitutive law*. The simplest type of constitutive law is that of *linear elasticity*, in which the strains are linear functions of the stress, and do not depend on the stress rate, or on the previous stress path, *etc*. This type of stress-strain law is often called "Hooke's law", and is the one used in most seismic analysis.

The precise form of Hooke's law depends on the structure of the rock, and whether it contains bedding planes, fractures, *etc*. Instead of discussing the most general possible case, we will focus on the simplest and most common one: an *isotropic rock*. An isotropic solid can be loosely defined as one in which all directions are "equivalent". For example, in an isotropic rock, if we take a core oriented vertically, and a core that was oriented horizontally, and compress these two cores in a rock testing machine, we would obtain the same results.

We first consider the effect of a normal stress. Imagine that we have a cylindrical core of rock having radius *a*, and height *L*. We now subject this rock column to an axial stress τ, as in Fig. 4.5.

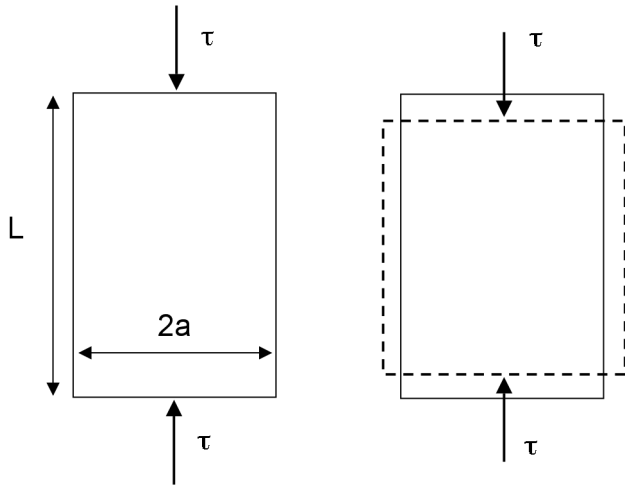

Fig. 4.5. A cylindrical rock specimen subject to a uniaxial compressive stress will contract in the direction of the load, and will expand in the two transverse directions.

In response to the axial stress, the rock core will get shorter, and will bulge outwards, as shown on the right. The fact that the core will bulge outward may not be obvious, but you can convince yourself that it occurs by doing this experiment using a rubber eraser.

The ratio between the axial stress and axial strain is called Young's modulus, E (named after Thomas Young, one of the scientists who derived the Young-Laplace equation of capillarity):

$$E = \frac{\tau_{axial}}{\varepsilon_{axial}}. \tag{1}$$

For most rocks, E lies between 10-100 GPa.

The outward bulging of the core is quantified by the *Poisson ratio*, which is usually denoted by ν, as follows:

$$\nu = \frac{-\varepsilon_{transverse}}{\varepsilon_{axial}}. \tag{2}$$

For natural materials, the Poisson ratio always lies between 0 (*i.e*., cork) and 0.5 (*i.e.,* rubber), but is usually between 0.1-0.3 for rocks.

If we combine (1) and (2), we see that the transverse strain is related to the axial stress by

$$\varepsilon_{transverse} = -\nu\varepsilon_{axial} = \frac{-\nu}{E}\tau_{axial}. \tag{3}$$

Now imagine that a piece of rock is subjected to three orthogonal stresses, $\{\tau_{xx},\tau_{yy},\tau_{zz}\}$. Let's try to find the strain ε_{xx} that would be caused by these stresses. From (1), we see that τ_{xx} will cause a strain of $\varepsilon_{xx} = \tau_{xx}/E$, whereas each of the two "transverse" stresses τ_{yy} and τ_{zz} will give rise to a strain $\varepsilon_{xx} = -\nu\tau_{yy}/E$, and similarly for τ_{zz}. Adding these strains together gives a total strain of

$$\varepsilon_{xx} = \frac{1}{E}\tau_{xx} - \frac{\nu}{E}\tau_{yy} - \frac{\nu}{E}\tau_{zz} = \frac{1}{E}[\tau_{xx} - \nu(\tau_{yy} + \tau_{zz})]. \tag{4}$$

The main idea here is that a normal stress τ causes a normal strain of τ/E in the direction of the stress, and normal strains of $-\nu\tau/E$ in the two other directions.

Using these ideas, we can write similar equations for the strains in the other two directions:

$$\varepsilon_{yy} = \frac{1}{E}\tau_{yy} - \frac{\nu}{E}\tau_{xx} - \frac{\nu}{E}\tau_{zz} = \frac{1}{E}[\tau_{yy} - \nu(\tau_{xx} + \tau_{zz})], \tag{5}$$

$$\varepsilon_{zz} = \frac{1}{E}\tau_{zz} - \frac{\nu}{E}\tau_{xx} - \frac{\nu}{E}\tau_{yy} = \frac{1}{E}[\tau_{zz} - \nu(\tau_{xx} + \tau_{yy})], \tag{6}$$

Now let's think about the effect of a shear stress. Imagine a cube of rock, subjected to tangential "shear" stresses τ_{xy}, as in the figure below. In response to this stress, the rock will be distorted, as shown on the right:

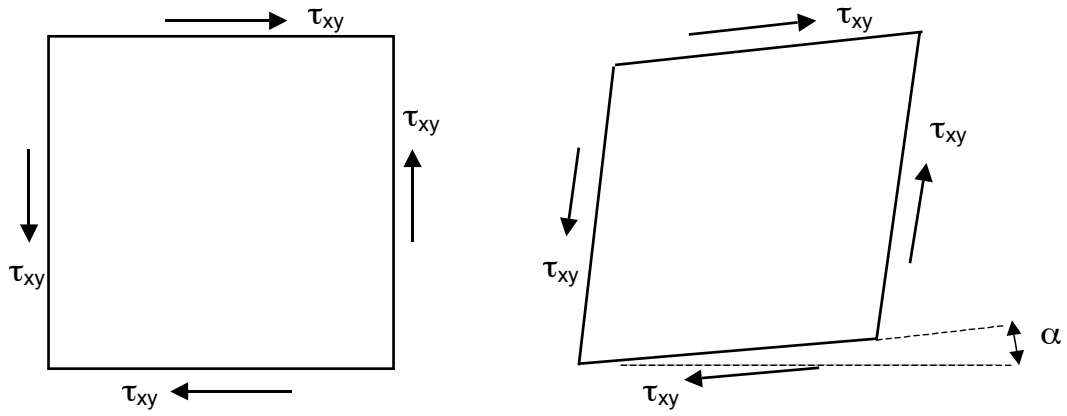

Fig. 4.6. A cubical rock specimen subject to a shear stress (left) will deform into the shape shown on the right. (Note that, according to our sign convention, the shear stresses shown here are *negative*).

The shear modulus, G, is the ratio of the amount of distortion, as measured by the angle of distortion α, to the applied shear stress, τ, as follows:

$$\frac{1}{G} = \frac{2\alpha}{\tau}. \tag{7}$$

But according to the discussion in §4.3, α is equal to the shear strain, ε_{xy}. Hence, $\varepsilon_{xy} = \tau_{xy}/2G$. (The appearance of the factor of 2 arises from the fact that before tensors were invented, the shear strains were originally defined without the factor of 1/2, in which case (7) took the more obvious form $\varepsilon = \tau/G$).

A normal stress will not cause the type of angular distortion shown in Fig. 4.6, and so there is no coupling between normal stresses and shear strains. So, in general, (4)-(6) are not affected by the presence of shear stresses, and the full expression of Hooke's law for an isotropic material consists of (4)-(6) along with the following equations for the shear stresses/strains:

$$\varepsilon_{xy} = \frac{\tau_{xy}}{2G}, \quad \varepsilon_{yz} = \frac{\tau_{yz}}{2G}, \quad \varepsilon_{zx} = \frac{\tau_{zx}}{2G}. \tag{8}$$

The shear modulus G is related to E and ν by $G = E/2(1+\nu)$, although this relation is not easy to derive without using concepts that we will not cover in this module (see Jaeger *et al.,* 2007).

If we invert (4)-(6) to express the normal stresses in terms of the strains, we find:

$$\tau_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{xx} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{yy} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{zz}, \tag{9}$$

$$\tau_{yy} = \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{xx} + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{yy} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{zz}, \tag{10}$$

$$\tau_{zz} = \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{xx} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{yy} + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{zz}. \quad (11)$$

The full inverse version of Hooke's law, including the normal strains and shear strains, is often written in the following simpler notation:

$$\tau_{xx} = (\lambda + 2G)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz}, \quad (12)$$

$$\tau_{yy} = \lambda\varepsilon_{xx} + (\lambda + 2G)\varepsilon_{yy} + \lambda\varepsilon_{zz}, \quad (13)$$

$$\tau_{zz} = \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda + 2G)\varepsilon_{zz}, \quad (14)$$

$$\tau_{xy} = 2G\varepsilon_{xy}, \quad \tau_{xz} = 2G\varepsilon_{xz}, \quad \tau_{yz} = 2G\varepsilon_{yz}, \quad (15)$$

where $\lambda = E\nu/(1+\nu)(1-2\nu)$. The two elastic moduli appearing in (12)-(15), λ and G, are also known as the *Lamé parameters*, named after the French elastician Gabriel Lamé. The parameter G is often denoted, particularly in mathematical elasticity treatments, by the symbol μ (note that λ and μ are the two Greek consonants in the name Lamé). In order to avoid confusion with the coefficient of friction, in rock mechanics the shear modulus is usually denoted by G. Although E and ν have a clearer physical interpretation, the mathematical development of the equations of elastodynamics is notationally simpler if we use λ and G instead of E and ν.

The *mean normal stress* can be found by summing up (12)-(14):

$$\tau_m = \frac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz}) = [\lambda + (2/3)G](\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}). \quad (16)$$

But the sum of the three normal strains is the volumetric strain, ε_v, so we see that the volumetric strain is related to the mean normal stress by

$$\tau_m = (\lambda + \frac{2}{3}G)\varepsilon_v \equiv K\varepsilon_v, \quad (17)$$

where $K = \lambda + (2/3)G$ is the *bulk modulus*. The multiplicative reciprocal of the bulk modulus, $1/K$, is known as the *bulk compressibility*, and is usually denoted by either β or C.

Although many elastic parameters can be defined for an isotropic material, only two of them are independent. A full listing of all thirty

relations that can be obtained by expressing any three of the set $\{\lambda,K,G,E,\nu\}$ in terms of the other two is given by Davis and Selvadurai (1996). Some of the more useful of these relations are

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad G = \frac{E}{2(1+\nu)}, \qquad K = \frac{E}{3(1-2\nu)}; \tag{18}$$

$$\lambda = \frac{2G\nu}{(1-2\nu)}, \qquad E = 2G(1+\nu), \qquad K = \frac{2G(1+\nu)}{3(1-2\nu)}; \tag{19}$$

$$\lambda = K - \frac{2}{3}G, \qquad E = \frac{9KG}{3K+G}, \qquad \nu = \frac{3K-2G}{6K+2G}. \tag{20}$$

In 1829 the French mathematical physicist Simeon Denis Poisson developed a simplified model for atomic interactions in an elastic solid, and concluded that $\lambda = G$. If this were the case, then we would also have

$$K = 5G/3, \quad E = 5G/2, \quad \nu = 1/4. \tag{21}$$

The approximation $\lambda = G$, known as "Poisson's relation", is actually not very accurate for most rocks. In fact, Poisson's ratio takes on a range of values for different rocks, usually in the range 0.1-0.3. Nevertheless, Poisson's relation is sometimes used in geophysics to simplify the equations of elasticity. However, the difficulties in solving elasticity problems are not caused by the appearance of two elastic parameters in the equations, but rather are due to the structure of the differential equations themselves. Hence, any advantages gained by putting $\lambda = G$ are outweighed by the resulting loss in generality.

Another particular case of an idealized isotropic elastic material is the *incompressible solid*, which has $\beta = 0$, and hence $K = 1/\beta \to \infty$. For such materials, (19) and (20) show that

$$K \to \infty, \qquad \lambda \to \infty, \qquad E \to 3G, \qquad \nu \to 1/2, \tag{22}$$

whereas E and G can remain finite. A completely *rigid* material, on the other hand, is not only incompressible, but also has infinite values of E and G.

The special case of a *compressible fluid* is that in which the shear modulus vanishes, but the bulk modulus remains finite. In this case (18)-(20) show that

$$G \to 0, \qquad \nu \to 1/2, \qquad E \to 0, \qquad \lambda = K. \tag{23}$$

4.4. Equations of elastodynamics for an isotropic rock

We are now ready to combine some of the equations that we have derived, to arrive at the equations of elastodynamics. First, if we substitute the strain-displacement equations, §4.2 (28) and (29), into the stress-strain equations, §4.3 (12)-(14), we find

$$\tau_{xx} = (\lambda + 2G)\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} + \lambda\frac{\partial w}{\partial z}, \tag{1}$$

$$\tau_{yy} = \lambda\frac{\partial u}{\partial x} + (\lambda + 2G)\frac{\partial v}{\partial y} + \lambda\frac{\partial w}{\partial z}, \tag{2}$$

$$\tau_{zz} = \lambda\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} + (\lambda + 2G)\frac{\partial w}{\partial z}. \tag{3}$$

$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \tau_{yz} = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad \tau_{zx} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right). \tag{4}$$

We now substitute these relations into the stress equilibrium equations, §4.1 (9)-(11), to arrive at

$$\lambda\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 w}{\partial x\partial z}\right) + G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 w}{\partial x\partial z} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{5}$$

$$\lambda\left(\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y\partial z}\right) + G\left(\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y\partial z} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\frac{\partial^2 v}{\partial t^2}, \tag{6}$$

$$\lambda\left(\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 v}{\partial z\partial y}+\frac{\partial^2 w}{\partial z^2}\right)+G\left(\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 v}{\partial z\partial y}+\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)=\rho\frac{\partial^2 w}{\partial t^2}. \tag{7}$$

This set of equations comprises three equations in three unknowns, and so they constitute a mathematically "complete" set of equations governing the motion of the rock. Note that they do not explicitly contain the stresses or the strains.

The displacement form of the equilibrium equations can also be expressed in a more compact vector-matrix notation. First, note the following two identities:

$$\nabla^{\mathrm{T}}\mathbf{u}\equiv\nabla\cdot\mathbf{u}\equiv\mathrm{div}\mathbf{u}=\begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}, \tag{8}$$

$$\nabla^{\mathrm{T}}\nabla\equiv\nabla^2=\begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}. \tag{9}$$

The Laplacian operator ∇^2 is a scalar operator that can operate on, by pre-multiplication, a scalar or a vector. Using these identities, (5)-(7) can be written in the following form:

$$(\lambda+G)\frac{\partial}{\partial x}\left(\nabla\cdot\mathbf{u}\right)+G\nabla^2 u=\rho\frac{\partial^2 u}{\partial t^2}, \tag{10}$$

$$(\lambda+G)\frac{\partial}{\partial y}\left(\nabla\cdot\mathbf{u}\right)+G\nabla^2 v=\rho\frac{\partial^2 v}{\partial t^2}, \tag{11}$$

$$(\lambda+G)\frac{\partial}{\partial z}\left(\nabla\cdot\mathbf{u}\right)+G\nabla^2 w=\rho\frac{\partial^2 w}{\partial t^2}. \tag{12}$$

From (9) we see that the three partial derivative operators appearing in these three equations form the components of the gradient row vector. Furthermore,

$$(\nabla^2 u, \nabla^2 v, \nabla^2 w) = \nabla^2 (u, v, w) = \nabla^2 \mathbf{u}. \tag{13}$$

Hence, (10)-(12) can be written as

$$(\lambda + G)\nabla(\nabla \cdot \mathbf{u}) + G\nabla^2 \mathbf{u} = \rho \ddot{\mathbf{u}}. \tag{14}$$

where the superposed dots represent derivatives with respect to time.

5. Elastic waves in unbounded media

In §2 we showed that elastic waves of long wavelength travel through thin rods at a speed of $c = (E/\rho)^{1/2}$. But elastic waves do not travel through a large rock mass at this speed. Moreover, we will show below that there are two different types of waves that can travel through unbounded rocks: transverse (shear) waves and longitudinal (compressional) waves.

Elastic waves in three-dimensional, unbounded, isotropic elastic media are governed by the full three-dimensional equations of motion, §4.4 (14):

$$(\lambda + G)\nabla(\nabla \cdot \mathbf{u}) + G\nabla^2\mathbf{u} = \rho\ddot{\mathbf{u}}. \tag{1}$$

Let's assume for now that the solution to (1) is wave-like, as in §2. So we consider a planar wave travelling at some (as yet unknown) speed c along a direction that is represented by the unit vector $\mathbf{n} = (n_x, n_y, n_z)$. Such a wave can be represented by

$$\mathbf{u} = f(\mathbf{x} \cdot \mathbf{n} - ct)\mathbf{d}, \tag{2}$$

where the vector **d,** which represents the direction of particle motion, is taken to be a constant. In the one-dimensional theory of §2, the particle displacement must necessarily be in the same direction as the wave motion, but this does not have to be the case in three dimensions. We will consider the direction of **d** later.

First, note that the dot product $\mathbf{x} \cdot \mathbf{n}$ represents the projection of the vector **x** onto the direction **n**. The phase variable $\eta = \mathbf{x} \cdot \mathbf{n} - ct$ therefore has the same value for all points **x** that lie on a given plane perpendicular to **n.** So for simplicity, we can consider a point that lies on the vector **n**, *i.e.,* $\mathbf{x} = \zeta\mathbf{n}$. Along direction **n**, the phase is equal to $\eta = (\zeta\mathbf{n}) \cdot \mathbf{n} - ct = \zeta - ct$, since **n** is a unit vector. The velocity at which the wavefront moves along **n** is then given, as in §2(8), by

$$\mathbf{v}(\textit{wavefront}) = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_\eta = \left(\frac{\partial(\zeta\mathbf{n})}{\partial t}\right)_\eta = \left(\frac{\partial \zeta}{\partial t}\right)_\eta \mathbf{n} = c\mathbf{n}. \tag{3}$$

Hence, the wave propagates in the **n** direction, and the parameter c that appears in (2) is the phase velocity of this disturbance.

The time derivatives of **u** are found by applying the chain rule to (2):

$$\dot{\mathbf{u}} = -cf'(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}, \qquad \ddot{\mathbf{u}} = c^2 f''(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}. \tag{4}$$

The phase of this wave is explicitly given by $\eta = xn_x + yn_y + zn_z - ct$, so the spatial derivatives of **u** are, for example,

$$\frac{\partial \mathbf{u}}{\partial x} = f'(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}\frac{\partial \eta}{\partial x} = f'(\mathbf{x}\cdot\mathbf{n} - ct)\mathbf{d}n_x. \tag{5}$$

Hence, it follows that

$$\nabla\cdot\mathbf{u} = f'(\eta)\mathbf{d}\cdot\mathbf{n}, \tag{6}$$

$$\nabla(\nabla\cdot\mathbf{u}) = f''(\eta)(\mathbf{d}\cdot\mathbf{n})\mathbf{n}, \tag{7}$$

$$\nabla^2\mathbf{u} = f''(\eta)\mathbf{d}, \tag{8}$$

and (1) reduces to

$$[(\lambda + G)(\mathbf{d}\cdot\mathbf{n})\mathbf{n} + (G - \rho c^2)\mathbf{d}]f''(\eta) = 0. \tag{9}$$

One obvious solution to this equation is the trivial case $f''(\eta) = 0$, which leads to either a rigid-body motion or a state of uniform strain that is independent of time. If we assume that $f''(\eta) \neq 0$, (9) is equivalent to

$$(\lambda + G)(\mathbf{d}\cdot\mathbf{n})\mathbf{n} + (G - \rho c^2)\mathbf{d} = 0. \tag{10}$$

Now we must think about the specific direction of the particle motion, which is given by **d**. If **d** is perpendicular to the direction **n** of wave propagation, then $\mathbf{d}\cdot\mathbf{n} = 0$, and (10) can only be satisfied for a non-zero vector **d** if $G - \rho c^2 = 0$, *i.e.,* if

$$c \equiv c_T = \sqrt{G/\rho}, \tag{11}$$

where c_T is the velocity of *transverse waves,* for which the particle motion is transverse to the direction of wave propagation. The direction of propagation **n** is completely arbitrary, as is the amplitude of the

particle displacement. Hence, transverse plane waves can travel in any direction of an isotropic elastic medium, but can only travel at a speed given by (11). This speed is independent of the frequency or wavelength of the disturbances, so these waves are non-dispersive.

If, on the other hand, **d** is *parallel* rather than *perpendicular* to the direction of wave propagation, then $\mathbf{d} = d\mathbf{n}$, where d is a scalar, and (10) reduces to

$$(\lambda + 2G - \rho c^2) d\mathbf{n} = 0, \tag{12}$$

which can only be satisfied by non-zero particle displacement d if

$$c \equiv c_L = \sqrt{(\lambda + 2G)/\rho}. \tag{13}$$

Longitudinal plane waves, in which the particle velocity is in the same direction as the wave propagation, can therefore travel through an isotropic elastic medium in any direction, but only at a velocity given by (13). These longitudinal waves are also non-dispersive, because c_L does not vary with frequency.

Use of relations §4.3 (18)-(20) allows the longitudinal wavespeed to be written in the following forms:

$$c_L = \sqrt{\frac{K + (4G/3)}{\rho}} = \sqrt{\frac{(1-\nu)E}{(1+\nu)(1-2\nu)\rho}} = \sqrt{\frac{2(1-\nu)G}{(1-2\nu)\rho}}, \tag{14}$$

from which it follows that the ratio of the two wavespeeds is

$$\frac{c_L}{c_T} = \sqrt{\frac{2(1-\nu)}{(1-2\nu)}}. \tag{15}$$

In any elastic medium, longitudinal waves travel faster than transverse waves. The ratio of the two wavespeeds is $\sqrt{2} \approx 1.41$ when $\nu = 0$, increases with increasing ν, and becomes unbounded when $\nu \to 0.5$.

Now let's examine transverse waves more closely. Without loss of generality, in an isotropic medium we can take the direction of propagation **n** to be the x-axis, and the direction of the particle displacement vector **d** to be the y-axis. The displacement vector for a transverse wave then has the form $\mathbf{u} = f(x - c_T t) d_y \mathbf{e_y}$, from which it

follows that the only non-zero strain component is $\varepsilon_{xy} = f'(x - c_T t)d_y / 2$, and the non-zero stress is $\tau_{xy} = Gf'(x - c_T t)d_y$. A transverse wave is therefore a *shear wave,* because the only non-zero stresses and strains in a transverse wave are shear stresses and shear strains. This is also consistent with the fact that the shear modulus is the only elastic modulus that affects c_T.

Similarly, by proper alignment of the co-ordinate system a longitudinal wave can be represented by $u = f(x - c_L t)d_x e_x$, for which the only non-zero strain component is $\varepsilon_{xx} = f'(x - c_L t)d_x$. Hence, a longitudinal wave is a wave of *uniaxial strain*, consistent with the fact that, according to §4.3 (12), $\lambda + 2G$ is the uniaxial strain modulus. However, due to the Poisson effect, longitudinal waves are *not* waves of uniaxial stress, as the normal stresses on planes perpendicular to the direction of wave propagation must satisfy $\tau_{yy} = \tau_{zz} = \nu\tau_{xx}/(1-\nu)$ in order to maintain a state of uniaxial strain. This is in contrast to the situation that occurs when a long-wavelength longitudinal wave travels along a thin bar, as in §2. In that case, the *stress* is essentially uniaxial, and the two lateral strains are non-zero.

In geophysics, the transverse wave velocity is usually denoted by V_s, and the longitudinal velocity by V_p. The subscript *s* can be thought of as signifying a *shear wave*, or it can be thought of as signifying a *secondary wave*, as these waves arrive at a receiver later than the faster-moving longitudinal waves. The subscript *p* can similarly be thought of as standing for *primary wave* or *pressure wave.* These two types of waves are usually referred to as P-waves and S-waves.

The two wave velocities depend on the elastic moduli and density of the rock, which in turn depend not only on mineral composition, pore structure, fluid properties (see §8 and §10), but also vary with stress, temperature, pore pressure, *etc*. For example, a decrease in pore fluid pressure would tend to allow cracks and grain-boundary pores to close up, thereby increasing the elastic moduli and the wavespeeds. Given the great variability in rock properties, even within the same rock type, it is difficult, and not very meaningful, to cite specific values for specific rocks. Table 5.1, adapted from Bourbié *et al.* (1987), gives ranges of representative values for several types of rock.

Table 5.1. Wavespeeds and densities of various rock types.

Rock Type	V_p (m/s)	V_s (m/s)	ρ (kg/m^3)
Vegetal soil	300-700	100-300	1700-2400
Dry sands	400-1200	100-500	1500-1700
Wet sands	1500-2000	400-600	1900-2100
Saturated shales or clays	1100-2500	200-800	2000-2400
Marls	2000-3000	750-1500	2100-2600
Saturated shale/sand sections	1500-2200	500-750	2100-2400
Porous saturated sandstones	2000-3500	800-1800	2100-2400
Limestones	3500-6000	2000-3300	2400-2700
Chalk	2300-2600	1100-1300	1800-2300
Salt	4500-5500	2500-3100	2100-2300
Anhydrite	4000-5500	2200-3100	2900-3000
Dolomite	3500-6500	1900-3600	2500-2900
Granite	4500-6000	2500-3300	2500-2700
Basalt	5000-6000	2800-3400	2700-3100
Gneiss	4400-5200	2700-3200	2500-2700
Coal	2200-2700	1000-1400	1300-1800
Water	1450-1500	-	1000
Ice	3400-3800	1700-1900	900
Oil	1200-1250	-	600-900

A useful mathematical tool for the analysis of elastodynamic problems is the Helmholtz decomposition of the displacement vector into the gradient of a scalar potential, φ, plus the curl of a vector potential, $\boldsymbol{\Psi}$ (Sternberg, 1960):

$$\mathbf{u} = \mathrm{div}\varphi + \mathrm{curl}\boldsymbol{\Psi} = \nabla\varphi + \nabla \times \boldsymbol{\Psi}. \tag{16}$$

The displacement corresponding to $\boldsymbol{\Psi}$ is divergence-free, and hence has no volumetric strain, and is therefore a state of pure shear. To prove this, consider

$$\mathbf{u} = \text{curl}\boldsymbol{\Psi} \equiv -2\,\text{asym}(\nabla\boldsymbol{\Psi}) = (\nabla\boldsymbol{\Psi})^{\mathsf{T}} - \nabla\boldsymbol{\Psi}$$

$$= \left[\left(\frac{\partial\psi_z}{\partial y} - \frac{\partial\psi_y}{\partial z}\right), \quad \left(\frac{\partial\psi_x}{\partial z} - \frac{\partial\psi_z}{\partial x}\right), \quad \left(\frac{\partial\psi_y}{\partial x} - \frac{\partial\psi_x}{\partial y}\right)\right], \qquad (17)$$

from which it readily follows that

$$\varepsilon_V = \nabla\cdot\mathbf{u} = \left(\frac{\partial^2\psi_z}{\partial x\partial y} - \frac{\partial^2\psi_z}{\partial y\partial x}\right) + \left(\frac{\partial^2\psi_y}{\partial z\partial x} - \frac{\partial^2\psi_y}{\partial x\partial z}\right) + \left(\frac{\partial^2\psi_x}{\partial y\partial z} - \frac{\partial^2\psi_x}{\partial z\partial y}\right) = 0, \quad (18)$$

in which case (1) reduces to

$$G\nabla^2\mathbf{u} = \rho\ddot{\mathbf{u}}. \qquad (19)$$

Eqn. (19) represents three uncoupled wave equations, one for each of the displacement components, each with wavespeed $c_T = (G/\rho)^{1/2}$:

$$\nabla^2 u = \frac{1}{c_T^2}\frac{\partial^2 u}{\partial t^2}, \quad \nabla^2 v = \frac{1}{c_T^2}\frac{\partial^2 v}{\partial t^2}, \quad \nabla^2 w = \frac{1}{c_T^2}\frac{\partial^2 w}{\partial t^2}. \qquad (20)$$

Taking the partial derivative of (20a) with respect to *y*, and adding it to the partial derivative of (20b) with respect to *x*, and similarly for the other two pairs of equations that can be chosen from (20), shows that each of the components of the strain tensor also satisfies this same wave equation:

$$\nabla^2\varepsilon_{xy} = \frac{1}{c_T^2}\frac{\partial^2\varepsilon_{xy}}{\partial t^2}, \quad \nabla^2\varepsilon_{xz} = \frac{1}{c_T^2}\frac{\partial^2\varepsilon_{xz}}{\partial t^2}, \quad \nabla^2\varepsilon_{yz} = \frac{1}{c_T^2}\frac{\partial^2\varepsilon_{yz}}{\partial t^2}. \qquad (21)$$

Alternatively, subtracting the partial derivative of (20a) with respect to *y* from the partial derivative of (20b) with respect to *x*, *etc.,* which is essentially equivalent to applying the curl operator to (19), shows that the three independent components of the rotation tensor also satisfy this same wave equation:

$$\nabla^2\omega_{xy} = \frac{1}{c_T^2}\frac{\partial^2\omega_{xy}}{\partial t^2}, \quad \nabla^2\omega_{xz} = \frac{1}{c_T^2}\frac{\partial^2\omega_{xz}}{\partial t^2}, \quad \nabla^2\omega_{yz} = \frac{1}{c_T^2}\frac{\partial^2\omega_{yz}}{\partial t^2}. \qquad (22)$$

Hence, "rotation" is also propagated through the medium at speed c_T. Consequently, shear waves are also sometimes called *rotational waves*.

The other part of the decomposed displacement vector, $\mathbf{u} = \nabla\varphi$, corresponds to a deformation in which there is *no rotation*. For example,

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial}{\partial y}\left[\frac{\partial \varphi}{\partial x}\right] - \frac{\partial}{\partial x}\left[\frac{\partial \varphi}{\partial y}\right]\right) = 0, \tag{23}$$

and similarly for ω_{xz} and ω_{yz}. In this case the term $\nabla(\nabla\cdot\mathbf{u})$ in (1) reduces to $\nabla^2\mathbf{u}$, because, for example,

$$\frac{\partial(\nabla\cdot\mathbf{u})}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \nabla^2 u, \tag{24}$$

and similarly for the other two components of $\nabla(\nabla\cdot\mathbf{u})$, in which case (1) takes the form

$$(\lambda + 2G)\nabla^2\mathbf{u} = \rho\ddot{\mathbf{u}}. \tag{25}$$

This is equivalent to the three uncoupled scalar wave equations

$$\nabla^2 u = \frac{1}{c_L^2}\frac{\partial^2 u}{\partial t^2}, \quad \nabla^2 v = \frac{1}{c_L^2}\frac{\partial^2 v}{\partial t^2}, \quad \nabla^2 w = \frac{1}{c_L^2}\frac{\partial^2 w}{\partial t^2}. \tag{26}$$

Hence, irrotational waves travel at the longitudinal wave velocity, c_L; equivalently, longitudinal waves are associated with irrotational motions.

Finally, differentiating (26a) with respect to *x*, (26b) with respect to *y*, and (26c) with respect to *z,* and adding the results, which is equivalent to taking the divergence of (25), yields

$$\nabla^2(\nabla\cdot\mathbf{u}) = \nabla^2\varepsilon_V = \frac{1}{c_L^2}\frac{\partial^2\varepsilon_V}{\partial t^2}. \tag{27}$$

Hence, changes in the bulk volumetric strain also propagate at the speed c_L.

More generally, the scalar potential φ satisfies the wave equation with wavespeed c_L. This is proven by substituting $\nabla \cdot \mathbf{u} = \nabla \cdot (\nabla \varphi) = \nabla^2 \varphi$ into (27), and noting that the operator ∇^2 commutes with the time derivatives, leading to

$$\nabla^2 \varphi = \frac{1}{c_L^2} \frac{\partial^2 \varphi}{\partial t^2}. \tag{28}$$

Similarly, it can be shown that each of the three Cartesian components of the vector potential $\boldsymbol{\Psi}$ satisfies the wave equation with wavespeed c_T:

$$\nabla^2 \psi_x = \frac{1}{c_T^2} \frac{\partial^2 \psi_x}{\partial t^2}, \quad \nabla^2 \psi_y = \frac{1}{c_T^2} \frac{\partial^2 \psi_y}{\partial t^2}, \quad \nabla^2 \psi_z = \frac{1}{c_T^2} \frac{\partial^2 \psi_z}{\partial t^2}. \tag{29}$$

The full explicit relationships between the displacements and the potentials are

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi_z}{\partial y} - \frac{\partial \psi_y}{\partial z}, \quad v = \frac{\partial \varphi}{\partial y} + \frac{\partial \psi_x}{\partial z} - \frac{\partial \psi_z}{\partial x}, \quad w = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y}. \tag{30}$$

6. Reflection and refraction of waves at an interface

In §2 the transmission of a wave across an interface between two possibly different elastic media was studied in the one-dimensional case. It was seen that, in general, some portion of the energy is transmitted through to the second medium, and the remaining portion is reflected back into the medium from which the wave came. The amplitudes of the reflected and transmitted waves depended on the ratios of the acoustic impedances of the two media, where the acoustic impedance is the product of the density and the wavespeed.

When an elastic wave impinges upon an interface between two rock types at an *oblique* angle, the behaviour is more complicated. This problem was first studied by Knott (1899), although the resulting equations are usually attributed to Zoeppritz (1919). Other early studies were made by Jeffreys (1926), Muskat and Meres (1940) and Ott (1942). Detailed results are given by Ewing, Jardetzky and Press (1957) and Brekhovskikh (1980). In general, regardless of whether the impinging wave is a shear wave or a compressional wave, four waves are created: both a shear and a compressional wave are transmitted (refracted) across the interface, and a shear and compressional wave are reflected back into the first medium. The amplitudes of these four waves, and the angles that they make with the interface, will depend not only on the acoustic impedances of the two media, but also on the angle of incidence of the incident wave.

An exception to this general behaviour occurs when the incident wave is a shear wave whose displacement vector is parallel to the interface. Such waves are called "SH" waves, referring to the fact that if the interface is horizontal, the displacement of such a wave will lie in the horizontal plane. Although interfaces between two rock types need not be horizontal, waves in which the displacement has no component normal to the interface are known as SH-waves. In this case, the transmitted and reflected wave will both be of the SH type, and no compressional waves will be generated. As this is the simplest case, we will study it first, in detail.

Consider two semi-infinite half-spaces, with their interface coinciding with the *x*-*y* plane. The region $z < 0$ is labelled with superscript 1, and

the region $z > 0$ is labelled with superscript 2. A plane shear wave propagates through medium 1, with its direction of propagation (**n** in the notation of §5) lying in the *x*-*z* plane, making an angle θ_0 with the *z*-axis (Fig. 6.1a). The particle displacement is in the *y*-direction; this motion therefore represents an "SH" wave.

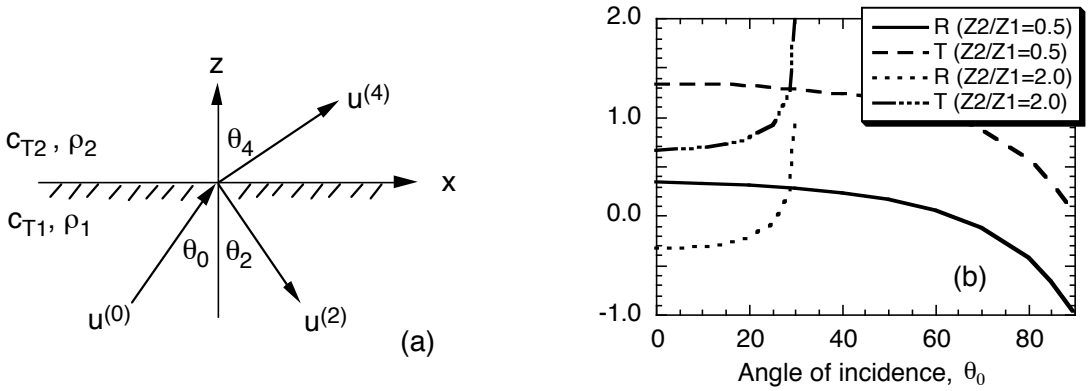

Fig. 6.1. (a) Incident, transmitted and reflected SH-wave impinging on a plane interface; (b) Reflection and transmission coefficients as a function of θ_0, for the case of $\rho_1 = \rho_2$.

It is traditional (and simpler) to use complex numbers to represent the displacements, stresses, *etc*., when solving such problems, bearing in mind that only the real component has physical significance. Consider (Fig. 6.1a) an incoming wave travelling in direction **n**, where **n** = ($\sin\theta$, 0, $\cos\theta$). In this notation, the only non-zero displacement component associated with the incoming wave is

$$v^{(0)} = A_0 \exp[ik_0(x\sin\theta_0 + z\cos\theta_0 - c_{T1}t)]. \qquad (1)$$

The wave that is reflected back into medium 1 is taken to be of the form

$$v^{(2)} = A_2 \exp[ik_2(x\sin\theta_2 - z\cos\theta_2 - c_{T1}t)], \qquad (2)$$

and the wave that is refracted into medium 2 is taken to be of the form

$$v^{(4)} = A_4 \exp[ik_4(x\sin\theta_4 + z\cos\theta_4 - c_{T2}t)]. \qquad (3)$$

The term “refracted” is used instead of “transmitted”, to emphasise that, in general, the wave changes its velocity of propagation as well as its direction of propagation when it enters the second medium.

The total displacement in medium 1 will be the sum of the displacement from the incoming wave and the reflected wave. In general, all three components of the total displacement vector must be continuous across a “welded” interface between two rock types. [If the rocks are separated by a fracture or other type of mechanical discontinuity, different boundary conditions must be used; see Jaeger *et al.* (2007) for details.]

For welded interfaces, the stresses acting *on the interface* must also be continuous. As the unit normal vector of the interface is $\mathbf{e_z}$, the stresses that act on the interface are $\{\tau_{zx},\tau_{zy},\tau_{zz}\}$. In general there are six continuity conditions, corresponding to the three displacement components and three interface stresses. In the present case, the only non-zero displacement component is *v*, and the only relevant non-zero stress component is τ_{zy}.

From (1)-(3), the condition for continuity of *v* across the interface $z = 0$ takes the form

$$\begin{aligned} A_0 \exp[ik_0(x\sin\theta_0 - c_{T1}t)] + A_2 \exp[ik_2(x\sin\theta_2 - c_{T1}t)] \\ = A_4 \exp[ik_4(x\sin\theta_4 - c_{T2}t)] \end{aligned} . \qquad (4)$$

As this equation must hold for all values of *x*, the factors multiplying *x* in each of the three exponential terms must be identical, *i.e.,*

$$k_0 \sin\theta_0 = k_2 \sin\theta_2 = k_4 \sin\theta_4 . \qquad (5)$$

The same must be true for the time-dependent terms, so

$$k_0 c_{T1} = k_2 c_{T1} = k_4 c_{T2} . \qquad (6)$$

Solving (5) and (6) for the angles and wave numbers of the reflected and refracted waves gives

$$k_2 = k_0, \quad k_4 = (c_{T1}/c_{T2})k_0, \quad \theta_2 = \theta_0, \quad \sin\theta_4 = (c_{T2}/c_{T1})\sin\theta_0, \qquad (7)$$

after which (4) reduces to

$$A_0 + A_2 = A_4. \tag{8}$$

Equation (7) shows that the angle of the reflected wave is always equal to the angle of the incident wave, whereas the refracted wave is "bent" towards the normal when medium 2 is "slower" than medium 1, and bent away from the normal if medium 2 is faster than medium 1. These facts, as implied by (7), are equivalent to Snell's law of optics.

The stress τ_{zy} that corresponds to a displacement of the form (1)-(3) is, suppressing the subscripts,

$$\tau_{zy} = \pm ik \cos\theta GA \exp[ik(x \sin\theta \pm z \cos\theta - c_T t)], \tag{9}$$

where the + sign is used for the incident and refracted wave, and the - sign is used for the reflected wave. From (9) and (7), the condition of continuity of τ_{zy} across the interface takes the form

$$\cos\theta_0 G_1 A_0 - \cos\theta_0 G_1 A_2 = (c_{T1}/c_{T2}) \cos\theta_4 G_2 A_4. \tag{10}$$

Solving (8) and (10) for the amplitudes of the reflected and transmitted waves gives

$$R = \frac{A_2}{A_0} = \frac{\rho_1 c_{T1} \cos\theta_0 - \rho_2 c_{T2} \cos\theta_4}{\rho_1 c_{T1} \cos\theta_0 + \rho_2 c_{T2} \cos\theta_4}, \tag{11}$$

$$T = \frac{A_4}{A_0} = \frac{2\rho_1 c_{T1} \cos\theta_0}{\rho_1 c_{T1} \cos\theta_0 + \rho_2 c_{T2} \cos\theta_4}. \tag{12}$$

The product of the density and shear-wave velocity is the shear-wave impedance (see §2), and so the *displacement* reflection and transmission coefficients, $R = A_2/A_0$ and $T = A_4/A_0$, depend on the shear-wave impedance ratio, Z_{T2}/Z_{T1}, and the angle of incidence of the incoming wave, θ_0. However, these coefficients also depend on the angle of the transmitted wave, θ_4, which in turn depends on the ratio of wavespeeds. Hence, the reflection and transmission coefficients depend on two material property ratios, impedance and wavespeed. From (9), it is clear that if the reflection and transmission coefficients were defined in terms of the amplitudes of the stresses, rather than the displacements, they would have somewhat different forms (Daehnke and Rossmanith, 1997).

The wavelength or frequency of the incident wave does not appear in the expressions for the reflection and transmission coefficients, so these expressions hold for arbitrary superposition of waves of different frequencies.

The reflection and transmission ratios are plotted in Fig. 6.1b, for two different impedance ratios. As there is somewhat more variability in wavespeed than in density between different rocks (see Table 5.1), for the purposes of illustration and discussion we take the densities of the two media to be equal, in which case the impedance ratio coincides with the velocity ratio. Aside from cases in which one of the media is a fluid, impedance ratios typically lie within the range shown in Fig. 6.1b, namely 0.5-2.0. The following observations can be made:

(1) If the incoming wave is normal to the interface, then $\theta_0 = \theta_2 = \theta_4 = 0$, and the reflection and transmission coefficients reduce to those given in §2 (18) and (19) and Fig. 2.2b for the one-dimensional wave model.

(2) If the second medium has zero shear impedance, such as occurs when a wave impinges on an interface between rock and fluid, the reflection coefficient is unity. Although the transmission coefficient approaches 2, this coefficient refers to the amplitudes, not the energies. If either the wavespeed or the density goes to zero, §2 (21) shows that no energy will be transmitted across the interface; the energy is entirely reflected back into medium 1. This apparent paradox of having a non-zero transmission coefficient can be eliminated by utilising the *stress* transmission coefficient, which would vanish in this case.

(3) There will be one particular angle of incidence for which there is no reflected wave. This angle is found by simultaneously solving, from (7), $\sin\theta_4 = (c_{T2}/c_{T1})\sin\theta_0$ and, from (11), $\cos\theta_4 = (Z_{T1}/Z_{T2})\cos\theta_0$. For the case in which the rocks on either side of the interface have the same density, this occurs for $\sin\theta_0 = [1+(c_{T2}/c_{T1})^2]^{-1/2}$.

(4) If $(c_{T2}/c_{T1})\sin\theta_0 > 1$, which can only occur if $c_{T2} > c_{T1}$, Snell's law yields an imaginary value for $\sin\theta_4$. The transmitted wave then has the form (Miklowitz, 1973, p. 184)

$$v^{(4)} = A_4 \exp(-bz)\exp[ik_4(x\sin\theta_4 - c_{T2}t)], \qquad (13)$$

where $b = k_0[(c_{T2}/c_{T1})^2 \sin^2\theta_0 - 1]^{1/2}$. Instead of representing a wave that propagates into medium 2, this displacement propagates only in the *x*-direction, parallel to the interface, with an amplitude (in medium 2) that decays exponentially with distance from the interface. Hence, this wave carries no energy into medium 2, and all of the incoming energy is reflected back into medium 1, with the reflected wave. This situation is referred to as *total internal reflection*.

If the incident wave is either an SV-wave or a P-wave, with particle motion in the *x*-*z* plane, the conditions of continuity for the displacement and stresses across the interface will in general only be satisfied by a combination of a pair of reflected P and SV-waves, and a pair of refracted P and SV-waves. An incident P-wave propagating towards the interface at an angle θ_o to the *z*-axis can be represented by

$$\mathbf{u}^0 = \{u^{(0)}, v^{(0)}, w^{(0)}\} = A_0 \exp[ik_0(x\sin\theta_0 + z\cos\theta_0 - c_{L1}t)]\{\sin\theta_0, 0, \cos\theta_0\}. \tag{14}$$

The reflected P-wave is denoted by 1, the "reflected" SV-wave by 2, the transmitted P-wave by 3, and the transmitted SV-wave by 4, as follows:

$$\mathbf{u}^1 = \{u^{(1)}, v^{(1)}, w^{(1)}\} = A_1 \exp[ik_1(x\sin\theta_1 - z\cos\theta_1 - c_{L1}t)]\{\sin\theta_1, 0, -\cos\theta_1\} \tag{15}$$

$$\mathbf{u}^2 = \{u^{(2)}, v^{(2)}, w^{(2)}\} = A_2 \exp[ik_2(x\sin\theta_2 - z\cos\theta_2 - c_{T1}t)]\{\sin\theta_2, 0, -\cos\theta_2\} \tag{16}$$

$$\mathbf{u}^3 = \{u^{(3)}, v^{(3)}, w^{(3)}\} = A_3 \exp[ik_3(x\sin\theta_3 + z\cos\theta_3 - c_{L2}t)]\{\sin\theta_3, 0, \cos\theta_3\} \tag{17}$$

$$\mathbf{u}^4 = \{u^{(4)}, v^{(4)}, w^{(4)}\} = A_4 \exp[ik_4(x\sin\theta_4 + z\cos\theta_4 - c_{T2}t)]\{\sin\theta_4, 0, \cos\theta_4\} \tag{18}$$

Matching the phases of the five waves along the interface $z = 0$ leads to the following equations that define the four angles $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ and four wave numbers $\{k_1, k_2, k_3, k_4\}$:

$$k_0 \sin\theta_0 = k_1 \sin\theta_1 = k_2 \sin\theta_2 = k_3 \sin\theta_3 = k_4 \sin\theta_4, \tag{19}$$

$$k_0 c_{L1} = k_1 c_{L1} = k_2 c_{T1} = k_3 c_{L2} = k_4 c_{T2}. \tag{20}$$

A set of four algebraic equations for the amplitudes of the four waves generated by the incident wave are found by imposing continuity conditions on the non-zero displacement components *u* and *w*, and the non-zero stresses τ_{zz} and τ_{zx}. The resulting equations are (Achenbach, 1973, p. 186)

$$\begin{bmatrix} -\sin\theta_1 & -\cos\theta_2 & \sin\theta_3 & -\cos\theta_4 \\ \cos\theta_1 & -\sin\theta_2 & \cos\theta_3 & \sin\theta_4 \\ \sin 2\theta_1 & \dfrac{c_{L1}}{c_{T1}}\cos 2\theta_2 & \dfrac{G_2}{G_1}\dfrac{c_{L1}}{c_{L2}}\sin 2\theta_3 & -\dfrac{G_2}{G_1}\dfrac{c_{L1}}{c_{T2}}\cos 2\theta_4 \\ -\cos 2\theta_2 & \dfrac{c_{T1}}{c_{L1}}\sin 2\theta_2 & \dfrac{G_2}{G_1}\dfrac{c_{L2}}{c_{L1}}\left(\dfrac{c_{T1}}{c_{T2}}\right)^2\cos 2\theta_4 & \dfrac{G_2}{G_1}\dfrac{c_{T1}}{c_{T2}}\dfrac{c_{T1}}{c_{L1}}\sin 2\theta_4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = A_0 \begin{bmatrix} \sin\theta_0 \\ \cos\theta_0 \\ \sin 2\theta_0 \\ \cos 2\theta_2 \end{bmatrix}. \tag{21}$$

Closed-form solutions to these equations have been given by Ewing *et al*. (1957), and extensive numerical tables have been generated by Muskat and Meres (1940). The results display complicated and often non-monotonic behaviour, and, as mentioned by the latter authors, “no simple physical interpretation or explanation can be given for the manifold variations of the coefficients with the parameters”.

One important special case, which can be solved and interpreted easily, is that of a P-wave impinging on a stress-free surface (Fig. 6.2a). In general, both a P-wave and an SV-wave will be reflected back into the rock. This case is obtained from (21) by setting $A_3 = A_4 = 0$, so as to ignore “transmitted” waves, and also ignoring the first two equations in (21), because the “continuity of displacement” boundary conditions are not relevant at a free surface. The third and fourth equations in (21), representing the stress-free boundary conditions, remain relevant. These two equations, corresponding to the lower-left 2×2 sub-matrix in (21), can be solved for

$$\frac{A_1}{A_0} = \frac{\sin 2\theta_0 \sin 2\theta_2 - (c_L/c_T)^2 \cos^2 2\theta_2}{\sin 2\theta_0 \sin 2\theta_2 + (c_L/c_T)^2 \cos^2 2\theta_2}, \tag{22}$$

$$\frac{A_2}{A_0} = \frac{2(c_L/c_T)\sin 2\theta_0 \cos 2\theta_2}{\sin 2\theta_0 \sin 2\theta_2 + (c_L/c_T)^2 \cos^2 2\theta_2}. \tag{23}$$

These coefficients are shown in Fig. 6.2b for two values of Poisson's ratio. When $\nu < 0.26$, there will be two angles of incidence for which the amplitude of the reflected P-wave is zero. In this case, known as mode conversion, all of the incident energy is reflected as an SV-wave. The two angles at which mode conversion occurs are equal to 38˚ and 90˚ when $\nu = 0$, and coalesce to 68˚ when $\nu = 0.26$ (Arenberg, 1948).

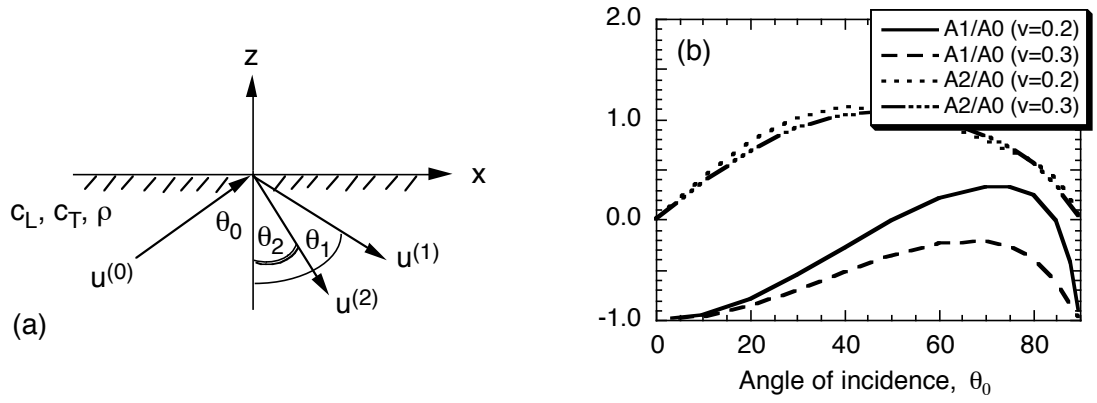

Fig. 6.2. (a) Incident P-wave, and reflected P and SV-waves, at a free surface. (b) Amplitude ratios for the reflected P-wave, A_1/A_0, and the reflected SV-wave, A_2/A_0.

7. Anisotropy

7.1. Elastic Anisotropy

Most rocks are anisotropic to one extent or another. If cylindrical cores are cut from a rock in the horizontal and a vertical direction, and P-wave speeds are measured along the length of these cores, the two values thus measured will in general differ from one another. Common cases of anisotropic rocks include sedimentary rocks that have different elastic properties in and perpendicular to the bedding planes, or metamorphic rocks such as slates that have a well-defined plane of cleavage. In contrast to an isotropic rock, in an anisotropic rock the *generalised Hooke's law* will have more than two independent elastic coefficients.

As both the stress and strain are second-order tensors, with nine components each, the most general linear relationship between the stresses and strains can be expressed via a fourth-order tensor that has $9 \times 9 = 81$ components. This relationship is often written as $\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\varepsilon}$, where $\mathbf{C}$ is a fourth-order tensor whose eighty-one components are known as the *elastic stiffnesses*. However, there is no straightforward way to write out the eighty-one components of a fourth-order tensor in the form of a matrix, so use of the tensor notation causes us to lose the convenience of matrix multiplication. A more concise approach is that due to Voigt (1928), in which the stress and strain tensors are each converted into 1×6 column vectors, and the elastic stiffnesses are then represented by thirty-six stiffness coefficients that can be written as a 6×6 matrix. Hooke's law can be written in the Voigt notation as

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{bmatrix}, \tag{1}$$

where the factors of "2" arise because Voigt originally worked in terms of the "engineering" shear strains, rather than the tensor shear strains. Eq. (1) can also be written symbolically as $\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\varepsilon}$, although now $\boldsymbol{\tau}$ and $\boldsymbol{\varepsilon}$

must be interpreted 6×1 row vectors rather than 3×3 matrices, and $\mathbf{C}$ must be interpreted as a 6×6 matrix.

The inverse version of Hooke's law, in which the strains are expressed as linear functions of the stresses, can be symbolically written as $\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\tau}$, where $\mathbf{S} = \mathbf{C}^{-1}$ is the inverse matrix of $\mathbf{C}$. The components of the matrix $\mathbf{S}$ are referred to as the *elastic compliances*.

The stiffness matrix that appears in (1) must necessarily be symmetric, so in fact at most only twenty-one of the stiffness coefficients can be independent (Jaeger *et al.,* 2007, §5.8), and Hooke's law can be written in the Voigt notation as

$$\begin{bmatrix}\tau_{xx}\\ \tau_{yy}\\ \tau_{zz}\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix}c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36}\\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46}\\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56}\\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ 2\varepsilon_{yz}\\ 2\varepsilon_{xz}\\ 2\varepsilon_{xy}\end{bmatrix}. \qquad (2)$$

The greatest possible number of independent stiffnesses is twenty-one, and some materials (triclinic crystals) do indeed fall into this category. However, if a material exhibits any physical symmetry, the number of independent stiffnesses will be less than 21. Fumi (1952a,b) devised a systematic method for deducing the number of independent components of the stiffness matrix from the symmetry elements of the material, using group theory. These results can also be found in an *ad hoc* manner, as explained in §5.8 of Jaeger *et al*. (2007). This method, however, requires use of the rules for transforming tensors from one co-ordinate system to another, and so will not be used here. Instead, we will only summarise the results of this analysis.

Consider a rock mass that contains three mutually perpendicular sets of fractures. Such a rock mass is called an *orthotropic* (or *orthorhombic*) *material*, and has nine independent components of the elastic moduli tensor. The stress-strain law of an orthotropic rock can be written as

$$\begin{bmatrix}\tau_{xx}\\ \tau_{yy}\\ \tau_{zz}\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0\\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ 2\varepsilon_{yz}\\ 2\varepsilon_{xz}\\ 2\varepsilon_{xy}\end{bmatrix}; \tag{3}$$

or, in explicit form:

$$\tau_{xx} = c_{11}\varepsilon_{xx} + c_{12}\varepsilon_{yy} + c_{13}\varepsilon_{zz}, \tag{4}$$

$$\tau_{yy} = c_{12}\varepsilon_{xx} + c_{22}\varepsilon_{yy} + c_{23}\varepsilon_{zz}, \tag{5}$$

$$\tau_{zz} = c_{13}\varepsilon_{xx} + c_{23}\varepsilon_{yy} + c_{33}\varepsilon_{zz}, \tag{6}$$

$$\tau_{yz} = 2c_{44}\varepsilon_{yz}, \quad \tau_{xz} = 2c_{55}\varepsilon_{xz}, \quad \tau_{xy} = 2c_{66}\varepsilon_{xy}. \tag{7}$$

The compliance matrix **S** of an orthotropic rock can be found by inverting the stiffness matrix **C**. As **C** is "block-diagonal", its inverse is readily found to be

$$s_{11} = \frac{c_{22}c_{33} - c_{23}^2}{D}, \quad s_{22} = \frac{c_{11}c_{33} - c_{13}^2}{D}, \quad s_{33} = \frac{c_{11}c_{22} - c_{12}^2}{D}, \tag{8}$$

$$s_{12} = \frac{c_{12}c_{23} - c_{12}c_{33}}{D}, \quad s_{13} = \frac{c_{12}c_{23} - c_{22}c_{13}}{D}, \quad s_{23} = \frac{c_{12}c_{13} - c_{11}c_{23}}{D}, \tag{9}$$

where D is the determinant of the upper-left-corner block of **C**, *i.e.*,

$$D = \det\begin{bmatrix} c_{11} & c_{12} & c_{13}\\ c_{12} & c_{22} & c_{23}\\ c_{13} & c_{23} & c_{33}\end{bmatrix}, \tag{10}$$

and all other components of **S** are zero. The inverse form of Hooke's law for an orthotropic material is therefore

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{22} & s_{23} & 0 & 0 & 0 \\ s_{13} & s_{23} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{bmatrix} \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}, \tag{11}$$

where the *s* coefficients are given by (8)-(10).

A special case of an orthotropic material is one in which all three directions (*x,y,z*) are elastically equivalent. This might be the case if the rock mass contains three sets of orthogonal fractures that each have the same spacing. This type of material, which is said to possess *cubic symmetry*, and will have three independent elastic moduli. Hence, even though all three orthogonal directions are elastically equivalent, a cubic material is *not* the same as an isotropic material, which has only two independent moduli. A physical explanation for this fact is that although, for example, the Young's modulus of a cubic material will be the same in the *x* and *y* directions, there is no reason for *E* to have the same value in a direction that is oriented at an arbitrary angle between the *x* and *y* axes.

The stress-strain law of a rock mass that has cubic symmetry can be written as

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{bmatrix}; \tag{12}$$

or, in non-matrix form:

$$\tau_{xx} = c_{11}\varepsilon_{xx} + c_{12}\varepsilon_{yy} + c_{12}\varepsilon_{zz}, \tag{13}$$

$$\tau_{yy} = c_{12}\varepsilon_{xx} + c_{11}\varepsilon_{yy} + c_{12}\varepsilon_{zz}, \tag{14}$$

$$\tau_{zz} = c_{12}\varepsilon_{xx} + c_{12}\varepsilon_{yy} + c_{11}\varepsilon_{zz}, \tag{15}$$

$$\tau_{yz} = 2c_{44}\varepsilon_{yz}, \quad \tau_{xz} = 2c_{44}\varepsilon_{xz}, \quad \tau_{xy} = 2c_{44}\varepsilon_{xy}. \tag{16}$$

The most important form of anisotropy for seismic exploration purposes is the case when one of the three orthogonal axes is an axis of rotational symmetry, in the sense that all directions perpendicular to this axis are elastically equivalent. In this case, the rock is isotropic within any plane normal to the axis of rotational symmetry. A rock possessing this type of symmetry is known as "transversely isotropic" (a slightly misleading term, as a *transversely isotropic* rock is actually *anisotropic*). This type of anisotropy is common in layered rocks.

A transversely isotropic rock has five independent coefficients in its elastic moduli tensor. Hooke's law for a transversely isotropic rock takes the form

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} = \frac{1}{2}(c_{11} - c_{12}) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{bmatrix}. \tag{17}$$

Finally, we note that using the Voigt notation, Hooke's law for an isotropic rock can be written as

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} = \frac{1}{2}(c_{11} - c_{12}) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{bmatrix}. \tag{18}$$

where $c_{11} = \lambda + 2G$, $c_{12} = \lambda$, and $c_{44} = G$.

7.2. Elastic Waves in Anisotropic Rocks

In an isotropic rock, there are three modes of wave propagation: one longitudinal mode that travels at a speed of $V_p = \text{sqrt}\{[K + (4/3)G]/\rho\}$, and two shear modes, SH and SV, both of which travel at the same speed, $V_S = \text{sqrt}(G/\rho)$. In the longitudinal mode, the particle motion is parallel to the direction of wave propagation, whereas in the transverse (shear) modes, the particle motion is perpendicular to the direction of wave propagation. These two shear modes are considered to be different modes because the particle motion vectors of these two modes are orthogonal to each other.

For anisotropic rocks, the situation is much more complicated. Full details can be found in the monographs by Musgrave (1970) and Carcione (2001). In general, we can say the following about elastic wave propagation in an unbounded anisotropic medium:

a. Planar elastic waves can propagate in any direction.

b. In any given direction, three wave modes are possible. Each of these three modes will have its own velocity, and its own direction of particle motion. These three directions of particle motion will always be mutually orthogonal.

c. In general, the velocities of any of these three modes will vary with the direction of propagation of the wave.

d. Unlike the situation for isotropic rocks, the particle velocity vectors will not generally be precisely parallel or perpendicular to the direction of wave propagation. However, for most rocks the particle velocity vectors are “almost” parallel or perpendicular to the direction of wave propagation. Hence, we can refer to these waves as “quasi-longitudinal” or “quasi-shear”.

The most important type of anisotropy is so-called transverse isotropy. Consider a transversely isotropic rock mass in which the axis of rotational symmetry is vertical. Now consider a wave whose direction of propagation lies in some vertical plane, making an angle θ to the vertical axis. As discussed above, there will be three possible wave modes that can propagate in this direction.

(a) One of these modes is a quasi-longitudinal mode whose velocity is given by (Mavko *et al*., 2009):

$$V_p = \text{sqrt}[(c_{11}\sin^2\theta + c_{33}\cos^2\theta + c_{44} + M)/2\rho], \quad (19)$$

where

$$M = \text{sqrt}\{[(c_{11} - c_{44})\sin^2\theta - (c_{33} - c_{44})\cos^2\theta]^2 + (c_{13} + c_{44})^2\sin^2 2\theta\}. \quad (20)$$

Note that if θ = 0, $M \to c_{33} - c_{44}$, and so $V_p \to \sqrt{c_{33}/\rho}$, as we would expect for a longitudinal wave travelling in the vertical (3) direction.

(b) There is a quasi-shear wave that travels at a speed give by

$$V_{SV} = \text{sqrt}[(c_{11}\sin^2\theta + c_{33}\cos^2\theta + c_{44} - M)/2\rho], \quad (21)$$

where M is given by (20). Note that (21) differs from (19) only by the sign in front of the coefficient M. If θ = 0, $M \to c_{33} - c_{44}$, and $V_{SV} \to \sqrt{c_{44}/\rho}$, as we would expect for a shear wave travelling in the vertical (3) direction.

(c) There is a pure-shear wave that travels at a speed given by

$$V_{SH} = \text{sqrt}[(c_{66}\sin^2\theta + c_{44}\cos^2\theta)/\rho]. \quad (22)$$

In this mode, the particle velocity lies in the horizontal plane. If θ = 0, the speed of this wave also reduces to $V_{SH} \to \sqrt{c_{44}/\rho}$.

The above expressions for the three velocities of waves that travel in a vertical plane in a transversely isotropic medium are cumbersome to use. Thomsen (1986) developed simplified expressions that are accurate if the degree of elastic anisotropy is not “too large”.

In this “weak elastic anisotropy” model, Thomsen first defined the following three parameters:

$$\varepsilon = \frac{c_{11} - c_{33}}{2c_{33}}, \quad (23)$$

$$\gamma = \frac{c_{66} - c_{44}}{2c_{44}}, \quad (24)$$

$$\delta = \frac{(c_{13} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})}. \tag{25}$$

Note that for an isotropic medium, $c_{11} = c_{33}$, and so (23) shows that ε = 0; $c_{66} = c_{44}$, and so (24) show that γ = 0; and $c_{11} = \lambda + 2G$, $c_{12} = \lambda$, and $c_{44} = G$, in which case (25) shows that δ = 0. Hence, each of these parameters in some sense quantifies the degree of anisotropy.

If each of these three parameters are small, Thomsen showed that the three wave speeds were approximately given by the following expressions:

$$V_p \approx \sqrt{\frac{c_{33}}{\rho}}\left[1 + \delta \sin^2\theta\cos^2\theta + \varepsilon \sin^4\theta\right], \tag{26}$$

$$V_{SV} \approx \sqrt{\frac{c_{44}}{\rho}}\left[1 + \frac{c_{33}}{c_{44}}(\varepsilon - \delta)\sin^2\theta\cos^2\theta\right], \tag{27}$$

$$V_{SH} \approx \sqrt{\frac{c_{44}}{\rho}}\left[1 + \gamma \sin^2\theta\right]. \tag{28}$$

If the anisotropy parameters vanish, then $V_p \to \sqrt{c_{33}/\rho}$, and $V_{SH}, V_{SV} \to \sqrt{c_{44}/\rho}$, as would be the case in an isotropic medium.

For a wave travelling vertically, θ = 0, and $V_p = \sqrt{c_{33}/\rho}$, whereas for a wave travelling horizontally, θ = 90°, and $V_p = (1+\varepsilon)\sqrt{c_{33}/\rho}$. Hence, ε represents the fractional difference between the velocities of horizontally travelling and vertically travelling P-waves, *i.e.,*

$$\varepsilon = \frac{V_p(\theta = 90^o) - V_p(\theta = 0^o)}{V_p(\theta = 0^o)}. \tag{29}$$

Similarly, (28) shows that γ is the fractional difference between the velocities of horizontally travelling and vertically travelling SH-waves, *i.e.,*

$$\gamma = \frac{V_{SH}(\theta = 90^o) - V_{SH}(\theta = 0^o)}{V_{SH}(\theta = 0^o)}. \tag{30}$$

8. Effect of Fluid Saturation

The theory of wave propagation presented in previous sections of these notes applies to a homogeneous, single-phase elastic material. When the void space of a porous or fractured rock is saturated with a fluid, the rock can no longer be considered to be a homogeneous, single-phase material. Several approaches, having differing degrees of rigour and complexity, can be used to account for the effect of fluid saturation on seismic velocities.

Consider a rock with porosity ϕ, saturated with a fluid having density ρ_f and bulk modulus K_f; recall that the shear modulus of a fluid is zero. Although the rock is usually composed of a mixture of different mineral grains, the effective "upscaled" moduli of the minerals, denoted by $\{K_m, G_m\}$, can be accurately estimated from the mineral composition using the Voigt-Reuss-Hill average (see §10). The effective density ρ_m of the mineral phase of the rock is given by the volumetric average of the densities of the individual mineral components.

The simplest model for the wavespeeds in a fluid-saturated rock is the so-called time-average model proposed by Wyllie *et al.* (1956), in which the travel time of a wave passing through the rock is approximated by the volume-weighted average of the travel times through a layer of solid rock and a layer of pore fluid (Fig. 8.1).

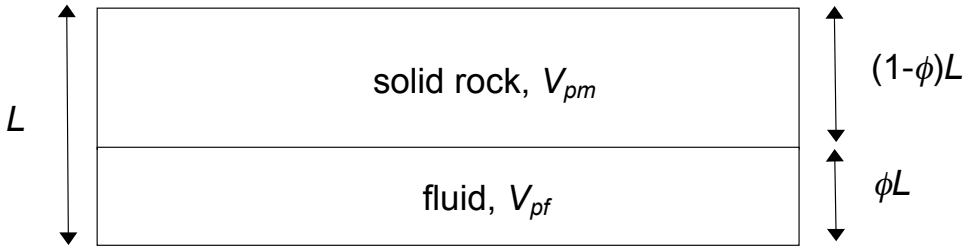

Fig. 8.1. Layer of a solid rock and a fluid, as used in the Wyllie time-average model.

In order for this model to have the correct porosity, the rock layer must have thickness $(1-\phi)L$, and the fluid layer should have thickness ϕL. The compressional wave speed in the rock is $V_{pm} = [(3K_m + 4G_m)/3\rho_m]^{1/2}$, and the compressional wave speed in the fluid is $V_{pf} = (K_f/\rho_f)^{1/2}$. The time required for the wave to pass through the layer of rock is $L(1-\phi)/V_{pm}$,

and the time required for the wave to pass through the layer of fluid is $L\phi/V_{pf}$. The total time required for the wave to pass through both layers is $t = L(1-\phi)/V_{pm} + L\phi/V_{pf}$. Since the total thickness is L, the effective velocity of the fluid-saturated rock will be $V_p = L/t$, which is to say,

$$\frac{1}{V_p} = \frac{\phi}{V_{pf}} + \frac{1-\phi}{V_{pm}}. \tag{1}$$

This model usually under-estimates wavespeeds, but is sometimes reasonably accurate for cemented and consolidated sandstones, particularly at high pressures, when the crack-like pores have been shut. This approach cannot be applied to shear waves, because the shear speed in a fluid is zero.

The next level of sophistication is to treat the fluid-saturated rock as an "effective elastic medium". This approach works well in the low-frequency, quasi-static limit, and is more accurate than the Wyllie time average. In this approach, if we can calculate the effective (upscaled) elastic moduli, K and G, of the fluid-saturated rock, and an effective density, ρ, then the two wavespeeds can be calculated from the usual equations, $V_p = [(3K + 4G)/3\rho]^{1/2}$, $V_s = (G/\rho)^{1/2}$.

First, we note that if the porous rock is dry, it will have some effective elastic moduli $\{K_d, G_d\}$. These moduli will be *less* than the moduli of the mineral phase, $\{K_m, G_m\}$, due to the presence of pores and cracks. The precise relation between the mineral moduli and the "dry porous rock" moduli will depend on the amount of porosity, and the shapes of the pores, as discussed in §10. For now, we just denote the dry rock moduli as $\{K_d, G_d\}$.

Now imagine that we add fluid to the pore space. In principle, the presence of the pore fluid will cause changes in the elastic moduli and density of the rock. Within the context of the effective medium approach, the effective density of the fluid-saturated rock is assumed to be given by the volumetrically-weighted average of the mineral and fluid densities, *i.e.,*

$$\rho = (1-\phi)\rho_m + \phi\rho_f. \tag{2}$$

Since the pore fluid has zero shear modulus, it seems reasonable to assume that the pore fluid cannot contribute any additional shear stiffness to the rock. Hence, we assume that the shear modulus of the fluid-saturated rock must equal G_d, the shear modulus of the dry rock. This is rigorously true in the static, zero-frequency limit, and is thought to be a good approximation at low frequencies.

As mentioned above, the relationship between the bulk modulus of the dry porous rock and the bulk modulus of the minerals depends strongly on pore structure. Fortunately, however, in the "quasi-static" limit of low frequencies, an exact expression can be found for the effective bulk modulus of the fluid-saturated rock, if we know the bulk modulus of the mineral phase, K_m, the bulk modulus of the dry porous rock, K_d, the bulk modulus of the pore fluid, K_f, and the porosity, ϕ.

This expression was first derived by Gassmann in 1951, in a complicated manner; a much simpler derivation can be found in Zimmerman (1991), but will not be repeated here, where we merely state Gassmann's equation without derivation. The Gassmann equation requires two crucial but subtly different assumptions. One assumption is that the pressure in the pore fluid is locally uniform, in the sense that the pressure in two adjacent pores will be the same. This will be true if the wave is of sufficiently low frequency, in which case the pore pressure will have time locally equilibrate within the time needed for the stress pulse to pass through a region of the rock.

Another assumption implicit in the Gassmann (1951) model is that, as the wave passes through the rock, the stress changes so rapidly that the fluid cannot "escape". The fluid is effectively trapped in the pore, and consequently plays a role in resisting being compressed by the passing stress wave. This type of process is known as "undrained" compression. Under these two assumptions, Gassmann was able to use superposition arguments to derive the following expression for the undrained elastic moduli:

$$K_u = \frac{\phi(1/K_f - 1/K_m) + (1/K_d - 1/K_m)}{\phi(1/K_f - 1/K_m)/K_d + (1/K_d - 1/K_m)/K_m}, \quad G_u = G_d. \qquad (3)$$

Finally, according to the Gassmann model, the wavespeeds in the fluid-saturated rock are given by

$$V_P(\text{low } \omega) = \{[K_U + (4/3)G_U]/\rho\}^{1/2}, \quad V_S(\text{low } \omega) = (G_U/\rho)^{1/2}, \tag{4}$$

where the effective moduli and density are given by (2) and (3).

Although we will not derive the Gassmann equation, we can verify that it makes intuitive sense in various limiting cases. First, if the porosity goes to zero, we see from (3) that the undrained bulk modulus equals that of the mineral phase, K_m.

If the pore fluid is a very compressible gas, *i.e.,* $K_f = 0$, then (3) predicts that the undrained bulk modulus will coincide with the drained (or "dry") bulk modulus, K_d .

If the pore fluid has the same bulk modulus as the mineral phase, *i.e.*, $K_f = K_m$, then (3) predicts that $K_u = K_m$, as would be expected. (Although this case is physically unlikely to ever occur, we can nevertheless use it to test the plausibility of the Gassmann equation).

Although it is not easy to see from (3), the Gassmann equation predicts, as we would expect, that the undrained modulus will increase as the bulk modulus of the pore fluid increases. In other words, the undrained bulk compressibility increases as the fluid compressibility increases, all other parameters being constant. The variation of $C_{bu} \equiv 1/K_u$ with $C_f \equiv 1/K_f$, for the Fort Union sandstone described by Murphy (1984), is shown in Fig. 8.2. This sandstone had a porosity of 0.085, a matrix bulk modulus of $K_m = 35.0$ GPa, and a dry/drained bulk modulus of $K_d = 7.63$ GPa. If the pores were filled with air at atmospheric pressure, which has a bulk modulus of $K_f = 0.101$ MPa, then the undrained bulk modulus would equal (to three significant figures) the drained value, 7.63 GPa. If the rock were saturated with water, which has a bulk modulus of $K_f = 2.0$ GPa, the undrained bulk modulus would be $K_U = 17.4$ GPa.

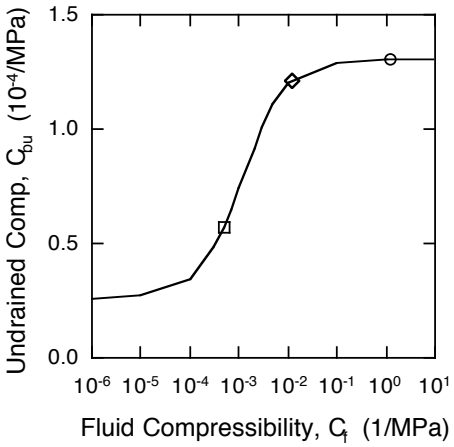

Fig. 8.2. Undrained bulk compressibility, C_{bu} = 1/K_u, of Fort Union sandstone, as a function of the compressibility of the pore fluid, C_f (Murphy, 1984; Zimmerman, 1985). If the pore fluid is a mixture of water and air at atmospheric pressure, the symbols (●,◆,■,) show the three cases of 0%, 99% and 100% water saturation, respectively.

Under quasi-static loading, the effective compressibility of the pore fluid is the volumetrically weighted average of the compressibilities of the different fluid phases. Hence, a small amount of air greatly increases the effective value of C_f, and would thereby cause the undrained bulk compressibility to increase.

According to the Gassmann model, saturating the pore space with fluid will increase the density and leave the shear modulus unchanged, leading to a slightly lower shear wave velocity. For example, in a sandstone of 20% porosity, with mineral density 2.65 g/cm^3, saturating the pore space with water having density 1.00 g/cm^3 will lower the shear wave speed by about 6%. The increase in bulk modulus usually overshadows the increase in density, so that compressional wave speeds are greater under saturated conditions than under dry conditions. The Gassmann approach usually works very well at seismic frequencies, below about 100 Hz.

At higher frequencies, in the logging (10 kHz) or laboratory ultrasonic (1 MHz) range, the pore fluid does not have sufficient time to redistribute itself so as to locally equilibrate the pore pressure. In this regime, the fluid-saturated rock cannot be treated as an effective single-phase continuum. Rather, the motion of the fluid, as distinguished from that of the solid phase, must somehow be accounted for. Biot (1956a,b)

developed a theory, based on the model of pores being long, cylindrical tubes, that allows for macroscopic flow of the fluid phases. Biot's theory reduces to the Gassmann approach at low frequencies, but predicts higher P- and S-wave velocities than does Gassmann in the high-frequency limit. At intermediate frequencies, the velocities are given by complicated expressions that involve the parameters appearing in (2) and (3), along with a "tortuosity" parameter and a characteristic grain size parameter (Berryman, 1980).

Although Biot's theory works reasonably well for very porous, high-permeability sediments (Stoll, 1989), it does not give accurate results for consolidated rocks. Wave propagation at logging or acoustic frequencies in rocks seems to involve small-scale "squirt-like" fluid flow that occurs on the length scale of individual pores and cracks (Mavko and Nur, 1975), not necessarily in the direction of propagation of the wave, as is implicitly assumed in Biot's model. Mavko and Jizba (1991) developed a procedure for estimating the wave speeds over the complete range of frequencies, which required knowledge of the wavespeeds in the dry rock at a given value of the effective stress, and in the high-stress limit, when all crack-like pores are closed. The computational procedure is summarised by Mavko *et al*. (2009, pp. 297-302).

9 Attenuation

When a wave travels through an elastic medium, the total energy contained in the wave, which (as explained in §2 and §3) is partitioned between elastic strain energy and kinetic energy, is conserved. Since the energy of a plane wave is related to its amplitude, a *plane* elastic wave will propagate without any change in amplitude. But for waves that spread out radially, such as those emanating from spherical cavities or cylindrical boreholes, the amplitude will decrease, because a finite amount of energy is spread out over a wavefront that has an ever-increasing area. This type of amplitude decay is known as *geometric attenuation*, and is not associated with a loss of overall kinetic energy.

However, rocks do not behave purely elastically under transient conditions. There are many mechanisms that cause the kinetic energy of seismic waves to be transformed into internal energy. This energy is not lost, but is transformed into internal energy, which is manifested by a slight increase in the temperature of the rock. But from a purely mechanical point of view, this energy appears to be “lost”, or “dissipated”.

The attenuation of a plane wave can be introduced by following the development in §2 for one-dimensional wave propagation, but using a simple viscoelastic constitutive model, such as the Kelvin-Voigt model. The Kelvin-Voigt model can be interpreted as an elastic element (a spring) in parallel with a viscous element (a dashpot). In an elastic spring, the stress is proportional to the strain, whereas in a viscous fluid, the stress is proportional to the strain rate. Hence, a Kelvin-Voigt material has the following stress-strain relationship:

$$\tau = E\varepsilon + \eta\dot{\varepsilon}\,, \tag{1}$$

where η is a viscosity-like parameter.

If we consider a process in which the strain varies according to $\varepsilon = \varepsilon_0 e^{i\omega t}$, the stress can be written as

$$\tau = (E + i\omega\eta)\varepsilon_0 e^{i\omega t} = (M_R + iM_I)\varepsilon, \tag{2}$$

where $M_R + iM_I$ is the complex modulus. Eq. (2) shows that E can be interpreted as the real part of the complex modulus, and $\omega\eta$ as the imaginary part.

Substituting (1) into the governing equation §2.2 yields

$$E\frac{\partial^2 u}{\partial x^2} + \eta\frac{\partial^3 u}{\partial t \partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}. \tag{3}$$

This differs from a standard elastic wave equation by the presence of the viscosity term.

Now consider a plane wave whose displacement is described by

$$u(x,t) = U_o \exp[i\{(k_R + ik_I)x - \omega t\}], \tag{4}$$

where we take the wave number k to have both a real and an imaginary part. Substitution of (4) into (3) yields the condition

$$(k_R + ik_I)^2(E - i\omega\eta) = \rho\omega^2, \tag{5}$$

the solution to which is (Kolsky, 1963, p. 117)

$$k_R = \left[\frac{\rho E\omega^2}{2(E^2 + \eta^2\omega^2)}\left\{\left(\frac{E^2 + \eta^2\omega^2}{E^2}\right)^{1/2} + 1\right\}\right]^{1/2}, \tag{6}$$

$$k_I = \left[\frac{\rho E\omega^2}{2(E^2 + \eta^2\omega^2)}\left\{\left(\frac{E^2 + \eta^2\omega^2}{E^2}\right)^{1/2} - 1\right\}\right]^{1/2}. \tag{7}$$

The positive root for k_R is chosen so that the wave propagates to the right, whereas the positive root must be chosen for k_I so as to yield a wave whose amplitude *decreases* with x as it propagates.

For a non-molten rock, we expect that the elastic part of the stress will dominate the viscous part, which is to say η must in some sense be "small". Expanding (6) and (7) for small values of η gives

$$k_R = \frac{\omega}{c_0}\left[1 - \frac{3}{8}\left(\frac{\eta\omega}{E}\right)^2\right], \qquad k_I = \frac{\eta\omega^2}{2Ec_0}, \tag{8}$$

where $c_0 = (E/\rho)^{1/2}$ is the velocity that the elastic wave would have in the absence of any dissipative mechanisms. The actual velocity, $c = \omega / k_R$, varies with frequency, as it must for any dissipative medium, as is required by the Kramers-Kronig relations (Mavko *et al*., 2009, pp. 127-128). However, to first-order in η, the wavespeed is unaffected by a small amount of viscous damping, and is given by $c = (E/\rho)^{1/2}$. Using this further simplification, the wave (3) can be expressed as

$$u(x,t) = U_0 \exp(-k_I x)\exp[i\{(\omega / c_0)x - \omega t\}], \tag{9}$$

where k_I is given by (8). Thus, the wave travels at velocity c_0, but with an amplitude that decays exponentially with distance. This represents actual viscous attenuation, rather than the geometrical attenuation that occurs in spherical or cylindrical waves.

According to this model, the attenuation seems to increase with the square of the frequency, according to (8b). However, there are various mechanisms in rocks that give rise to viscous-like behaviour, and each has, in effect, its own dependence of η on frequency. Thus, each mechanism predicts a frequency-dependence of attenuation that will reflect both the ω^2 term from (8), and the frequency-dependence of η, usually giving rise to an exponent that differs from 2. Before discussing these dissipative mechanisms, we discuss several standard definitions that are used to quantify attenuation.

The imaginary part of the wavenumber, k_I, is also denoted by α, the *attenuation coefficient*. Its inverse, $1/k_I$, is the length over which the amplitude will decay by a factor of $1/e \approx 0.37$. The mechanical energy (kinetic plus elastic strain energy) contained in a sinusoidal plane wave is proportional to the square of the amplitude, according to §3(8), so the fractional loss of energy over one wavelength is

$$\frac{\Delta \mathcal{T}}{\mathcal{T}} = \frac{\exp(-2\alpha x) - \exp(-2\alpha\{x+\lambda\})}{\exp(-2\alpha x)} = 1 - \exp(-2\alpha\lambda) \approx 2\alpha\lambda. \tag{10}$$

The *quality factor* Q is defined in terms of this fractional energy loss, as follows:

$$\frac{1}{Q} \equiv \frac{\Delta \mathcal{T}}{2\pi \mathcal{T}} = \frac{2\alpha\lambda}{2\pi} = \frac{2\alpha c}{\omega}, \tag{11}$$

where the last step makes use of the relation $\lambda = 2\pi c/\omega$. Substituting k_I from (8) into (11), and recalling that $\omega\eta = M_I$, and $E = M_R$, shows that Q can also be expressed as

$$\frac{1}{Q} = \frac{2\alpha c}{\omega} = \frac{2c\eta\omega^2}{2Ec\omega} = \frac{\eta\omega}{E} = \frac{M_I}{M_R}. \tag{12}$$

It can also be shown that, if α is small, $1/Q$ is equal to the phase shift (in radians) between the stress and the strain, under sinusoidal oscillations such as described in (2). Another parameter occasionally used to quantify attenuation in rocks is the *logarithmic decrement*, defined by $\delta = \pi/Q$.

Although α and Q contain essentially the same information, α measures the energy loss per distance travelled by the wave, whereas $1/Q$ measures the energy loss per wave cycle. Hence, as seen in (12), they will vary with frequency in different ways, a fact that should be remembered when viewing graphs of Q or α.

Versions of the relations (10)-(12) that do not require the assumption of small attenuation are given by Bourbié *et al*. (1987, p. 113). Relations similar to those described above for waves propagating along a thin bar can be derived for bulk P and S-waves in terms of the real and imaginary parts of K and G, and the P and S-wave quality factors (Winkler and Nur, 1979).

Toksöz and Johnston (1981) have collected many of the seminal papers on wave attenuation in rocks, and have provided several summary/overview chapters. Bourbié *et al*. (1987) present much data on attenuation measurements, and also review the various mechanisms. Measured values of Q for P-waves in various rocks are shown in Table 9.1. Porous rocks such as sandstones and limestones tend to have Q values in the range of 10-100, whereas igneous and metamorphic rocks will be in the range 100-1000 (Bradley and Fort, 1966).

Table 9.1. P-wave quality factors of some rocks

Rock	Condition	f (Hz)	Q	Source
Tennessee marble	dry	$0\text{-}2\times10^{4}$	480	Wyllie *et al*. (1962)
Quincy granite	air dry	$2\text{-}45\times10^{2}$	125	Birch & Bancroft (1938)
Solenhofen limestone	air dry	$3\text{-}15\times10^{6}$	112	Peselnick & Zeitz (1959)
Amherst sandstone	oven dry	$1\text{-}13\times10^{3}$	52	Born (1941)
Pierre shale	*in situ*	$5\text{-}45\times10^{1}$	32	McDonal *et al*. (1958)
Berea sandstone	brine saturated	$2\text{-}8\times10^{5}$	10	Toksöz *et al*. (1979)

Walsh (1966) developed a model in which attenuation is due to sliding friction along the faces of closed elliptical cracks. The model predicts $Q^{-1} = f(\mu, E/E_m)\varpi$, where μ is the coefficient of sliding friction along the crack faces, f is a dimensionless function whose values are on the order of 0.1, and ϖ is the crack density parameter (defined in §10) for those cracks whose faces are barely touching. There will be no frictional sliding along faces of open cracks, and the small stresses associated with seismic waves will be insufficient to cause sliding on crack faces that are tightly closed. Savage (1969) argued that there are unlikely to be a sufficient number of "barely closed" cracks to yield appreciable values of Q^{-1}.

Mavko (1979) considering a tapered crack, a portion of whose two faces will always be in contact, and for typical parameter values found $Q^{-1} \approx \varepsilon/\overline{\alpha}_i$, where ε is the incremental strain associated with the wave, and $\overline{\alpha}_i$ is some appropriate mean value of the initial crack aspect ratio. For values such as $\overline{\alpha}_i \approx 10^{-3}$, this attenuation may be appreciable under laboratory conditions, but would be negligible at the strains encountered in seismic waves, which would typically be $< 10^{-6}$, except very close to the source (Winkler *et al*., 1979).

Savage (1966) analysed attenuation arising from the conversion of mechanical energy into internal energy due to the coupling between strain and heat flow. The strain rises and falls sharply in the vicinity of cracks or pores, causing local heat flow, leading to wave attenuation described by

$$Q^{-1} = \left(\frac{\beta^2 K T_o}{\rho c_V} \right) g(\nu) \phi F(\omega), \tag{13}$$

where β is the *volumetric* thermal expansion coefficient, K is the bulk modulus, T_o is the ambient temperature, ρ is the rock density, c_V is the specific heat, ϕ is the porosity, g is a dimensionless function of the Poisson ratio (that takes on a slightly different form depending on whether the wave is compressional or shear), an F is a dimensionless function of frequency. The term in parenthesis in (13) is the thermoelastic coupling parameter that measures the extent to which strain induces heat flow, and is on the order of $10^{-3} - 10^{-4}$ for most rocks (Zimmerman, 2000). The function F is greatest at a characteristic frequency of about $\omega^* = k_T / \rho c_V a^2$, where a is the pore/crack size and k_T is the thermal conductivity; it increases with ω for lower frequencies, and drops off as ω^{-1} at higher frequencies. At the critical frequency, the diffusion length of the induced heat flux is on the order of the pore/crack size. By assuming a range of cracks sizes so as to smear out the function F over a range of frequencies, Savage found Q values of about 225 for longitudinal waves and 350 for transverse waves in granite, about twice as high as the measured values. This mechanism therefore seems to be important for cracked igneous rocks, and can lead to attenuations as large as about $Q^{-1} \approx 0.01$.

The Biot theory of wave propagation in fluid-saturated rocks predicts attenuation due to the viscous drag exerted by the pore walls on the pore fluid. At frequencies below the Biot critical frequency $\omega^* = \phi \mu / k \rho_f$, which for most rock/fluid systems exceeds 1 MHz, the theory predicts (Schön, 1996, p. 252)

$$Q^{-1} = \frac{g \rho_f^2 k \omega}{2 \rho \mu}, \tag{14}$$

where ρ_f is the density of the fluid, ρ is the density of the fluid-saturated rock, *k* is the permeability, and *g* is a numerical constant that equals 1 for shear waves, and is less than 1 for compressional waves. The inverse relation between attenuation and viscosity predicted by (14) is in contradiction to most measured data (Jones and Nur, 1983). Mochizuki (1982) made measurements on a Massilon sandstone, and found attenuations much greater than those predicted by Biot theory. For consolidated rocks with permeabilities below about 1 Darcy, Biot attenuation appears to be negligible at seismic and logging frequencies ($< 10^4$ Hz). Biot theory has been more successful when applied to highly permeable sediments, for which (14) yields appropriately high values of $1/Q$ (Stoll, 1989).

Whereas the Biot attenuation mechanism is based on "global flow" of the pore fluid, Mavko and Nur (1975), Murphy *et al*. (1984) and others have modelled the attenuation arising from the local flow of fluid squirting out of compliant cracks that are compressed by the passing elastic wave. This "squirt" flow is local, and is not necessarily aligned with the direction of wave propagation. Instead, it occurs locally in directions determined by the geometry of crack and pore intersections. The attenuation predicted by squirt flow models goes to zero at low frequencies, since in the limit of zero frequency there can be no viscous attenuation, and also goes to zero at very high frequencies, at which the fluid does not have sufficient time to move from one crack to a neighbouring crack within one period of the wave. The attenuation is peaked about a critical frequency that is roughly given by (Sams *et al*., 1997)

$$\omega^* \approx K\alpha^3 / \mu, \tag{15}$$

where *K* is the bulk modulus of the rock, and α is the aspect ratio of the cracks. Sams *et al*. (1997) measured wavespeeds and attenuations on a finely-layered sequence of limestones, sandstones, siltstones and mudstones in northeast England, over a range of methods/frequencies spanning vertical seismic profiling (30-280 Hz), crosshole surveys (0.2-2.3 kHz), sonic logging (8-24 kHz), and laboratory measurements (300-900 kHz), and found a bell-shaped curve for $1/Q$ vs. $\ln\omega$ that fit very well to the squirt-flow model.

One mechanism that produces "attenuation" without any conversion of mechanical energy into internal energy is elastic wave scattering. When

an elastic wave impinges upon an inhomogeneity, such as a pore, a crack, a fracture, *etc*., the inhomogeneity causes some portion of the energy to be scattered in directions other than the direction of propagation of the incident wave (Sato and Fehler, 1998), thereby effectively decreasing the amplitude of the original pulse. Yamakawa (1962) calculated the scattering from isolated spherical pores of radius a, and found

$$Q^{-1} = g\phi(\omega a/c)^3 = g\phi(ka)^3, \tag{16}$$

where ϕ is the porosity, and g is a dimensionless parameter of order 1 whose precise value depends on the moduli and densities of the rock and pore fluid. Other pore shapes lead to the same general form, with a being some characteristic dimension such as crack length, but with different values of g. Yamakawa's result applies asymptotically for wavelengths much larger than the inclusion size, the so-called *Rayleigh scattering* limit. As "typical" values of the parameters will be $a \approx 100\ \mu$m and $c \approx 5000$ m/s, attenuation due to Rayleigh scattering will be become appreciable only for frequencies greater than about 1 MHz, *i.e*., perhaps important at laboratory frequencies, but negligible at seismic frequencies.

As an approximation, it is usually assumed that the attenuation arising from different mechanisms, as quantified by Q^{-1}, is additive. Individual mechanisms that give rise to very high values of Q are therefore negligible in comparison with other, more dominant, mechanisms.

10. Effect of Microstructure on Seismic Velocities

The seismic velocities in a rock are controlled by the rock's elastic moduli. The specific numerical values of the elastic moduli are in turn controlled by the mineralogy, the pore structure of the rock, and the pore fluids. The effect of pore fluids was discussed to some extent in §8. In this section we will discuss the influence of mineralogy and pore structure. Much more discussion of this topic can be found in *Compressibility of Sandstones* by Zimmerman (Elsevier, 1991), and the *Rock Physics Handbook* by Mavko *et al*. (Cambridge, 2009).

10.1. Effect of Mineralogy

First consider a piece of rock that contains no cracks or pores. In some rare cases the rock may be composed of a single mineral, but in general it will be composed of a mixture of several minerals, each of which has its own elastic moduli, K and G. (To keep the discussion simple, we will ignore anisotropy on the scale of a single mineral crystal). But on a scale that is much larger than a single grain, this rock can be treated as a homogeneous elastic body that has some effective elastic moduli, K_{eff} and G_{eff}. Calculating these effective moduli is conceptually similar to the upscaling problem of calculating the effective permeability of a heterogeneous rock.

We start by considering a piece of a homogeneous rock that consists of a single mineral of bulk modulus K. If this piece of rock is subjected to a hydrostatic pressure P over its external surface, the volumetric strain inside the rock will be uniform, and equal to P/K. The total volume change of the rock will be $\Delta V = PV/K$.

Next, consider a rock that consists of an assemblage of N different minerals, each having its own bulk modulus K_i, where $i = 1,2,3,\ldots,N$. If a piece of this rock is subjected to a hydrostatic pressure P, the resultant strain in the rock will not be uniform. In general, the stiffer minerals will deform less than the more compliant minerals, although the precise deformation in each grain will depend on the geometrical configuration of the grains. For example, a compliant clay particle that is sitting inside a pore in a sandstone will undergo much less deformation than would a clay particle that is wedged between two sand grains. Nevertheless, this

rock will undergo some overall volume decrease, ΔV. From a purely macroscopic point of view, an "effective" bulk modulus K_{eff} can be defined by the relation $K_{eff} = PV/\Delta V$. This effective bulk modulus can be interpreted as the bulk modulus of a hypothetical homogeneous rock that would undergo the same mean volumetric strain as does the actual heterogeneous rock.

Precise calculation of K_{eff} would require exact knowledge of the microstructure of the rock, which in practice is never available. However, if the volume fractions χ_i and bulk moduli K_i of the various mineral components of the rock are known, the methods of Reuss and Voigt can be used to provide estimates of K_{eff}.

Reuss (1929) made the assumption that the *stresses* would be uniform throughout the heterogeneous rock. In reality this cannot be exactly true, as the different strains in the various mineral grains would then lead to displacement discontinuities at the grain boundaries. But if the stress within each mineral were indeed a hydrostatic compression P, then the volume change of each component would be $\Delta V_i = PV_i/K_i$. The total volume change would then be given by the sum of the volume changes of the individual minerals, *i.e.,*

$$\Delta V = \sum_{i=1}^{N} \Delta V_i = \sum_{i=1}^{N} \frac{PV_i}{K_i} = \sum_{i=1}^{N} \frac{P\chi_i V}{K_i} = PV \sum_{i=1}^{N} \frac{\chi_i}{K_i}. \tag{1}$$

Using the definition $K_{eff} = PV/\Delta V$,

$$K_{eff}^{Reuss} = \frac{PV}{\Delta V} = \left[\sum_{i=1}^{N} \frac{\chi_i}{K_i} \right]^{-1}. \tag{2}$$

The Reuss effective bulk modulus is therefore the *weighted harmonic mean* of the individual bulk moduli. In terms of the compressibilities,

$$C_{eff}^{Reuss} \equiv 1/K_{eff}^{Reuss} = \sum_{i=1}^{N} \frac{\chi_i}{K_i} = \sum_{i=1}^{N} \chi_i C_i, \tag{3}$$

and so the Reuss effective compressibility is the *weighted arithmetic mean* of the individual mineral compressibilities.

Voigt (1889), on the other hand, made the assumption that the *volumetric strains* are uniform throughout the heterogeneous body. Again, this cannot be precisely true, as equality of strains implies that the stresses in each mineral phase are different, and so the resulting stress field would be discontinuous, and would not satisfy the stress equilibrium equations. Under the assumption of equal volumetric strains, the mean normal stress in each component would be given by $\sigma_{m,i} = \varepsilon_V K_i$. The average value of the mean normal stress is simply the volumetric average of the mean normal stress in each component:

$$\left\langle \sigma_m \right\rangle = \sum_{i=1}^{N} \chi_i \sigma_{m,i} = \sum_{i=1}^{N} \chi_i \varepsilon_V K_i = \varepsilon_V \sum_{i=1}^{N} \chi_i K_i \,. \tag{4}$$

Making the obvious identification of the average value of the mean normal stress with the applied stress P, it follows from the definition $K_{eff} = PV / \Delta V = P / \varepsilon_V$ that

$$K_{eff}^{Voigt} = \frac{P}{\varepsilon_V} = \frac{\left\langle \sigma_m \right\rangle}{\varepsilon_V} = \sum_{i=1}^{N} \chi_i K_i \,. \tag{5}$$

Voigt's estimate of the effective bulk modulus is therefore simply the weighted arithmetic mean of the individual bulk moduli. The Voigt estimate of the effective compressibility is

$$C_{eff}^{Voigt} \equiv \frac{1}{K_{eff}^{Voigt}} = \left[\sum_{i=1}^{N} \chi_i K_i \right]^{-1} = \left[\sum_{i=1}^{N} \frac{\chi_i}{C_i} \right]^{-1} . \tag{6}$$

Since the harmonic mean of a set of numbers can never exceed the arithmetic mean, the Voigt estimate of K_{eff} will always exceed the Reuss estimate. More specifically, however, Hill (1952) used strain energy arguments to prove that the Voigt and Reuss estimates are rigorous upper and lower bounds on the true value of the effective bulk modulus, *i.e.,*

$$\left[\sum_{i=1}^{N} \frac{\chi_i}{K_i} \right]^{-1} = K_{eff}^{Reuss} \le K_{eff} \le K_{eff}^{Voigt} = \sum_{i=1}^{N} \chi_i K_i \,. \tag{7}$$

Hill then proposed using the average of the Voigt and Reuss bounds to find a best estimate of K_{eff}. The resulting value is known as the *Voigt-Reuss-Hill estimate* of the effective modulus:

$$K_{eff}^{VRH} = \frac{1}{2}[K_{eff}^{Reuss} + K_{eff}^{Voigt}]. \tag{8}$$

Although there is no particular justification for assuming that K_{eff} will lie exactly midway between the two bounds, Hill's assumption has the advantage of giving an estimate of the effective modulus that *a priori* will be guaranteed of having the minimum possible error.

The Voigt, Reuss and Hill arguments can also be applied to the estimation of the effective *shear* modulus of a heterogeneous rock. In this case,

$$\left[\sum_{i=1}^{N} \frac{\chi i}{G_i}\right]^{-1} = G_{eff}^{Reuss} \le G_{eff} \le G_{eff}^{Voigt} = \sum_{i=1}^{N} \chi_i G_i\,, \tag{9}$$

$$G_{eff}^{VRH} = \frac{1}{2}[G_{eff}^{Reuss} + G_{eff}^{Voigt}]. \tag{10}$$

Similar equations are often written for the Young's modulus. However, in general the arithmetic and harmonic means of the individual Young's moduli will *not* necessarily provide bounds on E_{eff} (Grimvall, 1986, p. 261). Instead, bounds on E_{eff} can be obtained from the bounds on K_{eff} and G_{eff} by using the identity $1/E = 1/(3G) + 1/(9K)$.

Values for the elastic moduli of various rock-forming minerals have been compiled by Clark (1966), Simmons and Wang (1971), and Mavko *et al*. (2009). Bulk moduli values of common minerals range from about 36-38 GPa for quartz, 63-77 GPa for calcite, 130 GPa for olivine, up to about 253 GPa for corundum. Shear moduli range from about 28-32 GPa for calcite, 44-46 GPa for quartz, 80 GPa for olivine, up to about 162 GPa for corundum. Hence, the range of values of K and G observed in common minerals span a range of less than one order of magnitude.

The relative lack of variability of the elastic moduli of different minerals causes the Voigt and Reuss bounds to usually be fairly close together. In many cases, the difference between the Voigt and Reuss estimates

will be within the experimental uncertainty of the moduli values of the individual minerals.

Brace (1965) measured the bulk moduli of several crystalline rocks at pressures up to 900 MPa. At such high pressures, it can be assumed that any cracks that may have been present at low pressures will be closed. As a representative example, consider the granite from Stone Mountain, Georgia, which was composed of 42% plagioclase, 30% quartz, 24% microcline, and 4% mica. In its unstressed state, it had a density of 2631 kg/m^3, and a micro-crack porosity of 0.3%. At 900 MPa, the individual mineral bulk moduli assumed by Brace were 62.7 GPa for plagioclase, 44.5 GPa for quartz, 60.0 GPa for microcline, and 50.1 GPa for mica. From (5), the Voigt estimate is 56.1 GPa, the Reuss estimate is 54.8 GPa, and the Voigt-Reuss-Hill estimate is 55.5 GPa. The measured bulk modulus was 56.8 GPa. Considering that the mineral compositions were reported only to within the nearest percent, the Voigt-Reuss-Hill average agrees with the measured value of the effective bulk modulus as nearly as one could expect. Brace found similar results for the other rocks.

10.2. Effect of Pores on the Elastic Moduli

The presence of pores or cracks will cause the moduli of a rock to be less than K_m and G_m. The extent to which the effective moduli of the actual porous rock are less than those of the mineral phase will depend on the amount of porosity, and the shape of the pores. Precise calculation of this effect is a difficult and still controversial problem in rock physics. We will review some of the models that are used to estimate the effective moduli, and show specific results for the two important cases of spherical pores and thin, penny-shaped cracks.

Numerous methods for accounting for the effect of pores on the elastic moduli have been proposed in the fields of rock physics, ceramics, and materials science. These methods are all applicable to the more general problem of predicting the effective elastic moduli of heterogeneous, multi-component materials (Christensen, 1991; Nemat-Nasser and Hori, 1993; Milton, 2002). Most of these methods can be discussed within the general formalism based on the stored elastic strain energy. Since the elastic energy stored in a homogeneous body subjected to a pressure P

will be $\mathcal{E} = P^2V/2K$, the effective bulk modulus K of a porous rock can be defined by

$$\frac{P^2V}{2K} = \frac{P^2V}{2K_m} + \Delta\mathcal{E}_{hydro}, \tag{1}$$

where $\Delta\mathcal{E}_{hydro}$ is the excess elastic strain energy that would be stored in a rock of bulk volume V, if the pores were introduced into the initially solid rock while maintaining a hydrostatic confining pressure P over its outer surface. The effective shear modulus can be defined in a similar way, with the rock assumed to be subjected to a shear stress of magnitude S:

$$\frac{S^2V}{2G} = \frac{S^2V}{2G_m} + \Delta\mathcal{E}_{shear}. \tag{2}$$

Utilization of (1) and (2) requires the estimation of the energy perturbation terms caused by the presence of pores, under hydrostatic and shear loading. The simplest approach is to neglect stress-field interactions between nearby pores, and calculate the energy change due to each pore as if were an isolated void in an infinite intact rock, and then sum up this energy for all the pores. The two energy perturbations, $\Delta\mathcal{E}_{hydro}$ and $\Delta\mathcal{E}_{shear}$, will depend on the moduli of the intact rock, $\{K_m, G_m\}$, and will be proportional to the porosity. The equations for the two effective moduli will be uncoupled, and can be solved explicitly for the effective moduli K and G.

For example, the energy terms for an isolated spherical pore are given by (Nemat-Nasser and Hori, 1993)

$$\Delta\mathcal{E}_{hydro} = \frac{P^2V}{2K_m}\left[\frac{3K_m + 4G_m}{4G_m}\right]\phi, \tag{3}$$

$$\Delta\mathcal{E}_{shear} = \frac{P^2V}{2G_m}\left[\frac{15K_m + 20G_m}{9K_m + 8G_m}\right]\phi. \tag{4}$$

If the energy terms (3) and (4) are used in (1) and (2), the effective moduli are predicted to be

$$\frac{K}{K_m} = \left[1 + \frac{3K_m + 4G_m}{4G_m}\phi\right]^{-1} = \left[1 + \frac{3(1-\nu_m)}{2(1-2\nu_m)}\phi\right]^{-1}, \tag{5}$$

$$\frac{G}{G_m} = \left[1 + \frac{15K_m + 20G_m}{9K_m + 8G_m}\phi\right]^{-1} = \left[1 + \frac{15(1-\nu_m)}{(7-5\nu_m)}\phi\right]^{-1}, \tag{6}$$

As these predictions ignore pore-pore interactions, they are correct only to first-order in ϕ, and increasingly overestimate the moduli as the porosity increases. This overestimation is also clear from the fact that they predict finite elastic moduli when the porosity reaches 100% (Fig. 10.1).

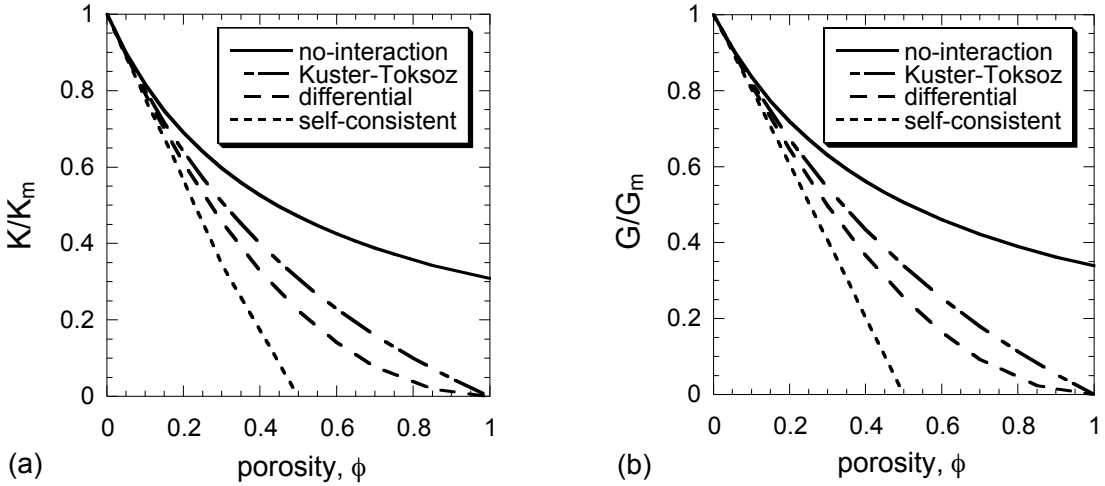

Fig. 10.1. Elastic moduli of a rock containing dry, randomly distributed spherical pores, according to various effective medium theories. The Poisson ratio of the intact rock is taken to be 0.25.

In the so-called "self-consistent" scheme of Hill (1965) and Budiansky (1965), the excess strain energy due to each single pore is calculated by assuming that it is introduced into a homogeneous medium that has the elastic properties of the actual porous material. This leads to the same functional forms for the two energy terms as does the "no-interaction" approach, except that $\{K_m, G_m\}$ are replaced by $\{K, G\}$. In general, (1) and (2) become two coupled nonlinear algebraic equations that require numerical solution. For spherical pores, the self-consistent method yields

$$\frac{1}{K} = \frac{1}{K_m} + \frac{1}{K}\left[\frac{3K + 4G}{4G}\right]\phi, \tag{7}$$

$$\frac{1}{G} = \frac{1}{G_m} + \frac{1}{G}\left[\frac{15K + 20G}{9K + 8G}\right]\phi. \tag{8}$$

This method yields much lower effective moduli than does the no-interaction method, and predicts that the moduli vanish at some finite porosity (Fig. 10.1).

Bruner (1976) suggested that the self-consistent method implicitly takes interactions between pairs of pores into account twice, since the "typical" pore is assumed to be imbedded in an effective medium whose elastic moduli already reflect in part the interactions between this typical pore and all other pores. To avoid this double-counting, one can introduce the pores into the rock sequentially, with pore $n+1$ considered to be placed into a homogeneous medium which has the effective elastic properties of the body with *n* pores, *etc*. In this way, pore $n+1$ feels the effect of pore *n*, but not *vice versa*. Each new pore is assumed to be randomly placed, and so if the "current" porosity is ϕ, the next pore "replaces" intact rock with probability 1-ϕ, and replaces existing pore space with probability ϕ (McLaughlin, 1977). In the limit in which each new addition of pores is infinitesimal, this method gives two coupled ordinary differential equations for the two effective moduli.

In the case of spherical pores, these equations are

$$-\frac{(1-\phi)}{K}\frac{dK}{d\phi} = \frac{3K + 4G}{4G}, \tag{9}$$

$$-\frac{(1-\phi)}{G}\frac{dG}{d\phi} = \frac{15K + 20G}{9K + 8G}. \tag{10}$$

The initial conditions are that $K = K_m$ and $G = G_m$ when $\phi = 0$. These equations can be integrated to give the following implicit expressions for the effective moduli (Norris, 1985):

$$\frac{G}{G_m} = (1-\phi)^2\left[\frac{1+\beta(G/G_m)^{3/5}}{1+\beta}\right]^{1/3}, \tag{11}$$

$$\frac{K}{K_m} = \frac{G}{G_m}\left[\frac{1+2\beta}{1+2\beta(G/G_m)^{3/5}}\right], \tag{12}$$

where $\beta = (1-5\nu_m)/2(1+\nu_m)$. In general, these two equations must be solved numerically for K and G. Two-digit accuracy can be achieved by substituting $G/G_m = (1-\phi)^2$ into the right-hand sides of (11) and (12).

The problem of computing the effective moduli of a porous rock can also be approached using a wave-scattering formalism; the relationships between elastic moduli and seismic wavespeeds are discussed in §5. Kuster and Toksöz (1974) calculated the sum of the elastic waves that have been scattered once from each of an assemblage of pores in a body with moduli $\{K_m, G_m\}$, and equated this to the wave that would be scattered from an "equivalent homogeneous spherical inclusion" whose moduli are equal to the effective moduli of the porous rock, $\{K, G\}$. Toksöz *et al*. (1976) and Wilkens *et al.* (1986) used this approach to model seismic velocities in reservoir sandstones, and Zimmerman and King (1986) used it to study the effect of the ice/water ratio on seismic velocities in permafrost. For a rock containing dry spherical pores, the method of Kuster and Toksöz gives (Fig. 10.1)

$$\frac{K}{K_m} = \frac{1-\phi}{1+(3K_m/4G_m)\phi} = \frac{1-\phi}{1+[(1+\nu_m)/2(1-2\nu_m)]\phi}, \tag{13}$$

$$\frac{G}{G_m} = \frac{1-\phi}{1+[(6K_m+12G_m)/(9K_m+8G_m)]\phi} = \frac{1-\phi}{1+[2(4-5\nu_m)/(7-5\nu_m)]\phi}. \tag{14}$$

The predicted moduli lie between those of the no-interaction method and the differential method, and vanish at a porosity of 100%. The Kuster-Toksöz predictions for a rock containing spherical pores also coincide *exactly* with the upper bounds of Hashin and Shtrikman (1961), which are valid regardless of the geometry of the pores.

The predictions of these four methods are plotted in Fig. 10.1, for $\nu_m = 0.25$. Each approach predicts similar, although in general not identical, behavior for G as it does for K. Consider the curves for the effective bulk modulus as a function of spherical porosity. If expanded in Taylor series in ϕ, all four methods agree to first-order, but give different values for the higher-order coefficients. The predictions diverge from each other markedly for porosities greater than about 0.10. As an indication of the validity of these approaches, consider the suite of porous glass specimens that were fabricated by Walsh *et al*. (1965) to have pores that were as nearly spherical as possible. The measured

bulk moduli, for porosities ranging from 0.05-0.70, generally fell about midway between the predictions of the Kuster-Toksöz and the differential schemes (Zimmerman, 1991, p. 120).

Each of the sets of predicted effective moduli take on a particularly simple form when $K_m = 4G_m/3$, which corresponds to $\nu_m = 0.2$. In this case the no-interaction method gives $K/K_m = G/G_m = (1+2\phi)^{-1}$, the self-consistent method gives $K/K_m = G/G_m = 1-2\phi$, the differential method gives $K/K_m = G/G_m = (1-\phi)^2$, and the method of Kuster and Toksöz gives $K/K_m = G/G_m = (1-\phi)/(1+\phi)$.

10.3. Effect of Cracks on the Elastic Moduli

In order to apply the effective moduli theories to a body with penny-shaped cracks, the two strain energy perturbation terms are needed. The energy perturbation $\Delta\mathcal{E}_{hydro}$ for a single penny-shaped crack of radius *a* under hydrostatic loading is given by §8.13 (4) of Jaeger *et al.* (2007):

$$\Delta\mathcal{E}_{hydro} = \frac{4(1-\nu_m)a^3P^2}{3G_m} = \frac{8(1-\nu_m^2)a^3P^2}{9(1-2\nu_m)K_m}. \qquad (1)$$

The strain energy perturbation $\Delta\mathcal{E}_{shear}$ for an isolated crack subjected to a shear stress of magnitude *S* will depend on the orientation of the shear stress with respect to the crack plane; this issue does not arise for hydrostatic loading, nor does it arise for shear loading in the case of a spherical pore. If the orientations of the cracks within the rock are randomly distributed, the strain energy must be averaged over all possible angles of inclination with respect to the direction of shear. The average excess strain energy per crack, for a random distribution of crack planes, is (see Jaeger *et al.,* 2007, §10.5 (60) for details):

$$\Delta\mathcal{E}_{shear} = \frac{16(1-\nu_m)(5-\nu_m)a^3S^2}{45(2-\nu_m)G_m}. \qquad (2)$$

According to the no-interaction scheme, the effective elastic moduli of a randomly/isotropically-cracked body can be found by inserting the energy perturbations (1) and (2) into the general expressions §10.2 (1) and (2), to yield

$$\frac{K}{K_m} = \left[1 + \frac{16(1-\nu_m^2)}{9(1-2\nu_m)}\Gamma\right]^{-1}, \tag{3}$$

$$\frac{G}{G_m} = \left[1 + \frac{32(1-\nu_m)(5-\nu_m)}{45(2-\nu_m)}\Gamma\right]^{-1}, \tag{4}$$

where $\Gamma = Na^3/V$ is the "crack density parameter", and N/V is the number of cracks per unit volume. The predictions of the self-consistent method are also found by inserting (1) and (2) into §10.2 (1) and (2), but with $\{K,G,\nu\}$ used in the excess energy terms. This leads to the following implicit expressions for K and G (O'Connell and Budiansky, 1974):

$$\frac{K}{K_m} = 1 - \frac{16(1-\nu^2)}{9(1-2\nu)}\Gamma, \tag{5}$$

$$\frac{G}{G_m} = 1 - \frac{32(1-\nu)(5-\nu)}{45(2-\nu)}\Gamma. \tag{6}$$

These equations can be partially inverted by using the identity $3K(1-2\nu) = 2G(1+\nu)$ to eliminate G and K, to arrive at

$$\Gamma = \frac{45(\nu_m - \nu)(2-\nu)}{16(1-\nu^2)(10\nu_m - 3\nu_m\nu - \nu)}. \tag{7}$$

After (7) is solved numerically for ν as a function of Γ, this value of ν can be used in (5) and (6) to find K and G. The curves of K and G as functions of Γ are somewhat nonlinear, but the effective Young's modulus that follows from (5) and (6) is very nearly a linear function of crack density. For all $0 < \nu_m < 1/2$, the following expression is accurate to within 1%:

$$\frac{E}{E_m} = 1 - \frac{16}{9}\Gamma. \tag{8}$$

The equations of the differential scheme, for a rock containing randomly distributed and oriented cracks, are (Salganik, 1973)

$$\frac{1}{K}\frac{dK}{d\Gamma} = -\frac{16(1-\nu^2)}{9(1-2\nu)}, \tag{9}$$

$$\frac{1}{G}\frac{dG}{d\Gamma} = -\frac{32(1-\nu)(5-\nu)}{45(2-\nu)}. \tag{10}$$

Using the initial conditions that $K = K_m$ and $G = G_m$ when $\Gamma = 0$, these equations can be integrated to yield (Zimmerman, 1991)

$$e^{\Gamma} = \left(\frac{3-\nu}{3-\nu_m}\right)^{5/128}\left(\frac{1-\nu}{1-\nu_m}\right)^{30/128}\left(\frac{1+\nu}{1+\nu_m}\right)^{45/128}\left(\frac{\nu}{\nu_m}\right)^{-80/128}, \tag{11}$$

$$\frac{K}{K_m} = \left(\frac{\nu}{\nu_m}\right)^{10/9}\left(\frac{3-\nu}{3-\nu_m}\right)^{-1/9}\left(\frac{1-2\nu}{1-2\nu_m}\right)^{-1}. \tag{12}$$

A simple and accurate approximate solution to (9) and (10) is (Bruner, 1976):

$$\frac{E}{E_m} = e^{-16\Gamma/9}, \qquad \frac{\nu}{\nu_m} = e^{-8\Gamma/5}. \tag{13}$$

Kuster and Toksöz (1974) presented equations for the effective moduli of a body containing oblate spheroidal pores of arbitrary aspect ratio, but did not explicitly consider the limiting case of infinitely thin cracks. If we examine their solution in the limit as the aspect ratio goes to zero, the effective moduli predicted by their method are found to be

$$\frac{K}{K_m} = \frac{1-[32(1+\nu_m)/27]\Gamma}{1+[16(1+\nu_m)^2/27(1-2\nu_m)]\Gamma}, \tag{14}$$

$$\frac{G}{G_m} = \frac{1-[32(5-\nu_m)(7-5\nu_m)/675(2-\nu_m)]\Gamma}{1+[64(5-\nu_m)(4-5\nu_m)/675(2-\nu_m)]\Gamma}. \tag{15}$$

The various predictions for the effective moduli of a rock containing a random distribution of cracks are plotted in Fig. 10.2. Note that the relative positions of the curves are not quite the same as for spherical pores; for cracks, the Kuster-Toksöz model predicts lower moduli than does the differential scheme. All four methods agree to first-order in

crack density, but begin to diverge appreciably for crack densities greater than about 0.15. The no-interaction method predicts that the moduli decay to zero roughly as $1/\Gamma$, whereas the differential method predicts a faster, exponential decay. Nevertheless, both of these methods predict substantial moduli values for crack densities as high as 1.0. The self-consistent method, on the other hand, predicts that the moduli both vanish at a crack density of 0.5625. The Kuster-Toksöz method also predicts that the moduli vanish at some finite crack density, although this critical density differs for *K* and *G*, and varies with the Poisson ratio of the intact rock.

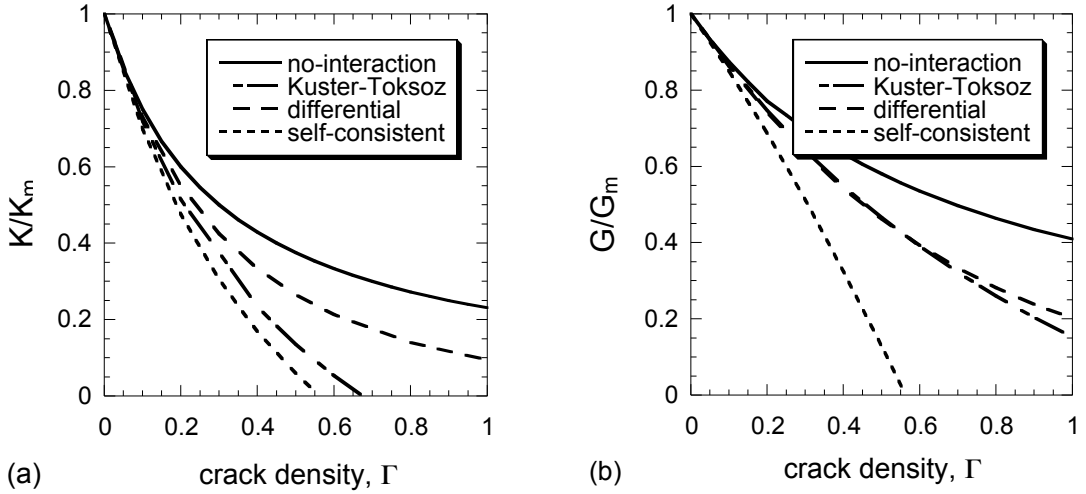

Fig. 10.2. (a) Effective bulk modulus and (b) effective shear modulus of a rock containing dry, randomly distributed and randomly oriented penny-shaped cracks, according to various effective medium theories. The Poisson ratio of the intact rock is taken to be 0.25.

All three energy-based methods predict that as the moduli decay to zero due to an increase in crack density, the effective Poisson's ratio goes to zero. However, as the Kuster-Toksöz method predicts that *K* decays faster than *G*, there is a range of crack densities for which this method predicts positive values for both of the elastic moduli, but a *negative* value for Poisson's ratio. For example, if $\nu_m = 0.25$, the Kuster-Toksöz method predicts negative values of the Poisson ratio for the range of crack densities $0.420 < \Gamma < 0.675$. This is clearly physically unrealistic.

Budiansky and O'Connell (1976) showed that the results for cracks with circular planforms can be applied to cracks having elliptical planforms, if the crack density is defined as $\Gamma = 2NA^2/P\pi V$, where *A* is the area of

the crack in its plane, and P is its perimeter. Various special cases of two and three-dimensional bodies containing systems of aligned cracks are discussed by Hashin (1988), Kachanov (1994), Nemat-Nasser and Hori (1993), and Mavko *et al*. (2009). The effective moduli of a body containing randomly-oriented needle-shaped pores are discussed by Berryman (1995).

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Homework Questions

As we discussed in the lectures, the coursework questions consist of filling in some missing steps in the derivations of some of the equations in the notes. In doing so, please make use of any already-derived equations in the notes.

1. In a one-dimensional elastic wave, prove that the stress and the strain satisfy the same partial differential equation as does the displacement, i.e., derive eq. (4) of p. 3.

2. Starting with eqs. (9) and (10) of. p. 5, derive the d'Alembert solution of the wave equation, eq. (11). If you can't derive it, then work backwards from eq. (11), and verify that it satisfies eqs. (9) and (10).

Hint: recall that $$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(s)ds\right] = f(b)b'(x) - f(a)a'(x).$$

3. Verify that reflected and transmitted waves, as given by expressions (16) and (17) on p. 7, contain the same total energy that was contained in the original incident wave. Hint: the energy of a one-dimensional wave is given by eq. (24) of p. 9.

4. Starting with eqs. (4-6) on pp. 33-34, for the strains in terms of the stresses, derive eqs. (9-11) of p. 35, which give the stresses in terms of the strains. Hint: adding up eqs. (4-6) will give you a useful relation between the sum of the three normal stresses and the sum of the three normal strains. Then, add and subtract $\nu\tau_{xx}/E$ from the right side of eq. (4), so that the term $\tau_{xx} + \tau_{yy} + \tau_{zz}$ appears, etc.

5. Derive eq. (8) on p. 41. Hint: recall the definition of ∇^2 given in eq. (9) on p. 38.

6. Prove the assertion made on p. 41: choosing $f''(\eta) = 0$ in eq. (9) leads to a rigid-body motion or a steady-state uniform strain, but not to a wave-like motion.

7. Using eqs. (18-20) on p. 36, and eq. (13) on p. 42, show that the compressional wave speed can also be written in the forms given by eq. (14) of p. 42.

8. Consider an SH-wave impinging on an interface between two rock types that have the same density but different shear moduli (i.e., different wavespeeds). Starting with the reflection coefficient given by eq. (11) on p. 51, find the angle of incidence for which the amplitude of the reflected wave is zero.

9. Starting with eqs. (20,21) on p. 62, derive Thomsen's approximate expression, eq. (27), for the speed of the quasi-shear wave.

10. Solve eq. (5) on p. 71 to find the real and imaginary parts of the wave number of the viscoelastic wave, as given by eqs. (6) and (7).

Please turn in your solutions to Shashi Luther by 5 pm, Monday, 31 January.